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EXISTENCE AND UNIQUENESS RESULTS FOR SOLUTIONS TO INITIAL VALUE PROBLEMS IN SCALES OF BANACH SPACES

A THESIS PRESENTED IN PARTIAL FULFILMENT FOR
THE DEGREE OF MASTER OF SCIENCE IN
MATHEMATICS AT
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Abstract

This thesis addresses existence and uniqueness of solutions to certain classes of initial-value problems with functional differential equations. The technique of scales of Banach spaces is used. A scale of Banach spaces is a collection of Banach spaces varying on a real parameter. A scale consisting of function spaces can be used to suppress one variable in an initial-value problem in a partial differential equation of two independent variables, therefore enabling local existence and uniqueness of a solution to the problem to be shown with the classical method of successive approximations from the Picard-Lindelöf Theorem of ordinary differential equations. Tuschke's presentation (c.f. [7]) of this technique and a related theorem has been adapted in Chapters 1 and 2. Chapters 3 and 4 present original theorems, stating existence and uniqueness of solutions to more general initial-value problems, having a retarded character.

Acknowledgements

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Contents

List of Figures

Chapter 1

Scales of Banach Spaces

1.1 Introduction

The existence and uniqueness of solutions for certain classes of differential equations are investigated in this thesis, using the technique of scales of Banach spaces. This technique has been extended to a certain class of functional partial differential equations. Related equations arise in geodesic problems, including some in optics. Work in this area has been done by Friedmann and McLeod [2], Rogers [4], and van-Brunt and Ockendon [8]. The differential equations investigated in these references have real-valued solutions depending on only one variable, and were treated without the use of scales of Banach spaces. The differential equation dealt with in Chapter 3, however, requires a complex valued solution depending on one real and one complex variable. The presence of derivatives on the right-hand side of the differential equation results, after a functional reformulation, in an unbounded operator on the right-hand side. A scale of Banach spaces is instrumental in dealing with such an unbounded operator.

Dependence on the complex variable is suppressed, and the solution is represented by a function valued in the scale of holomorphic function spaces, and depending on only the real variable. Then the method of successive approximations, from the standard proof of the Picard-Lindelöf theorem, can be applied. The method of successive approximations is illustrated in its simplest case in the immediately following section, since it will be used repeatedly throughout the rest of the thesis.

Existence and uniqueness results for functional differential equations in scales of

Banach spaces have been established by Sekine and Yamanaka [5], Walter [9], and Yamanaka and Tamaki [10]. The simplest existence and uniqueness result of this nature, generalising the Cauchy-Kowalevski theorem, appears in Tutschke [7], and has been included in Chapter 2 to illustrate the method of proof, much of which carries over into the existence and uniqueness proofs in Chapter 3. The differential equation investigated in Chapter 3 is notable in that retarded values of the timelike derivative are involved on the right-hand side. Chapter 4 extends the results in Chapter 3, by generalising the operator affecting the retardations appearing on the right-hand side of the differential equation.

1.2 The Picard-Lindelöf Theorem in a Banach Space

Techniques appearing in this thesis are developments on the method of successive approximations used in the standard proof of Picard's Theorem for ordinary differential equations (cf. [1]). Picard's Theorem concerns the initial value problem

$$\frac{dw}{dt}(t) = f(t, w(t)), \tag{1.1}$$

$$w(0) = w_0, \tag{1.2}$$

where w_0 is some prescribed real number. A solution w is a function mapping from some interval containing $0 \in R$ to R which satisfies both of these equations. If f is Lipschitz continuous on an open set around $(0, w_0)$, then the theorem states that there is a unique solution to initial value problem (1.1), (1.2) on some open interval containing 0. As the generalisation to the case in which the solution w is valued in a Banach space is straightforward, the method of successive approximations will be illustrated in this context.

Theorem 1 *Let B be a Banach space with norm $\|\cdot\|$. Let $w_0 \in B$, and suppose that f is a Lipschitz continuous mapping from a neighbourhood of $(0, w_0)$ into B. Then the initial-value problem (1.1), (1.2) has a unique solution on the interval $[0, T^*]$ for some positive number T^*.*

The initial-value problem (1.1), (1.2) is equivalent to the integral equation

$$w(t) = w_0 + \int_0^t f(\tau, w(\tau))d\tau. \tag{1.3}$$

Successive approximations are defined recursively by substitution into the right-hand side of this equation, that is, the approximations w_1, w_2, ... map the interval $[0, T]$ into B according to

$$w_1(t) = w_0 + \int_0^t f(\tau, w_0) d\tau \tag{1.4}$$

$$w_{k+1}(t) = w_0 + \int_0^t f(\tau, w_k(\tau)) d\tau. \tag{1.5}$$

These approximations are then shown to converge uniformly on some subinterval $[0, T^*]$, and the limit function w thus defined is shown to solve integral equation (1.3), and hence also initial-value problem (1.1), (1.2).

1.2.1 Proof of Existence

Initially the conditions in the statement of the theorem must be quantified. Suppose that the domain of f is the cylinder

$$[0, T] \times \{w \in B : \|w - w_0\| \leq R\}, \tag{1.6}$$

where T and R are positive numbers. Clearly T^* will be some positive number not exceeding T. Also, since f is Lipschitz continuous, there is a positive constant L such that whenever $t \in [0, T]$ and w, $v \in \{w \in B : \|w - w_0\| \leq R\}$,

$$\|f(t, w) - f(t, v)\| \leq L\|w - v\|. \tag{1.7}$$

Further, since f is continuous and the set $\{(t, w_0) : t \in [0, T]\}$ is compact, there is some positive number K such that

$$\|f(t, w_0)\| \leq K \tag{1.8}$$

whenever $t \in [0, T]$.

The initial-value problem (1.1), (1.2) is equivalent to the integral equation (1.3). Define the *successive approximations* $w_k : [0, T] \to B$, $k = 1, 2, \ldots$ by equations (1.4) and (1.5). Inequality (1.8) yields

$$\|w_1(t) - w_0\| \leq Kt \tag{1.9}$$

for $0 \leq t \leq T$, and therefore for $0 \leq t \leq T^*$.

If w_{k+1} is to be defined by equation (1.5), then the integrand on the right-hand side of that equation must be defined, so that it is necessary for τ to be no greater than T, as mentioned before, and similarly for $\|w_k(\tau) - w_0\| \leq R$. This will be satisfied if it can be shown that, for $k \geq 1$,

$$\|w_k(\tau) - w_{k-1}(\tau)\| \leq \epsilon_k R, \tag{1.10}$$

where $\{\epsilon_k\}$ is some sequence of positive numbers satisfying

$$\sum_{k=1}^{\infty} \epsilon_k \leq 1, \tag{1.11}$$

since then

$$\|w_k(\tau) - w_0\| \leq \|w_k(\tau) - w_{k-1}(\tau)\| + \cdots + \|w_1(\tau) - w_0\| \leq \sum_{j=1}^{k} \epsilon_k R \leq R. \tag{1.12}$$

Inequality (1.10) in the case $k = 1$ is implied by inequality (1.9) if Kt is restricted to be no greater than $\epsilon_1 R$, i.e., if t is restricted to be no greater than $\frac{\epsilon_1 R}{K}$; to this end choose $T^* \leq \frac{\epsilon_1 R}{K}$. In the general case $k = j > 1$, the definition (1.5) and the inequalities (1.7) and (1.10) give

$$\begin{aligned} \|w_j(t) - w_{j-1}(t)\| &\leq \left\| w_0 + \int_0^t f(\tau, w_{j-1}(\tau))d\tau - w_0 - \int_0^t f(\tau, w_{j-2}(\tau))d\tau \right\| \\ &\leq \int_0^t \|f(\tau, w_{j-1}(\tau)) - f(\tau, w_{j-2}(\tau))\| d\tau \\ &\leq L \int_0^t \|w_{j-1}(\tau) - w_{j-2}(\tau)\| d\tau \\ &\leq L \int_0^t \epsilon_{j-1} R d\tau \leq LT^* \epsilon_{j-1} R. \end{aligned} \tag{1.13}$$

Thus, inequality (1.10) with $k = j - 1$ implies the inequality (1.10) with $k = j$ if

$$\epsilon_j \geq LT^* \epsilon_{j-1}. \tag{1.14}$$

Since LT^* is independent of j, a geometric sequence is obtained by defining

$$\epsilon_j = LT^* \epsilon_{j-1}, \tag{1.15}$$

and the series (1.11) will converge if $LT^* < 1$. Merely choosing $\epsilon_1 \leq 1 - LT^*$ will yield inequality (1.11).

It has been proven that T^* can be chosen sufficiently small that the successive approximations $\{w_k\}$ are all defined on the interval $[0, T^*]$, and satisfy inequality (1.11). Inequality (1.10) implies that the sequence $\{w_k(t)\}$ is Cauchy for each $t \in [0, T^*]$. Since B is a Banach space, there must exist a limit

$$w(t) = \lim_{k \to \infty} w_k(t). \tag{1.16}$$

Now it remains to show that the limit function w thus formed is a solution to initial-value problem (1.1), (1.2), i.e., that integral equation (1.3) is satisfied. The left-hand side $w(t)$ of (1.3) is the limit of the points $w_k(t)$, and if its left-hand side, $w_0 + \int_0^t f(\tau, w(\tau))d\tau$, can also be shown to be the limit of the points $w_k(t)$, then w is a solution to initial-value problem (1.1), (1.2), and the existence result is thus established. By the definition (1.5),

$$\begin{aligned} &\left\| w_0 + \int_0^t f(\tau, w(\tau))d\tau - w_k(t) \right\| \\ &= \left\| w_0 + \int_0^t f(\tau, w(\tau))d\tau - \left(w_0 + \int_0^t f(\tau, w_{k-1}(\tau))d\tau \right) \right\| \\ &\leq \int_0^t \| f(\tau, w(\tau)) - f(\tau, w_{k-1}(\tau)) \| d\tau. \end{aligned} \tag{1.17}$$

By the Lipschitz condition,

$$\begin{aligned} \int_0^t \| f(\tau, w(\tau)) - f(\tau, w_{k-1}(\tau)) \| d\tau &\leq \int_0^t L \| w(\tau) - w_{k-1}(\tau) \| d\tau \\ &\leq Lt \sum_{j=k}^{\infty} \epsilon_j R. \end{aligned} \tag{1.18}$$

As $k \to \infty$, $\sum_{j=k}^{\infty} \epsilon_j R \to 0$; thus w is a solution.

■

1.2.2 Proof of Uniqueness

Suppose that w_1 and w_2 are two solutions to the initial-value problem (1.1), (1.2), both defined on the interval $[0, U]$. Here it will be demonstrated that, on the interval $[0, U]$, the two solutions w_1 and w_2 are equal. Since the solutions are both differentiable, they are *a fortiori* continuous, and therefore so is their difference; thus, for any $t_* \in (0, U]$, the supremum

$$\sup_{t \in [0, t_*]} \| w_1(t) - w_2(t) \| \tag{1.19}$$

is finite. As observed above, the initial-value problem (1.1), (1.2) is equivalent to the integral equation (1.3), so that the difference $w_1(t) - w_2(t)$ satisfies

$$w_1(t) - w_2(t) = \int_0^t \left(f(\tau, w_1(\tau)) - f(\tau, w_2(\tau)) \right) d\tau. \tag{1.20}$$

Applying the Lipschitz condition, if $t \leq t_*$, then

$$\begin{aligned} \| w_1(t) - w_2(t) \| &\leq \int_0^t L \| w_1(\tau) - w_2(\tau) \| d\tau \\ &\leq Lt \sup_{\tau \in [0, t_*]} \| w_1(\tau) - w_2(\tau) \|, \end{aligned} \tag{1.21}$$

so that if $Lt_* < 1$, then $\sup_{\tau\in[0,t_*]} \|w_1(\tau) - w_2(\tau)\|$ must vanish, i.e., $w_1(t) = w_2(t)$ on $[0, t_*]$. Dividing the interval $[0, U]$ into subintervals of length smaller than $\frac{1}{L}$, it can now inductively be shown that $w_1(t) = w_2(t)$ on the whole interval $[0, U]$.
■

1.3 Complex Differentiation as an Unbounded Operator

Consider the initial-value problem

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial z}, \tag{1.22}$$
$$w(0, z) = w_0(z). \tag{1.23}$$

The solution w will be a complex-valued function, differentiable with respect to the real independent variable t, and holomorphic with respect to the complex independent variable z. The initial condition w_0 is a holomorphic function defined on the disc

$$\Omega = \{z \in C : |z| \leq s_0\},$$

where s_0 is some positive number. Let B be the Banach space of all holomorphic functions defined on the disc Ω, with the norm

$$\|f\| = \sup_{z\in\Omega} |f(z)|.$$

Let D denote the complex differentiation operator $\frac{d}{dz}$ on B, that is, whenever $f \in B$, let

$$(Df)(z) = \frac{df}{dz}(z)$$

for each $z \in \Omega$. Then the initial value problem (1.22), (1.23) can be reformulated as the following initial-value problem in the space B:

$$w'(t) = D(w(t)) \tag{1.24}$$
$$w(0) = w_0. \tag{1.25}$$

Since, for any given t, the dependence of w upon z is described by the one point $w(t)$ in the function space B, the variable z in equations (1.22), (1.23) can be suppressed in this formulation.

The reformulation of initial-value problem (1.22), (1.23) as (1.24), (1.25) does not rely on the properties of R, C, or holomorphic functions. If U, V and W are arbitrary sets, and $f : \mathsf{U} \times \mathsf{V} \to \mathsf{W}$, and Π is the set of functions mapping V into W, then we can naturally define a function $F : \mathsf{U} \to \Pi$ representing f, by letting $F(u)$ be the function which maps v into $f(u,v)$, whenever $u \in \mathsf{U}$ and $v \in \mathsf{V}$, i.e.,

$$(F(u))(v) = f(u,v). \tag{1.26}$$

This is the approach taken here to initial-value problem (1.22), (1.23); the solution is naturally a function of two variables, but can be expressed as a function of one variable and the method of successive approximations can thus be applied.

The initial-value problem (1.24), (1.25) is in the form of problem (1.1), (1.2), so that, were the operator D Lipschitz continuous, the Picard-Lindelöf theorem in a Banach space would apply, ensuring, locally, the existence of a unique solution to the initial-value problem. However, D is *not* Lipschitz continuous. To see this, suppose L is a Lipschitz constant for D, satisfying, for any two holomorphic functions f and g,

$$\|Df - Dg\| \leq L\|f - g\|, \tag{1.27}$$

that is,

$$\sup\left\{\frac{df}{dz}(z) - \frac{dg}{dz}(z) : |z| \leq s_0\right\} \leq L \sup\{f(z) - g(z) : |z| \leq s_0\}. \tag{1.28}$$

The functions f and g are arbitrary, and the operation of differentiation is linear, so that inequality (1.28) must be satisfied for the function $h = f - g$ holomorphic on Ω. Now any hypothetical Lipschitz constant L would have to satisfy

$$\|Dh\| \leq L\|h\|, \tag{1.29}$$

that is,

$$\sup\left\{\frac{dh}{dz}(z) : |z| \leq s_0\right\} \leq L \sup\{h(z) : |z| \leq s_0\}. \tag{1.30}$$

(In the language of functional analysis, the existence of such a finite positive constant L would imply that the operator D is bounded.) Let Λ be any integer greater than Ls_0, and suppose

$$h(z) = \left(\frac{z}{s_0}\right)^{\Lambda}$$

for z in the disc (1.3). Then

$$Dh(z) = \frac{dh}{dz}(z) = \frac{\Lambda}{s_0}\left(\frac{z}{s_0}\right)^{\Lambda-1}.$$

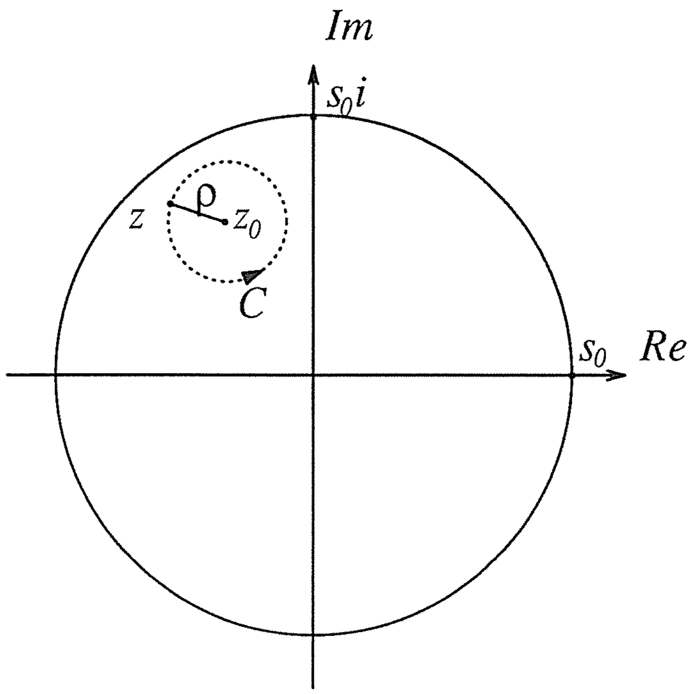

Figure 1.1: A nonuniform bound on the derivative of a holomorphic function

Clearly $L \sup\{h(z) : |z| \leq s_0\} = L$, but $\sup\left\{\frac{dh}{dz}(z) : |z| \leq s_0\right\} = \frac{\Lambda}{s_0} > L$, so that inequality (1.30) cannot be satisfied. Thus, the right-hand side of differential equation (1.24) is not Lipschitz continuous with respect to the argument $w(t)$, and consequently Picard's Theorem cannot be applied directly to this initial-value problem.

It is useful to consider in detail the unbounded nature of the operator D. A nonuniform bound describing the behaviour of D can be derived from the Cauchy integral formula. Let f be some holomorphic function on the disc (1.3), and let z_0 be some point on the interior of the disc (see Figure (1.1)). The distance from z_0 to the boundary of the disc is $s_0 - |z_0|$, so if $0 < \rho < s_0 - |z_0|$, then the circle C of radius ρ with centre z_0 lies entirely within the disc (1.3). The Cauchy integral formula states that

$$\frac{df}{dz}(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)dz}{(z - z_0)^2}, \tag{1.31}$$

and leads immediately to

$$\left|\frac{df}{dz}(z_0)\right| \leq \frac{1}{2\pi} \frac{\|f\|}{\rho^2} 2\pi\rho = \frac{\|f\|}{\rho}. \tag{1.32}$$

Letting $\rho \to s_0 - |z_0|$ gives

$$\left|\frac{df}{dz}(z_0)\right| \leq \frac{\|f\|}{s_0 - |z_0|}. \tag{1.33}$$

This inequality is known as Nagumo's Lemma (*c.f.* Tutschke [7]).

1.4 Scales of Banach Spaces

In the last section, it was shown that the complex differentiation operator D, mapping B, the space of functions holomorphic on the disc $\{z \in C : |z| \leq s_0\}$, into itself, is unbounded, so that the Picard-Lindelöf theorem in a Banach space cannot be applied directly to initial-value problem (1.24), (1.25); however, a nonuniform bound on D was found, viz. inequality (1.33). The use of this bound requires the definition of a scale of Banach spaces. Instead of just the one Banach space B, for each $r \in (0, r_0)$ define B_r to be the space of bounded functions holomorphic on the disc $\{z \in C : |z| < r\}$, with the norm

$$\|f\|_r = \sup\{|f(z)| : |z| < r\}.$$

Now let $0 < s < r < s_0$. An argument similar to that in the last section shows that if $f \in B_r$ and $z_0 \in C$ satisfies $|z_0| < s$, then

$$\left|\frac{df}{dz}(z_0)\right| \leq \frac{\|f\|_r}{r - s}, \tag{1.34}$$

that is,

$$\|Df\|_s \leq \frac{\|f\|_r}{r - s}.$$

In the language of functional analysis, D is a bounded operator from B_r to B_s with norm no greater than $\frac{1}{r-s}$.

The collection of these spaces $\{B_r : 0 < r < s_0\}$ forms what is called a *scale of Banach spaces.* For $0 < s < r < s_0$, any function holomorphic on the disc $\{z \in C : |z| < r\}$ is certainly holomorphic on the disc $\{z \in C : |z| < s\}$, and the domain restriction operator

$$I_{r,s} : B_r \to B_s$$

can be defined by

$$I_{r,s} f(z) = f(z) \qquad \forall |z| < s$$

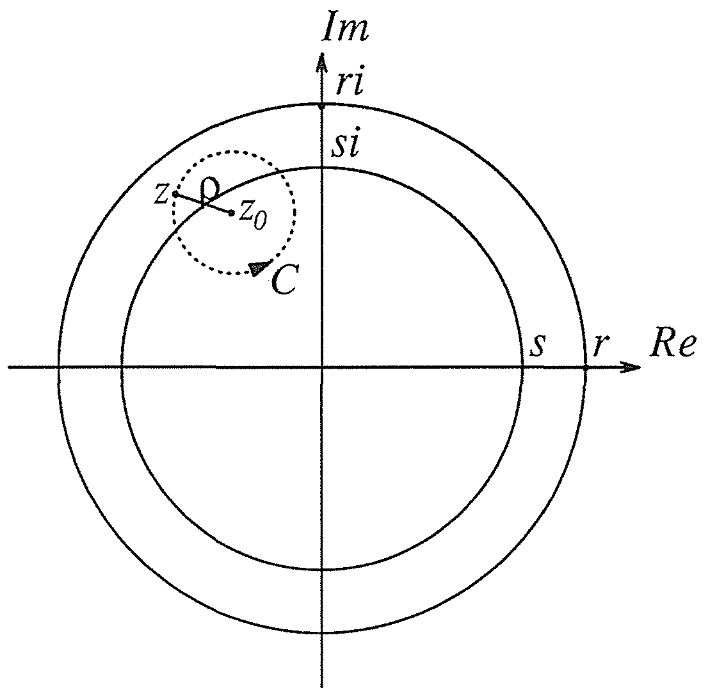

Figure 1.2: Complex differentiation as a bounded operator between two function spaces

whenever $f \in B_r$. Thus the action of $I_{r,s}$ is merely to shrink the domain of f from $\{z \in C : |z| < r\}$ to $\{z \in C : |z| < s\}$. This is clearly a linear operator with $||I_{r,s}|| \leq 1$, where

$$||I_{r,s}|| = \sup\{||I_{r,s}f||_s : ||f||_r = 1\}.$$

Note also that the domain restriction operator $I_{r,s}$ is an injection, that is, if f and g are holomorphic functions belonging to B_r, such that $I_{r,s}f = I_{r,s}g$, then $f = g$, since holomorphic functions identically equal on an open set are equal throughout the intersection of their domains.

Generalising the scale $\{B_r\}$, we define a *scale of Banach spaces* to be a set of Banach spaces $\{H_r : 0 < r < s_0\}$, each space H_r having norm $|| \cdot ||_r$, together with a set of *scale operators* $\{J_{r,s} : 0 < s < r < s_0\}$, each linear, injective, and satisfying $||J_{r,s}|| \leq 1$. Then the set

$$\{B_r : 0 < r < s_0\}$$

of function spaces defined above is a scale of Banach spaces, and the set of domain restiction operators $I_{r,s}$ are the scale operators.

Chapter 2

Initial Value Problems in Scales of Banach Spaces

2.1 Mappings into Scales of Banach Spaces

In the last chapter the reformulation of initial-value problem (1.22), (1.23), using the function space B to suppress the complex variable z, was considered. The inital-value problem (1.24), (1.25) resulting cannot immediately be shown to have a solution using the method of successive approximations, because the operator D, representing complex differentiation, is unbounded. However, with the scale of function spaces B_r, a more sophisticated approach can be considered, viz. to seek a solution to initial-value problem (1.24), (1.25) valued in a scale of Banach spaces. A solution whose domain varies in width, or extent in the z-plane, as t changes, can thus be considered. Specifically, it is reasonable to expect the extent in the z-plane over which existence can be proven to contract as t increases, yielding a conical domain. That is, the desired solution $w(t)$ will be defined for $t \in [0, T^*]$ for some positive T^*, but for each given t, the point $w(t)$ will belong only to some of the function spaces B_r. Figure (2.1) illustrates this geometry. If $w(t) \in B_r$ whenever $0 \leq t < a(s_0 - r)$, then $w(t, z)$ is specified for each (t, z) lying in the cone, where $T^* = as_0$, whereas, if only the one function space B_{s_0} is defined, and $w(t)$ is required to belong to B_{s_0} whenever $0 \leq t < T^*$, then $w(t, z)$ is specified in the cylinder. Intuitively, existence and uniqueness of a solution within the cone are more plausible than within the cylinder, which contains the cone.

An alternative and slightly less gesticulative description of what is meant by a

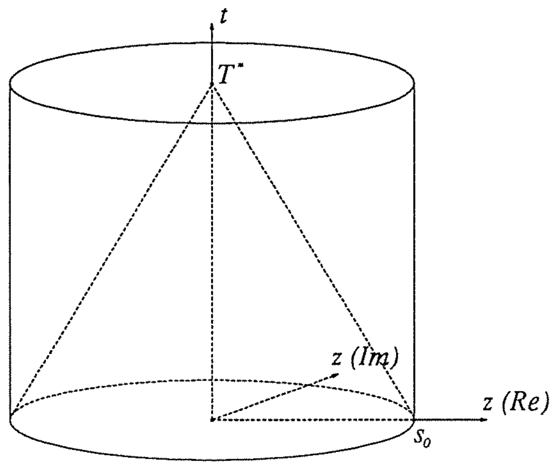

Figure 2.1: A scale of Banach spaces allows a conical domain to be considered

mapping into a Banach space is given for precision. Let $\{H_r : 0 < r < s_0\}$ be a scale of Banach spaces with scale operators $\{J_{r,s} : 0 < s < r < s_0\}$. For each $r \in (0, s_0)$, let w_r be a mapping from some interval Υ_r, possibly empty, into H_r. Suppose $0 < s < r < s_0$. If $\Upsilon_r \subseteq \Upsilon_s$ and, for each $t \in \Upsilon_r$,

$$w_s(t) = J_{r,s} w_r(t),$$

then w_r and w_s are said to be *compatible.* If w_r and w_s are compatible whenever $0 < s < r < s_0$, then the collection of mappings $\{w_r : 0 < r < s_0\}$ is said to constitute a *mapping into the scale of Banach spaces* $\{H_r\}$. The condition of compatibilty arises naturally in the case at hand, where the Banach spaces in the scale are the function spaces B_r. If w_r and w_s represent the same function $w(t, z)$, and $w_r(t)$ is defined and $|z| < r$, then

$$w(t, z) = (w_r(t))(z).$$

If also $|z| < s$, then

$$w(t, z) = (w_s(t))(z),$$

so that

$$(w_s(t))(z) = (w_r(t))(z).$$

In terms of the scale operator $I_{r,s}$, this can be written

$$w_s(t) = I_{r,s} w_r(t).$$

2.2 Generalised Cauchy-Riemann Operators

Let

$$\{H_r : 0 < r < s_0\}$$

be an arbitrary scale of Banach spaces with scale operators $\{J_{r,s} : 0 < s < r < s_0\}$, and consider the initial-value problem

$$u_s'(t) = F_{r,s}(t, u_r(t)) \tag{2.1}$$

$$u_s(0) = u_{0,s}. \tag{2.2}$$

Here u is to be valued in the scale $\{H_r\}$, in the sense described in the last paragraph, and specifically, u_s is valued in the space H_s. Whenever $0 < s < r < s_0$, the operator $F_{r,s}$ takes one real argument and one argument from the space H_r, and has a value in the space H_s. If this equation is to be soluble, it can immediately be seen that a compatibility condition must be met by the operators $F_{r,s}$. Let $0 < s < r < p < s_0$, so that

$$u_s'(t) = F_{r,s}(t, u_r(t)) = F_{r,s}(t, J_{p,r}u_p(t))$$

and

$$u_s'(t) = F_{p,s}(t, u_p(t)).$$

Thus it is required that

$$F_{r,s}(t, J_{p,r}u_p(t)) = F_{p,s}(t, u_p(t)). \tag{2.3}$$

As regards the initial condition (2.2), it is clearly required that

$$u_{0,s} = J_{r,s}u_{0,r}.$$

For brevity, then, the functionals $F_{r,s}$ will not be distinguished, and neither will the initial condition functions $u_{0,s}$ be, so that the initial-value problem will be written

$$u'(t) = F(t, u(t)) \tag{2.4}$$

$$u(0) = u_0. \tag{2.5}$$

If $\{H_r\}$ is the scale $\{B_r\}$ of holomorphic function spaces, then the conditions imposed here on the right-hand side operator F will allow for retardation-type displacements in the complex variable. Choose any $z_0 \in C$ with $|z_0| < s_0$. For every $s \in (|z_0|, s_0)$, differential equation (2.4) specifies $\frac{\partial u}{\partial t}(t, z_0)$ in terms of t and $u_r(t)$, where $r \in (s, s_0)$ is chosen arbitrarily. Since $u_r(t)$ encapsulates all of the values $u(t,z)$ such that $|z| < r$, to calculate $\frac{\partial u}{\partial t}(t, z_0)$, one must know the value of t and also of $u(t,z)$ for each $|z| < r$. However, one is free to choose r as any number exceeding $|z_0|$, so that any dependence of $\frac{\partial u}{\partial t}(t, z_0)$ on $u(t,z)$, if $|z_0| < |z|$, is spurious, only appearing formally. If any of the values $u(t,z)$ for $|z| < |z_0|$ are unknown, however, then $\frac{\partial u}{\partial t}(t, z_0)$ cannot be calculated, unless such values are first obtained using analytic continuation. Even if this approach is taken, it is necessary for *some* values $u(t,z)$ to be known. Such values are described as retarded due to the inequality $|z| < |z_0|$.

The property of example differential equation (1.24) which will allow proof of local existence and uniqueness of a solution to an initial-value problem is that the right-hand side $Dw(t)$ obeys Nagumo's Lemma, that is, that

$$\|D(w_1 - w_2)\|_s \leq \frac{\|w_1 - w_2\|_r}{r - s} \tag{2.6}$$

whenever $0 < s < r < s_0$ and $w_1, w_2 \in B_r$. Take the initial-value problem (2.4), (2.5), with the conditions that the operator F on the right-hand side is continuous with respect to t, and, generalising inequality (2.6), satisfies

$$\|F(t, u_1) - F(t, u_2)\|_s \leq C \frac{\|u_1 - u_2\|_r}{r - s} \tag{2.7}$$

whenever $0 < s < r < s_0$ and $u_1, u_2 \in H_r$, for some positive number C independent of t, u_1, and u_2. Because inequality (2.7) arises from the properties of complex differentiation, an operator satisfying this inequality is called a *generalised Cauchy-Riemann operator*. Proofs of local existence and uniqueness of a solution to initial-value problem (2.4), (2.5) now follow, adapted from similar material appearing in Tutschke (*op. cit.*).

Theorem 2 *Let $\{H_r : 0 < r < s_0\}$ be a scale of Banach spaces. Suppose there exist positive constants T and R such that, whenever $0 < s < r < s_0$, the right-hand side $F(t, u)$ of differential equation (2.4) is a continuous function from*

$$[0, T] \times \Theta_r \tag{2.8}$$

to H_s, where

$$\Theta_r = \{u \in H_r : \|u - u_0\|_r \leq R\}.$$

Suppose further that $F(t,u)$ is a generalised Cauchy-Riemann operator with respect to u, that is, that there is some positive constant C such that inequality (2.7) holds whenever $0 < s < r < s_0$ and u_1, $u_2 \in \Theta_r$. Moreover, assume that, for each $t \in [0,T]$, the function $F(t,u_0)$ satisfies the asymptotic bound

$$\|F(t,u_0)\|_r \leq \frac{K}{s_0 - r}, \tag{2.9}$$

for each $r \in (0, s_0)$, where K is some positive constant. Then there exists a unique solution u to initial-value problem (2.4), (2.5) on some conical domain of positive height, that is, there is a positive constant a and, for each $r \in (0, s_0)$, a unique function u_r mapping the interval $[0, a(s_0 - r)]$ into the space H_r such that u_r is a solution to initial-value problem (2.4), (2.5), and the functions $\{u_r\}$ are compatible.

2.3 Proof of Existence

This proof uses the method of successive approximations illustrated in Section 1.2. The initial-value problem (2.4), (2.5) is equivalent to the integral equation

$$u(t) = u_0 + \int_0^t F(\tau, u(\tau))d\tau. \tag{2.10}$$

In view of this, we define the successive approximations u_1, u_2, ... recursively:

$$\begin{aligned} u_1(t) &= u_0 + \int_0^t F(\tau, u_0)d\tau, & (2.11)\\ u_k(t) &= u_0 + \int_0^t F(\tau, u_{k-1}(\tau))d\tau, & (2.12) \end{aligned}$$

for each $k = 2, 3, \ldots$. This is analogous to equation (1.3) and definitions (1.4) and (1.5) in the case of the Picard-Lindelöf theorem for ordinary differential equations. The approximations u_k may not all be defined everywhere, and where they are, they may not belong to each of the spaces H_r. Certainly for $u_k(t)$ to be defined, it is required that $0 < t < T$, since the integrand defining $u_k(t)$ in equation (2.12) is undefined for $\tau \notin [0,T]$. The integrand will also be undefined if $u_{k-1}(\tau)$ lies outside the ball Θ_r. Note that this condition depends on r, so that if $0 < s < r < s_0$, it may happen that $u_k(t) \in H_s$ but $u_k(t) \notin H_r$, that is, if $\{u_{k,r} : 0 < r < s_0\}$ are the compatible functions constituting the function u_k, it may be that $u_{k,s}(t)$ is defined but $u_{k,r}(t)$ is not.

These approximations will be shown to converge absolutely and uniformly on the conical domain described above, that is, it will be shown that there exists a convergent series

$$\sum_{k=1}^{\infty} \omega_k = \Omega < \infty \tag{2.13}$$

of positive terms such that, for each $k \in N$, if $0 < r < s_0$ and $0 \leq t < a(s_0 - r)$, the inequality

$$\|u_k(t) - u_{k-1}(t)\|_r \leq \omega_k \tag{2.14}$$

holds (read u_0 for $u_{k-1}(t)$ in the case $k = 1$). From this follows the existence of a limit function u. To see this, let $0 < r < s_0$ and $0 \leq t < a(s_0 - r)$. The sequence $\{u_{k,r}(t)\}$ is Cauchy in H_r, since, assuming without loss of generality that $m < n$,

$$\begin{aligned} \|u_{n,r}(t) - u_{m,r}(t)\|_r &\leq \sum_{k=m+1}^{n} \|u_{k,r}(t) - u_{k-1,r}(t)\|_r \\ &\leq \sum_{k=m+1}^{\infty} \omega_k, \end{aligned}$$

and since

$$\sum_{k=1}^{\infty} \omega_k$$

is convergent,

$$\sum_{k=m+1}^{\infty} \omega_k \to 0$$

as $m \to \infty$. Thus, as H_r is complete, the sequence $\{u_{k,r}(t)\}$ converges to a limit $u_r(t) \in H_r$, that is,

$$\|u_{k,r}(t) - u_r(t)\|_r \to 0.$$

For fixed r, as t varies, we obtain a function u_r mapping the interval $[0, a(s_0 - r))$ into the space H_r. For this to define a function u mapping into the scale $\{H_r\}$ as r varies, the two functions u_s and u_r must be compatible, where $0 < s < r < s_0$. Clearly $[0, a(s_0 - r)) \subset [0, a(s_0 - s))$. Let $t \in [0, a(s_0 - r))$. For each $k \in N$,

$$\begin{aligned} \|u_s(t) - I_{r,s}u_r(t)\|_s &\leq \|u_s(t) - u_{k,s}(t)\|_s + \|u_{k,s}(t) - I_{r,s}u_{k,r}(t)\|_s \\ &\quad + \|I_{r,s}u_{k,r}(t) - I_{r,s}u_r(t)\|_s \\ &\leq \|u_s(t) - u_{k,s}(t)\|_s + 0 + \|u_{k,r}(t) - u_r(t)\|_r \\ &\to 0 \end{aligned}$$

as $k \to \infty$. Thus u_s and u_r are in fact compatible, and the limit functions $\{u_r : 0 < r < s_0\}$ define a limit function u mapping into the scale $\{H_r\}$.

The limit function u solves integral equation (2.10) and therefore initial-value problem (2.4), (2.5). If $0 < r < s_0$ and $0 \leq t < a(s_0 - r)$, then a number q can be chosen such that $0 < r < q < s_0$ and $0 \leq t < a(s_0 - q)$, for instance,

$$q = \frac{1}{2}\left(s_0 - \frac{t}{a} + r\right).$$

Then

$$\begin{aligned}
&\left\| u(t) - \left(u_0 + \int_0^t F(\tau, u(\tau))d\tau\right)\right\|_r \\
&\quad\leq \|u(t) - u_{k+1}(t)\|_r + \left\| u_{k+1}(t) - \left(u_0 + \int_0^t F(\tau, u_k(\tau))d\tau\right)\right\|_r \\
&\quad+ \left\| \left(u_0 + \int_0^t F(\tau, u_k(\tau))d\tau\right) - \left(u_0 + \int_0^t F(\tau, u(\tau))d\tau\right)\right\|_r \\
&\quad= \|u(t) - u_{k+1}(t)\|_r + \left\| \int_0^t \left(F(\tau, u_k(\tau)) - F(\tau, u(\tau))\right) d\tau \right\|_r \\
&\quad\leq \|u(t) - u_{k+1}(t)\|_q + \int_0^t \frac{\|u(\tau) - u_k(\tau)\|_q}{q - r} d\tau \\
&\quad\leq \sum_{j=k+2}^{\infty} \omega_j + \frac{t}{q - r} \sum_{j=k+1}^{\infty} \omega_j \to 0
\end{aligned}$$

as $k \to \infty$. Thus $u(t)$ and

$$u_0 + \int_0^t F(\tau, u(\tau))d\tau$$

are equal in H_r, and hence represent the same holomorphic function, so that integral equation (2.10) is solved. It has now been shown that if inequalities (2.13) and (2.14) can be satisfied, then the successive approximations u_k converge absolutely and uniformly to a function u solving initial-value problem (2.4), (2.5).

From the definition of the first approximation u_1 an idea of the shapes of the domains of the successive approximations can be obtained. Of course since the u argument of the operator F in the integrand in equation (2.11) is always simply u_0, the function u_1 may be defined and belongs to H_r whenever $0 < r < s_0$ and $t \in [0, T]$. From equation (2.11) and property (2.9),

$$\|u_1(t) - u_0\|_r \leq \int_0^t \|F(\tau, u_0)\| d\tau \leq \frac{Kt}{s_0 - r}. \tag{2.15}$$

Thus if a_1 is some positive number, and r and t are such that

$$0 \leq t < a_1(s_0 - r), \tag{2.16}$$

then

$$\|u_1(t) - u_0\|_r \leq K a_1. \tag{2.17}$$

If, further,

$$a_1 \leq \frac{T}{s_0}, \tag{2.18}$$

then the restriction (2.16) implies $0 \leq t < a_1(s_0 - r)$. If, further still, $Ka_1 < R$, then whenever $0 < r < s_0$, $0 \leq t < a_1(s_0 - r)$, and $0 \leq \tau \leq t$,

$$u_2(t) = u_0 + \int_0^t (\tau, u_1(\tau))d\tau, \tag{2.19}$$

and $u_1(\tau)$ in the integrand lies within the ball Θ_r, so that $u_2(t) \in H_s$ for any positive $s < r$.

Extrapolating, require a sequence $\{a_k\}$ such that not only is $u_k(t)$ defined and an element of H_r whenever $0 < r < s_0$ and $0 \leq t < a_k(s_0 - r)$, but also

$$\|u_k(t) - u_0\|_r < R. \tag{2.20}$$

Now, if the sequence $\{a_k\}$ is strictly decreasing, then

$$0 \leq t < a_{k+1}(s_0 - s) \quad \Rightarrow \quad \exists\, r \in (s, s_0) : 0 \leq t < a_k(s_0 - r), \tag{2.21}$$

so that if $0 < s < s_0$ and $0 \leq t < a_{k+1}(s_0 - s)$ then certainly $u_{k+1}(t)$ is defined and belongs to H_s, and considerations may be restricted to ensuring that $\|u_{k+1}(t) - u_0\|_s < R$. Inequality (2.20) will be satisfied if

$$\|u_k(t) - u_{k-1}(t)\|_r + \|u_{k-1}(t) - u_{k-2}(t)\|_r + \cdots + \|u_1(t) - u_0\|_r < R, \tag{2.22}$$

and thus we will require the existence of a sequence of positive reals $\{\epsilon_k\}$ such that for each $k \in N$, whenever $0 < r < s_0$ and $0 \leq t < a_k(s_0 - r)$,

$$\|u_k(t) - u_{k-1}(t)\|_r \leq \epsilon_k R, \tag{2.23}$$

and

$$\sum_{k=1}^{\infty} \epsilon_k \leq 1. \tag{2.24}$$

Note that inequality (2.17) implies inequality (2.23) in the case $k = 1$, provided that

$$a_1 \leq \frac{\epsilon_1 R}{K}. \tag{2.25}$$

Since the terms ϵ_n are all positive, inequality (2.24) implies

$$\sum_{i=1}^{n} \epsilon_i < 1 \tag{2.26}$$

for each positive integer n. Since the sequence $\{a_k\}$ is decreasing, inequality (2.23) holds whenever $k \leq n$ and $0 < r < s_0$ and $0 \leq t < a_n(s_0 - r)$. Thus

$$\begin{aligned} \|u_k(t) - u_0\|_r &< \sum_{i=1}^{k} \|u_i(t) - u_{i-1}(t)\|_r \\ &\leq \sum_{i=1}^{k} \epsilon_i R \\ &< R, \end{aligned}$$

so that inequality (2.20) is satisfied for each k.

Inequality (2.23) has another important consequence. Because $\{a_k\}$ is a decreasing sequence of positive numbers, it has a nonnegative limit a. Assume for the moment that a is positive. Since, for each k, $a < a_k$, inequality (2.23) will hold whenever $0 < r < s_0$ and $0 \leq t < a(s_0 - r)$. If $\omega_k = \epsilon_k R$, inequalities (2.13) and (2.14) are satisfied; thus the approximations u_1, u_2, ... converge to a limit function u which, as shown earlier, must be a solution to initial-value problem (2.4), (2.5). Therefore, the existence result will be established if it can be shown that the inequality

$$a = \lim_{k \to \infty} a_k > 0, \tag{2.27}$$

as well as, whenever $0 < r < s_0$ and $0 \leq t < a_k(s_0 - r)$, inequality (2.23), and inequality (2.24) are all satisfied.

For the present, it will be assumed merely that $\{\epsilon_k\}$ and $\{a_k\}$ are sequences of positive numbers, and that the sequence $\{a_k\}$ is strictly decreasing and bounded above by inequality (2.18). Inequality (2.23) will be established, dependent on certain relationships between the sequences $\{\epsilon_k\}$ and $\{a_k\}$, and it will subsequently be shown that such relationships can be satisfied for appropriate choices of the sequences $\{\epsilon_k\}$ and $\{a_k\}$ which also satisfy inequalities (2.24) and (2.27).

Inequality (2.23) will be established by induction, but not directly. A direct approach might begin by assuming that inequality (2.23) holds for $k = l$, so that

$$\begin{aligned} \|u_{l+1}(t) - u_l(t)\|_r &= \left\| \int_0^t (F(\tau, u_l(\tau)) - F(\tau, u_{l-1}(\tau)) d\tau \right\|_r \\ &\leq \int_0^t \|F(\tau, u_l(\tau)) - F(\tau, u_{l-1}(\tau)\|_r d\tau \\ &\leq \int_0^t \frac{C}{q-r} \|u_l(t) - u_{l-1}(t)\|_q d\tau \qquad \text{(by (2.7))} \\ &\leq \frac{Ct\epsilon_l R}{q-r}, \end{aligned} \tag{2.28}$$

where q is chosen such that $0 < r < q < s_0$ and $0 \leq t < a_l(s_0 - q)$. Any such q is less than $s_0 - \frac{t}{a_l}$, and t may be as large as $a_{l+1}(s_0 - r)$, so that the last expression in calculation (2.28) becomes at least as large as

$$\frac{Ca_{l+1}(s_0 - r)e_l R}{s_0 - \frac{a_{l+1}(s_0-r)}{a_l} - r} = \frac{Ca_{l+1}a_l\epsilon_l R}{a_l - a_{l+1}}.$$

If this is to imply inequality (2.23) for $k = l+1$, then ϵ_{l+1} must be defined in such a way that

$$\frac{Ca_{l+1}a_l\epsilon_l R}{a_l - a_{l+1}} \leq \epsilon_{l+1}R,$$

that is,

$$\frac{\epsilon_{l+1}}{\epsilon_l} \geq \frac{Ca_{l+1}a_l}{a_l - a_{l+1}}.$$

But, if the sequence $\{a_k\}$ converges to a positive number, then this fraction must tend to infinity, and therefore, if the numbers ϵ_k are positive, then $\epsilon_k \to \infty$, so that inequality (2.24) breaks down.

Earlier it was seen that the first approximation u_1 satisfies inequality (2.15); thus

$$\|u_1(t) - u_0\|_r \leq \frac{Kt}{s_0 - r} \leq \frac{Kt}{s_0 - r - \frac{t}{a_1}} = \frac{Ka_1 t}{a_1(s_0 - r) - t}, \tag{2.29}$$

whenever $0 < r < s_0$ and $0 \leq t < a_1(s_0 - r)$. The inequality

$$\|u_k(t) - u_{k-1}(t)\|_r \leq \frac{M_k t}{a_k(s_0 - r) - t} \tag{2.30}$$

is generalised from this for the case of u_k, where $\{M_k\}$ is some sequence of positive numbers yet to be specified. When $k = 1$, u_0 should be read for $u_{k-1}(t)$, and

$$M_1 = Ka_1. \tag{2.31}$$

Inequality (2.30) provides what can be thought of as a catalyst: although inequalities (2.23) and (2.30) cannot readily be proven by induction independently, considered together they can be established.

Inequality (2.23) is satisfied for $k = 1$ (read u_0 for $u_{k-1}(t)$) provided that a_1 satisfies inequalities (2.18) and (2.25). Inequality (2.29) and equation (2.31) show that inequality (2.30) holds in the case $k = 1$. Therefore suppose that inequalities (2.23) and (2.30) each hold in the cases $k = 1, 2, \ldots, j$, whenever $0 < r < s_0$ and $0 \leq t < a_k(s_0 - r)$. Inequality (2.30) will be established first, in the case $k = l+1$, whenever $0 < r < s_0$ and $0 \leq t < a_{l+1}(s_0 - r)$, and inequality (2.23) in the case $k = l+1$ will be shown shortly afterward. It is important to remind the reader that these inequalities are to be established on the condition that the sequences $\{\epsilon_k\}$, $\{a_k\}$ and $\{M_k\}$ satisfy certain relationships: the sequences $\{\epsilon_k\}$, $\{a_k\}$ and $\{M_k\}$ will then be defined so as to satisfy these relationships.

Let $0 < s < s_0$ and $0 \le t < a_{l+1}(s_0 - s)$. Then

$$\begin{aligned} \|u_{l+1}(t) - u_l(t)\|_s &= \left\| \int_0^t (F(\tau, u_l(\tau)) - F(\tau, u_{l-1}(\tau)))d\tau \right\|_s \\ &\le \int_0^t \|F(\tau, u_l(\tau)) - F(\tau, u_{l-1}(\tau))\|_s \\ &\le \int_0^t \frac{C}{r-s} \|u_l(\tau) - u_{l-1}(\tau)\|_r d\tau, \end{aligned} \tag{2.32}$$

where r is chosen so that $0 < s < r < s_0$ and $0 \le t < a_l(s_0 - s)$, by inequality (2.7). Note that the application of this inequality requires that u_l, $u_{l-1} \in \Theta_r$, but that this follows from inequality (2.23) in the cases $k = 1, 2, \ldots, k$, as was shown on page 19. There is no reason why r cannot depend on τ, so here we define

$$r = \frac{1}{2}\left(s_0 + s - \frac{\tau}{a_l}\right).$$

Substituting, and applying inequality (2.30) to the integrand of inequality (2.32) gives

$$\begin{aligned} \|u_{l+1}(t) - u_l(t)\|_s &\le \int_0^t \frac{2C}{s_0 - s - \frac{\tau}{a_l}} \|u_l(\tau) - u_{l-1}(\tau)\|_{\frac{1}{2}(s_0 + s - \frac{\tau}{a_l})} d\tau \\ &\le \int_0^t \frac{2C}{s_0 - s - \frac{\tau}{a_l}} \frac{2M_l\tau}{a_l(s_0 - s) - \tau} d\tau \\ &= 4CM_la_l \int_0^t \frac{\tau d\tau}{(a_l(s_0 - s) - \tau)^2} \\ &\le 4CM_la_lt \int_0^t \frac{d\tau}{(a_l(s_0 - s) - \tau)^2} \\ &= \frac{4CM_lt^2}{(a_l(s_0 - s) - t)(s_0 - s)}. \end{aligned}$$

Since it is assumed that $0 \le t < a_{l+1}(s_0 - s)$, it follows that

$$\|u_{l+1}(t) - u_l(t)\|_s \le \frac{4CM_la_{l+1}(s_0 - s)t}{(a_l(s_0 - s) - t)(s_0 - s)} \tag{2.33}$$

$$= 4Ca_{l+1}M_l \frac{t}{a_l(s_0 - s) - t}. \tag{2.34}$$

Since $a_{l+1} < a_l$, it follows that

$$\|u_{l+1}(t) - u_l(t)\|_s \le 4Ca_{l+1}M_l \frac{t}{a_{l+1}(s_0 - s) - t},$$

so that inequality (2.30) is satisfied in the case $k = l + 1$ provided that

$$M_{l+1} \geq 4Ca_{l+1}M_l. \tag{2.35}$$

Recalling equation (2.31), define

$$M_k = (4C)^{k-1}Ka_1^k \tag{2.36}$$

for each $k \in N$. Then $M_{l+1} = 4Ca_1M_l$, and since $a_{l+1} < a_1$, inequality (2.35) is satisfied. Inequality (2.34) becomes

$$\|u_{l+1}(t) - u_l(t)\|_s \leq (4C)^l a_1^l a_{l+1} K \frac{t}{a_l(s_0 - s) - t}. \tag{2.37}$$

Noting once more that $0 \leq t < a_{l+1}(s_0 - s)$, this yields

$$\|u_{l+1}(t) - u_l(t)\|_s \leq (4C)^l \frac{a_1^l a_{l+1}^2}{a_l - a_{l+1}} K,$$

so that inequality (2.23) is satisfied if

$$(4C)^l \frac{a_1^l a_{l+1}^2}{a_l - a_{l+1}} K \leq \epsilon_{l+1} R. \tag{2.38}$$

It has now been proven by induction, if $\{\epsilon_k\}$ and $\{a_k\}$ are sequences of positive numbers and $\{a_k\}$ is strictly decreasing, and inequalities (2.18), (2.24), (2.25), (2.27) and (2.38) hold, that, for each $k \in N$, whenever $0 < r < s_0$ and $0 \leq t < a_k(s_0 - s)$, inequalities (2.23) and (2.30) hold, and therefore that the approximations u_k converge absolutely to a solution to initial-value problem (2.4), (2.5) on a conical domain of positive height. For each $k \in N$, let

$$\epsilon_k = \mu^{k-1}(1 - \mu), \tag{2.39}$$

for any $\mu \in (0, 1)$, so that inequality (2.24) holds with equality. If

$$a_{k+1} = a_k - \frac{(4C)^k a_1^{k+2} K}{\epsilon_{k+1} R}, \tag{2.40}$$

then, since the sequence $\{a_k\}$ is decreasing,

$$\begin{aligned} (4C)^k \frac{a_1^k a_{k+1}^2}{a_k - a_{k+1}} K &\leq (4C)^k \frac{a_1^{k+2} K}{a_k - a_{k+1}} \\ &= (4C)^k \frac{a_1^{k+2} K}{(4C)^k a_1^{k+2} K / \epsilon_{k+1} R} \\ &= \epsilon_{k+1} R, \end{aligned}$$

so that inequality (2.38) holds. Repeated application of equation (2.40) gives

$$a = \lim_{k\to\infty} a_k = a_1 - \sum_{k=1}^{\infty} \frac{(4C)^k a_1^{k+2} K}{\epsilon_{k+1} R},$$

whence equation (2.39) yields

$$a = a_1 - \frac{4Ca_1^3 K\mu}{R(1-\mu)} \sum_{k=0}^{\infty} \left(\frac{4Ca_1}{\mu}\right)^k.$$

If

$$a_1 < \frac{\mu}{4C}, \tag{2.41}$$

then

$$a = a_1 - \frac{4Ca_1^3 K\mu}{R(1-\mu)\left(1 - \frac{4Ca_1}{\mu}\right)};$$

if this expression is positive then inequality (2.27) is satisfied. Thus, the inequalities remaining to be shown are (2.18), (2.25), (2.41), and

$$a_1 - \frac{4Ca_1^3 K\mu}{R(1-\mu)\left(1 - \frac{4Ca_1}{\mu}\right)} > 0. \tag{2.42}$$

The first three of these inequalities put together require that

$$0 < a_1 < \min\left\{\frac{T}{s_0}, \frac{\epsilon_1 R}{K}, \frac{\mu}{4C}\right\}, \tag{2.43}$$

and it is easily seen that this is compatible with the condition (2.42); one way is to note that at $a_1 = 0$, the expression in (2.42) has value 0 and a positive derivative, so that a positive value for a_1 can necessarily be chosen satisfying inequality (2.42) and small enough to satisfy inequality (2.43). Thus the existence result is shown.

■

2.4 Proof of Uniqueness

It will be shown that two solutions to initial-value problem (2.4), (2.5) mapping into the same space H_q must be equal on the intersection of their domains. From this it will follow that two mappings u and v into the scale of Banach spaces $\{H_q : 0 < q < s_0\}$ which each solve initial-value problem (2.4), (2.5) must be equal on the intersection of their domains.

Let $0 < q < s_0$, and suppose that $u : [0, \Delta] \to H_q$ and $v : [0, \Delta] \to H_q$ are two solutions to initial-value problem (2.4), (2.5), where Δ is some positive number not exceeding T. Because the initial-value problem is equivalent to the integral equation (2.10),

$$u(t) - v(t) = \int_0^t (F(\tau, u(\tau)) - F(\tau, v(\tau)))d\tau. \tag{2.44}$$

Thus, if $0 < s < q$, then

$$\begin{aligned} \|u(t) - v(t)\|_s &\leq \int_0^t \|F(\tau, u(\tau)) - F(\tau, v(\tau))\|_s d\tau \\ &\leq \int_0^t \frac{C}{r - s}\|u(\tau) - v(\tau)\|_r d\tau, \end{aligned} \tag{2.45}$$

where r is chosen to satisfy $0 < s < r < q$. Let α be an unspecified positive constant; it will first be shown that u and v are equal on the closed conical domain with base $\{z \in C : |z| \leq q\}$ and height αq. Or, if $\alpha q > \Delta$, then u and v will be shown to be equal on the same cone truncated at height Δ. In terms of the function spaces, this will amount to showing that whenever $0 < s \leq q$ and $0 \leq t \leq \min\{\alpha(q-s), \Delta\}$,

$$\|u(t) - v(t)\|_s = 0, \tag{2.46}$$

where, for notational convenience, we define

$$\|u\|_0 = |u(0)|.$$

Let

$$\Xi = \sup\left\{\|u(t) - v(t)\|_s \left(\frac{\alpha(q-s)}{t} - 1\right) : 0 < s \leq q \text{ and } 0 \leq t \leq \min\{\alpha(q-s), \Delta\}\right\}.$$

Assuming that Ξ is finite, it will be shown that α can be chosen sufficiently small that Ξ must vanish. Then the proof that Ξ must be finite will follow.

Let $0 < s < q$ and $0 \leq t < \min\{\alpha(q-s), \Delta\}$. The radius r in inequality (2.45) may be chosen to depend on τ; specifically it may be chosen as

$$r = \frac{1}{2}\left(q - \frac{\tau}{\alpha} + s\right).$$

Then inequality (2.45) becomes

$$\begin{aligned} \|u(t) - v(t)\|_s &\leq \int_0^t \frac{C}{\frac{1}{2}\left(q - \frac{\tau}{\alpha} - s\right)}\|u(\tau) - v(\tau)\|_r d\tau \\ &\leq 2C\alpha\Xi \int_0^t \frac{d\tau}{\left(\frac{\frac{1}{2}(\alpha(q-s)+\tau)}{\tau} - 1\right)(\alpha(q-s) - \tau)} \\ &= 4C\alpha\Xi \int_0^t \frac{\tau\, d\tau}{(\alpha(q-s) - \tau)^2} \\ &< 4C\alpha\Xi \frac{t}{\alpha(q-s) - t}, \end{aligned}$$

so that

$$\|u(t)-v(t)\|_s\left(\frac{\alpha(q-s)}{t}-1\right)\le 4C\alpha\Xi,$$

that is,

$$\Xi\le 4C\alpha\Xi.$$

Therefore, if $\alpha<1/4C$, then $\Xi=0$, assuming that Ξ is finite.

It will now be shown why Ξ must be finite. The difference $u-v$ is a continuous function mapping the closed interval $[0,\Delta]$ to H_q, so that it must be bounded; therefore, let A be the finite number

$$A=\sup\{\|u(t)-v(t)\|_q : t\in[0,\Delta]\}.$$

Now

$$\begin{aligned}
\Xi &= \sup\left\{\|u(t)-v(t)\|_s\left(\frac{\alpha(q-s)}{t}-1\right):\right.\\
&\qquad\qquad \left.0<s\le q \text{ and } 0\le t\le\min\{\alpha(q-s),\Delta\}\right\}\\
&\le \sup\left\{\int_0^t\frac{C}{q-s}\|u(\tau)-v(\tau)\|_q d\tau\left(\frac{\alpha(q-s)}{t}-1\right):\right.\\
&\qquad\qquad \left.0<s\le q \text{ and} 0\le t\le\min\{\alpha(q-s),\Delta\}\right\} \qquad \text{(by (2.45))}\\
&\le \sup\left\{\frac{tCA}{q-s}\left(\frac{\alpha(q-s)}{t}-1\right):\right.\\
&\qquad \left.0<s\le q \text{ and } 0\le t\le\min\{\alpha(q-s),\Delta\}\right\}\\
&= CA\sup\left\{\alpha-\frac{t}{q-s}:0<s\le q \text{ and } 0\le t\le\min\{\alpha(q-s),\Delta\}\right\}\\
&= CA\alpha.
\end{aligned}$$

Therefore the conclusion of the last paragraph is valid, so that in fact

$$\|u(t)-v(t)\|_s=0$$

whenever $0<s<q$ and $0\le t<\min\{\alpha(q-s),\Delta\}$, that is, whenever

$$t\in[0,\alpha q)\cap[0,\Delta]$$

and

$$0<s<q-\frac{t}{\alpha}.$$

For each

$$t \in [0, \alpha q) \cap [0, \Delta],$$

choose

$$s(t) \in \left(0, q - \frac{t}{\alpha}\right).$$

It has just been seen that

$$\|u(t) - v(t)\|_s = 0.$$

By the injective property of the scale operators, since

$$u(t) - v(t) \in H_q,$$

it follows that

$$\|u(t) - v(t)\|_q = 0.$$

Thus it has been shown that the solutions u and v are equal on the interval $[0, \alpha q) \cap [0, \Delta]$. Depending on the value of α, this interval may not be the full interval $[0, \Delta]$, but in that case an inductive approach can be taken.

Suppose that u and v are equal on the interval

$$[0, E) \subset [0, \Delta].$$

Assume for notational simplicity that

$$E \geq \frac{\alpha q}{2}.$$

To show that u and v are equal on the interval

$$\left[0, E + \frac{\alpha q}{2}\right) \cap [0, \Delta], \tag{2.47}$$

first note that u and v are equal on the closed interval

$$\left[0, E - \frac{\alpha q}{2}\right].$$

Now define

$$\Xi = \sup \left\{ \|u(t) - v(t)\|_s \left(\frac{\alpha(q - s)}{t - \left(E - \frac{\alpha q}{2}\right)} - 1 \right) : \right.$$
$$\left. 0 < s \leq q \text{ and } E - \frac{\alpha q}{2} \leq t \leq \min\left\{E + \alpha(\frac{q}{2} - s), \Delta\right\} \right\}.$$

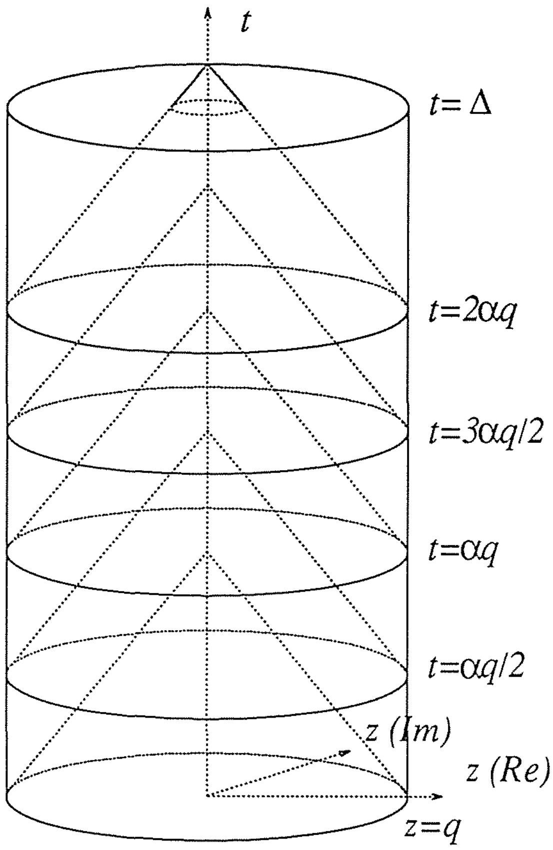

Figure 2.2: Regions within which uniqueness can successively be shown

The same method used above can be applied to show that Ξ must be finite, and thus that $\Xi = 0$, i.e.,

$$\|u(t) - v(t)\|_q = 0$$

whenever

$$E - \frac{\alpha q}{2} \leq t \leq \min\left\{E + \frac{\alpha q}{2}, T\right\};$$

consequently u and v are equal on the interval (2.47). Now clearly this procedure can be repeated to show that u and v are equal on $[0, \Delta]$.

Figure (2.2) illustrates the geometry of this induction, in the case where $\{H_r\}$ is the scale $\{B_r\}$ of holomorphic function spaces. It is first established that $u(t, z)$ and $v(t, z)$ are equal within the bottom cone depicted in the figure, i.e., where $0 \leq |z| < q$ and $0 \leq t < \alpha(q - |z|)$. Analytic continuation, corresponding to the injective property of the scale operators, immediately shows that $u(t, z) = v(t, z)$ within the cylinder, open at the top, circumscribing the cone, i.e., where $0 \leq |z| \leq q$ and $0 \leq t < \alpha q$. Now it can be shown that $u(t, z) = v(t, z)$ on the second cone, whose base is the cross-section midway up the last cylinder, using the same proöf as for the first cone. Repeating this, it is inductively shown that $u(t, z) = v(t, z)$ within the cones depicted, and thus within the cylinders circumscribing them. The top cone extends beyond the top of the cylinder of height Δ, so that $u(t, z) = v(t, z)$ for each $t \in [0, \Delta]$, and whenever $|z| \leq q$.

If u and v are two mappings into the scale $\{H_r\}$ of Banach spaces which solve initial-value problem (2.4), (2.5), then they must be equal on the intersection of their domains, in the sense that if $u(t)$ and $v(t)$ both belong to a space H_r, then

$$\|u(t) - v(t)\|_r = 0.$$

From the definition of a mapping into a scale of Banach spaces, it can be seen that if $u(t) \in H_r$, then $u(\tau) \in H_r$ for every $\tau \in [0, t]$; this is because the domain of u as a function into H_r must be an interval, and $u(0) = u_0 \in H_r$. Similarly v maps the interval $[0, t]$ into H_r. Since they are both solutions to initial-value problem (2.4), (2.5), u and v must be equal on this interval, so that $u(t) = v(t)$. ∎

In this chapter it was seen how a scale of Banach spaces could be used to reformulate initial-value problems of the type (2.4), (2.5) as a differential equation in one independent variable, and existence of a solution to such an initial-value problem was then established using the method of successive. Note that while the injective property of the scale operators was instrumental in showing uniqueness of a solution, the existence result made no appeal to it.

Results similar to this have been shown by others, using the same procedure of reformulating a partial differential equation as a differential equation of one independent variable, by suppressing the other variable or variables through the use of a scale of Banach spaces, defining successive approximations by integration, and showing that they converge uniformly to a solution to the reformulated differential equation. One such result appeared in a paper by Sekine and Yamanaka [5], and deals with retardations in the real variable remaining after reformulation. That is, they deal with a differential equation

$$u'(t) = F \tag{2.48}$$

where the independent variable t is real-valued, the solution u maps into a prescribed scale of Banach spaces, and the right-hand side F depends not only on t and $u(t)$, but on the behaviour of u over the whole interval $[t - d, t]$, where d is some positive constant. If t is thought of as time, this differential equation stipulates the rate of change of some system in terms of the current time together with the history of the system over the period of length d leading up to the present. An initial condition for this differential equation must therefore specify u over an interval of length d. Sekine and Yamanaka consider the initial-value problem combining equation (2.48) with the condition that

$$u(\tau) = 0 \tag{2.49}$$

whenever $\tau \in [-d, 0]$.

The precise formulation of the problem (2.48) requires some additional notation. Let

$$\{H_r : 0 < r < s_0\},$$

with the collection of scale operators

$$\{J_{r,s} : 0 < s < r < s_0\},$$

be the scale of Banach spaces into which u will map. Fix some positive constant d. For each $r \in (0, s_0)$, let D_r be the space of all differentiable mappings from the interval $[-d, 0]$ into H_r. If $\eta \in D_r$, then define the norm $\|\cdot\|_r$ by

$$\|\eta\|_r = \sup\{\|\eta(x)\|_r : x \in [-d, 0]\}.$$

Under this norm, D_r is a Banach space. The collection of these spaces forms another scale of Banach spaces

$$\{D_r : 0 < r < s_0\}.$$

A natural definition for the scale operator $Q_{r,s}$ associated with this scale states that

$$(Q_{r,s}\eta)(x) = J_{r,s}(\eta(x))$$

whenever $x \in [-d, 0]$.

If $u : [-d, T] \to H_r$, where T is some positive constant, then, for each $t \in [0, T]$, define $\overleftarrow{u}(t)$ to be that point in D_r such that

$$\left(\overleftarrow{u}(t)\right)(x) = u(t + x)$$

for each $x \in [-d, 0]$. There is no new information contained in this function; it is emphasised that the definitions of $\{D_r : 0 < r < s_0\}$ and $\overleftarrow{u}$ serve merely notational ends. For any $t \in [0, T]$, $\overleftarrow{u}(t)$ describes the behaviour of u over the whole interval $[t-d, t]$. Thus a function F can be defined, taking one argument from R and one argument from D_r, and mapping into H_r, so that the left-hand side of differential equation (2.48) can be described as

$$F\left(t, \overleftarrow{u}(t)\right).$$

The resulting differential equation

$$u'(t) = F\left(t, \overleftarrow{u}(t)\right) \tag{2.50}$$

does indeed give $u'(t)$ in terms t and not only $u(t)$, but the behaviour of u over the whole interval $[t-d, t]$.

Sekine and Yamanaka proved that if the right-hand side operator F is a continuous generalised Cauchy-Riemann operator, and the inequality

$$\|F(t, 0)\|_r \leq \frac{K}{s_0 - r}$$

holds for some positive constant K, then the initial-value problem (2.50), (2.49) has a unique solution defined on the cone of base

$$\{z \in C : |z| < s_0\}$$

and some positive height. The proof is similar to that of Theorem 2.

Chapter 3

An Integro-Differential Equation in a Scale of Banach Spaces

The integro-differential equation addressed in this chapter is similar to equation (2.50) in that it gives the solution's derivative in terms of retarded values of the solution itself, but not as general, because the retardation is affected by an integral, posing more stringent conditions on the right-hand side of the differential equation than in equation (2.50). In another sense, however, it is more general. The derivative of the solution is given in terms not only of the real independent variable t and retarded values of the solution, but also of retarded values of the derivative of the solution. In the next chapter an equation which combines the generality of both of these equations will be considered; the initial-value problem investigated in this chapter serves to illustrates the result of next chapter in the context of a specific scale of Banach spaces and a somewhat concrete retardation operator.

Let

$$\{B_r : 0 < r < s_0\}$$

be the scale of holomorphic function spaces introduced in Section (1.4), with the domain restriction scale operators

$$\{I_{r,s} : 0 < s < r < s_0\}.$$

Let $\Phi_{t,r}$, $\mathrm{X}_{t,r}$, and $\Psi_{t,r}$ be bounded operators which depend on a real parameter t and map the space B_r into itself, for each $r \in (0, s_0)$. For each r, the operator $\Phi_{t,r}$ maps B_s into itself; it is required, however, that

$$\Phi_{t,s} I_{r,s} = I_{r,s} \Phi_{t,r}, \tag{3.1}$$

so that in this sense $\Phi_{t,s}$ and $\Phi_{t,r}$ can be regarded as the same operator, and we can refer to either as Φ_t without risk of ambiguity. This compatibility condition will be more clearly motivated after the initial-value problem is presented. Similar compatibility conditions hold for the operators X_t and Ψ_t. It is also assumed that Φ_t, X_t, and Ψ_t depend continuously on the parameter t, in the sense that for every $r \in (0, s_0)$, $t \geq 0$, and bounded $Y \subset B_r$, the operators Φ_τ, X_τ and Ψ_τ converge uniformly on Y to Φ_t, X_t and Ψ_t, respectively, as $\tau \to t$, for $\tau > 0$.

The initial-value problem addressed in this chapter is the integro-differential equation

$$w'(t) = \int_0^t \left[\Phi_t(w(\tau)) + \mathrm{X}_t(D(w(\tau))) + \Psi_t(w'(\tau))\right] d\tau, \tag{3.2}$$

where the prime (′) denotes differentiation with respect to t, and the symbol D denotes the complex differentiation operator with respect to z, together with the initial condition

$$w(0) = w_0, \tag{3.3}$$

where $w_0 \in B_r$ for each $r \in (0, s_0)$, and satisfies an asymptotic bound akin to (2.9) from Theorem 2. A solution to this initial-value problem is a function mapping one real independent variable t into the complex function spaces B_r, so that the z dependence of the solution is suppressed in the notation.

It can now be seen how the compatibility condition (3.1) arises naturally. The mapping w into $\{B_r : 0 < r < s_0\}$ satisfies the compatibility conditions on a mapping into a scale of Banach spaces, so that, letting w_r and w_s be the mappings of w into the spaces B_r and B_s,

$$w_s = I_{r,s} w_r.$$

Differential equation (3.2) in detail is

$$w_r'(t) = \int_0^t \left[\Phi_{t,r}(w_r(\tau)) + \mathrm{X}_{t,r}(D_r(w_r(\tau))) + \Psi_{t,r}(w_r'(\tau))\right] d\tau, \tag{3.4}$$

where D_r is the complex differentiation operator mapping B_r to B_r. Of course differentiation with respect to t commutes with the scale operator, that is,

$$\begin{aligned} I_{r,s}w_r'(t) &= I_{r,s}\lim_{\tau\to t}\frac{w_r(t)-w_r(\tau)}{t-\tau} \\ &= \lim_{\tau\to t}\frac{I_{r,s}w_r(t)-I_{r,s}w_r(\tau)}{t-\tau} \\ &= \lim_{\tau\to t}\frac{w_s(t)-w_s(\tau)}{t-\tau} \\ &= w_s'(t), \end{aligned}$$

since $I_{r,s}$ is linear. Thus

$$w_s'(t) = I_{r,s}w_r'(t) = \int_0^t \left[I_{r,s}\Phi_{t,r}(w_r(\tau)) + I_{r,s}\mathrm{X}_{t,r}(D_r(w_r(\tau))) + I_{r,s}\Psi_{t,r}(w_r'(\tau))\right] d\tau, \tag{3.5}$$

again appealing to the linearity of $I_{r,s}$, this time in bringing it inside the integral. On the other hand, substituting s in for r in equation (3.4) gives

$$\begin{aligned} w_s'(t) &= \int_0^t \left[\Phi_{t,s}(w_s(\tau)) + \mathrm{X}_{t,s}(D_s(w_s(\tau))) + \Psi_{t,s}(w_s'(\tau))\right] d\tau \\ &= \int_0^t \left[\Phi_{t,s}I_{r,s}(w_r(\tau)) + \mathrm{X}_{t,s}I_{r,s}(D_r(w_r(\tau))) + \Psi_{t,s}I_{r,s}(w_r'(\tau))\right] d\tau. \end{aligned} \tag{3.6}$$

Comparison of equations (3.5) and (3.6) motivates the compatiblity condition (3.1).

An example of equation (3.2) is the following differential equation:

$$w'(t) = \int_0^t R_t(w(\tau) + Dw(\tau) + w'(\tau))d\tau. \tag{3.7}$$

Here the rotation operator

$$R_t : B_s \to B_s,$$

for arbitrary s, is defined by

$$(R_t(f))(z) = f(e^{it}z)$$

whenever $|z| < s$, for each $f \in B_s$. Equation (3.7) in detail is

$$\frac{\partial w}{\partial t}(t,z) = \int_0^t \left(w(\tau, e^{it}z) + \frac{\partial w}{\partial t}(\tau, e^{it}z) + \frac{\partial w}{\partial z}(\tau, e^{it}z)\right) d\tau. \tag{3.8}$$

In terms of this formulation, an initial condition would take the form

$$w(0,z) = w_0(z), \tag{3.9}$$

where w_0 is some prescribed function holomorphic on B_s for every $s \in (0, s_0)$. The differential equation (3.7) states that if, for a given t, the holomorphic functions given by

$$\int_0^t w(x)dx,$$
$$\int_0^t Dw(x)dx$$

and

$$w(t) - w_0$$

are added together, and the resulting function is rotated about 0 by and angle of t, the function $w'(t)$ is obtained.

Before the existence and uniqueness of a solution to initial value problem (3.2), (3.3) is addressed, the possibility that it is reducible to a simpler problem is considered. Consider the ostensibly similar integro-differential equation

$$w'(t) = \int_0^t \left[\Phi_\tau(w(\tau)) + \mathrm{X}_\tau(D(w(\tau))) + \Psi_\tau(w'(\tau))\right] d\tau. \tag{3.10}$$

Prima facie, this appears to define the $w'(t)$ in terms of the history of the solution and both of its first partial derivatives on the cylinder Ω. It can, however, be differented with respect to t, to give

$$w''(t) = \Phi_t(w(t)) + \mathrm{X}_t(D(w(t))) + \Psi_t(w'(t)), \tag{3.11}$$

so that integro-differential equation (3.10) is not actually retarded with respect to t. On the other hand, if the same reduction is attempted with equation (3.2) (even assuming that the derivatives Φ_t', X_t' and Ψ_t' of Φ_t, X_t and Ψ_t with respect to t can be defined) the resulting expression is

$$\begin{aligned} w''(t) &= \Phi_t(w(t)) + \mathrm{X}_t(D(w(t))) + \Psi_t(w'(t)) \\ &+ \int_0^t \left[\Phi_t'(w(\tau)) + \mathrm{X}_t'(D(w(\tau))) + \Psi_t'(w'(\tau))\right] d\tau, \end{aligned} \tag{3.12}$$

which still manifests a retarded nature with respect to t. To clarify this point, consider the example from last paragraph. Differentiating with respect to t yields

$$\begin{aligned} \frac{\partial^2 w}{\partial t^2}(t,z) &= w(t, e^{it}z) + \frac{\partial w}{\partial t}(t, e^{it}z) + \frac{\partial w}{\partial z}(t, e^{it}z) \\ &+ \int_0^t \left(e^{it}z\frac{\partial w}{\partial z}(\tau, e^{it}z) + e^{it}z\frac{\partial^2 w}{\partial t \partial z}(\tau, e^{it}z) + e^{it}z\frac{\partial^2 w}{\partial z^2}(\tau, e^{it}z)\right) d\tau, \end{aligned} \tag{3.13}$$

which still shows the retarded form of the original equation (3.7). On the other hand, the corresponding differential equation (3.10) is

$$\frac{\partial w}{\partial t}(t,z) = \int_0^t \left(w(\tau, e^{i\tau}z) + \frac{\partial w}{\partial t}(\tau, e^{i\tau}z) + \frac{\partial w}{\partial z}(\tau, e^{i\tau}z) \right) d\tau, \tag{3.14}$$

which can be differentiated to give

$$\frac{\partial^2 w}{\partial t^2}(t,z) = w(t, e^{it}z) + \frac{\partial w}{\partial t}(t, e^{it}z) + \frac{\partial w}{\partial z}(t, e^{it}z). \tag{3.15}$$

Although this equation is functional with respect to the z argument, it is no longer retarded in t, so that, essentially, equation (3.14) is not retarded in t.

Existence and uniqueness results now follow.

Theorem 3 *Suppose the operators Φ_t, X_t, and Ψ_t depend continuously on the real parameter t and are bounded uniformly with respect to t and r on a certain cylidrical domain; that is, suppose the existence of positive constants T, R_1, R_2, K, L and M such that $\forall t \in [0,T]$, $r \in (0, s_0)$, if u_1, u_2, v_1 and v_2 belong to B_r with*

$$\|u_1 - w_0\|_r \le R_1, \qquad \|u_2 - w_0\|_r \le R_1,$$
$$\|v_1\|_r \le R_2, \qquad \|v_2\|_r \le R_2,$$

then

$$\begin{aligned} \|\Phi_t u_1 - \Phi_t u_2\|_r &\le K\|u_1 - u_2\|_r, \\ \|\mathrm{X}_t Du_1 - \mathrm{X}_t Du_2\|_r &\le L\|Du_1 - Du_2\|_r, \\ \|\Psi_t v_1 - \Psi_t v_2\|_r &\le M\|v_1 - v_2\|_r. \end{aligned} \tag{3.16}$$

Suppose further that the initial value w_0 satisfies the compatibility condition

$$w_{0,s} = I_{r,s} w_{0,r}, \tag{3.17}$$

where $w_{0,s}$ and $w_{0,r}$ correspond to the initial value w_0 in the spaces B_s and B_r, respectively, and that there exist positive constants X_0, Y_0 and Z_0 such that the points $\Phi_t(u_0)$, $\mathrm{X}_t(Du_0)$ and $\Psi_t(0)$ obey the asymptotic bounds

$$\begin{aligned} \|\Phi_t(u_0)\|_r &\le \frac{X_0}{(s_0 - r)^2}, \\ \|\mathrm{X}_t(Du_0)\|_r &\le \frac{Y_0}{(s_0 - r)^2}, \\ \|\Psi_t(0)\|_r &\le \frac{Z_0}{(s_0 - r)^2}, \end{aligned}$$

uniformly in t. Then the initial value problem (3.2), (3.3) has a unique continuously differentiable solution w mapping into the scale $\{B_r\}$. A positive constant a can *be chosen such that the compatible functions w_r constituting the mapping w are defined on the intervals* $[0, a(s_0 - r)^2]$.

3.1 Proof of Existence

The existence of a solution to initial value problem (3.2), (3.3) can be established using a variation of the method of successive approximations similar to that used in the proof of Theorem 2. Note that any solution $w : [0, T_s] \to B_s$ to the integral equation

$$w(t) = w_0 + \int_0^t \int_0^\tau \left[\Phi_\tau(w(x)) + \mathrm{X}_\tau(D(w(x))) + \Psi_\tau(w'(x))\right] dx\, d\tau \tag{3.18}$$

also solves initial value problem (3.2), (3.3). Therefore, the successive approximations are defined by

$$\begin{aligned} w_1(t) &= w_0 + \int_0^t \int_0^\tau \left[\Phi_\tau(w_0) + \mathrm{X}_\tau(Dw_0) + \Psi_\tau(0)\right] dx\, d\tau \\ &= w_0 + \int_0^t \tau \left[\Phi_\tau(w_0) + \mathrm{X}_\tau(Dw_0) + \Psi_\tau(0)\right] d\tau \end{aligned} \tag{3.19}$$

and, for $k \geq 1$,

$$w_{k+1}(t) = w_0 + \int_0^t \int_0^\tau \left[\Phi_\tau(w_k(x)) + \mathrm{X}_\tau(D(w_k(x))) + \Psi_\tau(w_k'(x))\right] dx\, d\tau. \tag{3.20}$$

It will be shown that the sequence $\{w_k\}$ converges to a continuously differentiable function w. It is first shown, however, that such a limit function satisfies integral equation (3.18).

Recall from the proof of Theorem 2 that inequalities (2.23) and (2.24) serve the dual purpose of ensuring that each approximation is definable from the previous one, in view of the limited extent of the domain of the right-hand side operator F, and at the same time causing the sequence of approximations to converge absolutely and uniformly. A similar technique will be employed here. Since the right-hand side operator F now takes an argument of retarded values of the derivative of the solution, and its domain is bounded in this argument as well, it must be ensured that the derivative remains bounded, and not merely the solution. Let

$$S = \min\left\{\frac{R_1}{T}, R_2\right\}, \tag{3.21}$$

and suppose the existence of a sequence $\{\epsilon_k\}$ of positive numbers and a strictly decreasing sequence $\{a_k\}$ of positive numbers such that, whenever $0 < r < s_0$ and

$0 \leq t < a_k^2(s_0 - r)^2$,

$$\|w_k'(t) - w_{k-1}'(t)\|_r \leq \epsilon_k S, \tag{3.22}$$

and

$$\sum_{k=1}^{\infty} \epsilon_k \leq 1. \tag{3.23}$$

It is convenient for the condition $0 \leq t < a_k^2(s_0 - r)^2$ to imply $t \in [0, T]$, and it does if

$$a_k \leq \frac{\sqrt{T}}{s_0}.$$

Since the sequence $\{a_k\}$ is decreasing, it suffices to choose a_1 such that

$$a_1 \leq \frac{\sqrt{T}}{s_0}. \tag{3.24}$$

Now

$$\|w_k'(t) - w_{k-1}'(t)\|_r \leq \epsilon_k S \leq \epsilon_k R_2,$$

so that, after a telescoping argument similar to that used in the proof of Theorem 2,

$$\|w_k'(t)\|_r \leq R_2,$$

and therefore the bound in (3.16) on the operator Ψ_t can be applied. Further, definition (3.20) gives immediately

$$w_k(0) = w_0,$$

so that

$$\begin{aligned} \|w_k(t) - w_0\|_r &= \left\| \int_0^t w_k'(\tau) d\tau \right\|_r \\ &\leq \int_0^t \|w_k'(\tau)\|_r d\tau, \end{aligned}$$

whence the same telescoping argument gives

$$\|w_k(t) - w_0\|_r \leq tS \leq R_1,$$

so that the bounds in (3.16) on the operators Φ_t and X_t are also applicable.

As has been noted, in this proof the differences of the derivatives, as well as of the approximations themselves, are being bounded. Beyond this, the only difference between this and inequalities (2.23) and (2.24) is that the regions over which the bounds take effect are defined by a quadratic relationship between t and r, rather than a linear one. This is because the condition that the difference of the derivatives be bounded is stronger than the condition that the difference of the approximations themselves be bounded, and cannot be shown on as large a region.

Provided that inequalities (3.22) and (3.23) hold, for each $r \in (0, s_0)$ and $t \in [0, a^2(s_0 - r)^2)$, the sequence $\{w'_{k,r}(t)\}$ is Cauchy in the space B_r and thus converges to a point $w'_r(t) \in B_r$. Letting t vary gives a function w'_r mapping the interval $[0, a^2(s_0 - r)^2)$ into B_r. The functions w'_r produced in this way for differing values of r are compatible. To show this, it is first shown by induction that the facets $w'_{k,s}$ and $w'_{k,r}$ of the k^{th} approximation are compatible, where $0 < s < r < s_0$ and $k \in N$. Firstly, the constant functions mapping $[0, a^2(s_0 - s)^2)$ to $w_{0,s}$ and $[0, a^2(s_0 - r)^2)$ to $w_{0,r}$, which can loosely be called the 0^{th} approximations in B_s and B_r, are compatible, by equation (3.17). Clearly their derivatives with respect to t are also compatible, since they are simply the zero mappings into B_s and B_r. Now fix $k \in \{0\} \cup N$, and suppose that $w_{k,s}$ and $w_{k,r}$ are compatible, and also that their derivatives $w'_{k,s}$ and $w'_{k,r}$ are compatible. Then

$$\begin{aligned}
w'_{k+1,s}(t) &= \int_0^t \left(\Phi_{t,s}(w_{k,s}(\tau)) + \mathrm{X}_{t,s}(D_s w_{k,s}(\tau)) + \Psi_{t,s}(w'_{k,s}(\tau))\right) d\tau \\
&= \int_0^t \left(\Phi_{t,s} I_{r,s} w_{k,r}(\tau) + \mathrm{X}_{t,s} D_s I_{r,s} w_{k,r}(\tau) + \Psi_{t,s} I_{r,s} w'_{k,r}(\tau)\right) d\tau \\
&= \int_0^t \left(I_{r,s}\Phi_{t,r} w_{k,r}(\tau) + I_{r,s}\mathrm{X}_{t,r} D_r w_{k,r}(\tau) + I_{r,s}\Psi_{t,r} w'_{k,r}(\tau)\right) d\tau \\
&= I_{r,s}\int_0^t \left(\Phi_{t,r}(w_{k,r}(\tau)) + \mathrm{X}_{t,r}(D_r w_{k,r}(\tau)) + \Psi_{t,r}(w'_{k,r}(\tau))\right) d\tau \\
&= I_{r,s} w'_{k+1,r}(t),
\end{aligned}$$

and

$$\begin{aligned}
w_{k+1,s} &= w_{0,s} + \int_0^t w'_{k+1,s}(\tau) d\tau \\
&= I_{r,s} w_{0,r} + \int_0^t I_{r,s} w'_{k+1,r}(\tau) d\tau \\
&= I_{r,s}\left(w_{0,r} + \int_0^t w'_{k+1,r}(\tau) d\tau\right) \\
&= I_{r,s} w_{k+1,r}(t);
\end{aligned}$$

therefore, by induction, $w'_{k,s}$ and $w'_{k,r}$ are compatible, as are $w_{k,s}$ and $w_{k,r}$, for each $k \in N$. It follows that

$$\begin{aligned}
\|w'_s(t) - I_{r,s} w'_r(t)\|_s &= \left\| \lim_{k\to\infty} w'_{k,s} - I_{r,s} \lim_{k\to\infty} w'_{k,r} \right\|_s \\
&= \lim_{k\to\infty} \|w'_{k,s}(t) - I_{r,s} w'_{k,r}(t)\|_s \\
&= \lim_{k\to\infty} 0 = 0,
\end{aligned}$$

so that w'_s and w'_r are compatible, and thus constitute a mapping w' into the scale $\{B_r\}$.

Now w can be defined by

$$w(t) = w_0 + \int_0^t w'(\tau)d\tau. \tag{3.25}$$

This is composed of compatible functions, too, since

$$\begin{aligned} w_s(t) &= w_{0,s} + \int_0^t w_s'(\tau)d\tau \\ &= I_{r,s}w_{0,r} + \int_0^t I_{r,s}w_r'(\tau)d\tau \\ &= I_{r,s}\left(w_{0,r} + \int_0^t w_r'(\tau)d\tau\right) \\ &= I_{r,s}w_r(t). \end{aligned}$$

If $k \in N$, $0 < r < s_0$ and $0 \le t < a^2(s_0 - r)^2$, where

$$a = \lim_{k\to\infty} a_k > 0, \tag{3.26}$$

then

$$\begin{aligned} &\left\| w(t) - w_0 - \int_0^t \int_0^\tau \left[\Phi_\tau(w(x)) + \mathrm{X}_\tau(D(w(x))) + \Psi_\tau(w'(x))\right] dx\, d\tau \right\|_r \\ \le\ & \|w(t) - w_{k+1}(t)\|_r \\ +\ & \left\| w_{k+1}(t) - w_0 - \int_0^t \int_0^\tau \left[\Phi_\tau(w(x)) + \mathrm{X}_\tau(D(w(x))) + \Psi_\tau(w'(x))\right] dx\, d\tau \right\|_r \\ \le\ & \|w(t) - w_{k+1}(t)\|_r + \int_0^t \int_0^\tau \Big[\|\Phi_\tau(w(x)) - \Phi_\tau(w_k(x))\|_r \\ +\ & \|\mathrm{X}_\tau(D(w(x))) - \mathrm{X}_\tau(D(w_k(x)))\|_r + \|\Psi_\tau(w'(x)) - \Psi_\tau(w_k'(x))\|_r\Big]\, dx d\tau \\ \le\ & \|w(t) - w_{k+1}(t)\|_r + \int_0^t \int_0^\tau \Big[K\|w(x) - w_k(x)\|_r \\ +\ & L\|D(w(x)) - D(w_k(x))\|_r + M\|w'(x) - w_k'(x)\|_r\Big]\, dx d\tau, \end{aligned} \tag{3.27}$$

where the numbers K, L, and M are the bounds on Φ_t, X_t and Ψ_t from inequalities (3.16). If some q is chosen satisfying $0 < r < q < s_0$ and $0 \le t < a^2(s_0 - q)^2$, for instance

$$q = \frac{1}{2}\left(s_0 - \frac{\sqrt{t}}{a} + r\right),$$

then Nagumo's lemma can be applied to the middle term of the above integrand to give

$$\begin{aligned} &\left\| w(t) - w_0 - \int_0^t \int_0^\tau \left[\Phi_\tau(w(x)) + \mathrm{X}_\tau(D(w(x))) + \Psi_\tau(w'(x))\right] dx\, d\tau \right\|_r \\ \le\ & \|w(t) - w_{k+1}(t)\|_r + \int_0^t \int_0^\tau \Bigg[K\|w(x) - w_k(x)\|_r + \frac{L}{q-r}\|w(x) - w_k(x)\|_q \\ +\ & M\|w'(x) - w_k'(x)\|_r\Bigg]\, dx d\tau. \end{aligned} \tag{3.28}$$

Equation (3.20) gives $w_k(0) = w_0$, and similarly equation (3.25) gives $w(0) = w_0$;

thus, letting

$$I = [0, a^2(s_0 - q)^2),$$

$$\begin{aligned}
\|w(x) - w_k(x)\|_r &\leq \|w(x) - w_k(x)\|_q \\
&\leq \int_0^x \|w'(y) - w'_k(y)\|_q dy \\
&\leq \int_0^x \sup\{\|w'(\bar{y}) - w'_k(\bar{y})\|_q : \bar{y} \in I\} dy \\
&\leq T \sup\{\|w'(\bar{y}) - w'_k(\bar{y})\|_q : \bar{y} \in I\}.
\end{aligned}$$

Similarly,

$$\|w(x) - w_{k+1}(x)\|_r \leq T \sup\{\|w'(\bar{y}) - w'_{k+1}(\bar{y})\|_q : \bar{y} \in I\},$$

and

$$\begin{aligned}
\|w'(x) - w'_k(x)\|_r &\leq \|w'(x) - w'_k(x)\|_q \\
&\leq \sup\{\|w'(\bar{y}) - w'_k(\bar{y})\|_q : \bar{y} \in I\}.
\end{aligned}$$

Inequality (3.28) thus implies

$$\begin{aligned}
&\left\| w(t) - w_0 - \int_0^t \int_0^\tau \left[\Phi_\tau(w(x)) + \mathrm{X}_\tau(D(w(x))) + \Psi_\tau(w'(x)) \right] dx\, d\tau \right\|_r \\
&\leq T \sup\{\|w'(\bar{y}) - w'_{k+1}(\bar{y})\|_q : \bar{y} \in I\} \\
&+ \int_0^t \int_0^\tau \left(KT + \frac{LT}{q-r} + M \right) \sup\{\|w'(\bar{y}) - w'_k(\bar{y})\|_q : \bar{y} \in I\} dx d\tau \\
&\leq T \sup\{\|w'(\bar{y}) - w'_{k+1}(\bar{y})\|_q : \bar{y} \in I\} \\
&+ \frac{t^2}{2} \left(KT + \frac{LT}{q-r} + M \right) \sup\{\|w'(\bar{y}) - w'_k(\bar{y})\|_q : \bar{y} \in I\}.
\end{aligned}$$

Now since $w'_k(\bar{y}) \to w'(\bar{y})$ in B_q as $k \to \infty$, the suprema in the above expression tend to zero, so that in fact

$$\left\| w(t) - w_0 - \int_0^t \int_0^\tau \left[\Phi_\tau(w(x)) + \mathrm{X}_\tau(D(w(x))) + \Psi_\tau(w'(x)) \right] dx\, d\tau \right\|_r = 0,$$

that is, $w(t)$ and

$$w_0 - \int_0^t \int_0^\tau \left[\Phi_\tau(w(x)) + \mathrm{X}_\tau(D(w(x))) + \Psi_\tau(w'(x)) \right] dx\, d\tau$$

are equal in the space B_r, so that initial-value problem (3.2), (3.3) is solved.

The successive approximations w_k, as well as the solution w, are continuously differentiable mappings from the interval $[0, a^2(s_0 - r)^2]$ into B_r, for each $r \in (0, s_0)$. Consider the first approximation, w_1. Definition (3.19) implies

$$w'_1(t) = t \left[\Phi_\tau(w_0) + \mathrm{X}_\tau(Dw_0) + \Psi_\tau(0) \right],$$

which is continuous with respect to t. Now suppose that w_k is continuously differentiable with respect to t. If $[t, t+h] \subset [0, a^2(s_0 - r)^2]$, where h is some small number, then

$$\begin{aligned}
& w'_{k+1}(t+h) - w'_{k+1}(t) \\
&= \int_0^{t+h} \left[\Phi_{t+h}(w_k(\tau)) + \mathrm{X}_{t+h}(Dw_k(\tau)) + \Psi_{t+h}(w'_k(\tau))\right] d\tau \\
&- \int_0^{t} \left[\Phi_{t}(w_k(\tau)) + \mathrm{X}_{t}(Dw_k(\tau)) + \Psi_{t}(w'_k(\tau))\right] d\tau \\
&= \int_0^{t} \left[(\Phi_{t+h} - \Phi_t)(w_k(\tau)) + (\mathrm{X}_{t+h} - \mathrm{X}_t)(Dw_k(\tau)) + (\Psi_{t+h} - \Psi_t)(w'_k(\tau))\right] d\tau \\
&+ \int_t^{t+h} \left[\Phi_{t+h}(w_k(\tau)) + \mathrm{X}_{t+h}(Dw_k(\tau)) + \Psi_{t+h}(w'_k(\tau))\right] d\tau \\
&= I + J, \qquad (3.29)
\end{aligned}$$

say. The functions w_k, Dw_k and w'_k are, by hypothesis, continuous on the compact interval $[0, a^2(s_0 - r)^2]$, and therefore bounded by some positive constant Ξ. Since the operators $\Phi_{t+h} - \Phi_t$, $\mathrm{X}_{t+h} - \mathrm{X}_t$ and $\Psi_{t+h} - \Psi_t$ converge uniformly to the zero operator on bounded sets as $h \to 0$, the integral I tends to 0 as $h \to 0$. The integrand of J is bounded, because the operators Φ_{t+h}, X_{t+h} and Ψ_{t+h} are bounded uniformly with respect to $t + h$, and their arguments are bounded by Ξ. As $h \to 0$, the range of integration defining J approaches zero, so that $J \to 0$. Therefore $\|w'_{k+1}(t+h) - w'_{k+1}(t)\|_r \to 0$ as $h \to 0$, and w_{k+1} is thus continuously differentiable with respect to t; by induction, w_k is continuously differentiable for each $k \geq 1$. Now w' is the uniform limit of the sequence of continuous functions $\{w'_k\}$, and is thus continuous itself, so that w is continuously differentiable.

It has been shown that if sequences of positive numbers $\{\epsilon_k\}$ and $\{a_k\}$ can be found such that a_k decreases to a positive limit a, inequality (3.23) holds, and whenever $0 < r < s_0$ and $0 \leq t < a_k^2(s_0 - r)^2$, inequality (3.22) holds, then the approximations w_k, for $k = 1, 2, \ldots$, converge to a continuously differentiable mapping w solving initial-value problem (3.2), (3.3). The solution w comprises the functions w_r, for $0 < r < s_0$, and each w_r maps the interval $[0, a^2(s_0 - r)^2]$ into the space B_r.

It will be shown by induction that, for $j \geq 1$, the inequalities

$$\|w'_j(t) - w'_{j-1}(t)\|_s \leq \epsilon_j S \qquad (3.30)$$

and

$$\|w_j'(t) - w_{j-1}'(t)\|_s \leq \frac{M_j\left(a_j(s_0 - s) - \frac{\sqrt{t}}{2}\right)}{\left(a_j(s_0 - s) - \sqrt{t}\right)^2} \tag{3.31}$$

hold whenever $0 < s < s_0$ and $0 \leq t \leq a_j^2(s_0 - s)^2$, for some sequence of positive constants $\{M_j\}$, and assuming certain relationships between the sequences $\{\epsilon_k\}$ and $\{a_k\}$. Inequality (3.31) is a catalyst inequality facilitating the establishment of inequality (3.30). These two inequalities together are the analogue of inequalities (2.23) and (2.30) in the proof of Theorem 2. As in that proof, after it has been determined what relationships the sequences $\{\epsilon_k\}$, $\{a_k\}$ and $\{M_k\}$ must satisfy in order that this induction can work, the sequences will be specified so as to satisfy those relationships.

In the case $j = 1$, $w_{j-1}(t)$ is to be interpreted as the initial value w_0, and consequently $w_{j-1}'(t)$ is equal to 0. Recalling equation (3.19), we see that if t and r satisfy the relations $0 < r < s_0$ and $0 \leq t < a_1^2(s_0 - r)^2 \leq 1$, then

$$\begin{aligned}
\|w_1'(t)\|_r &= \left\| \int_0^t \left[\Phi_t(w_0) + \mathrm{X}_t(Dw_0) + \Psi_t(0)\right] d\tau \right\|_r \\
&= t\left\|\Phi_t(w_0) + \mathrm{X}_t(Dw_0) + \Psi_t(0)\right\|_r \\
&\leq t\frac{X_0 + Y_0 + Z_0}{(s_0 - r)^2} && (3.32) \\
&\leq a_1^2(X_0 + Y_0 + Z_0). && (3.33)
\end{aligned}$$

Therefore, if $M_1 = a_1^3 s_0(X_0 + Y_0 + Z_0)$, then

$$\|w_1'(t)\|_s \leq \frac{M_1}{a_1 s_0} \leq \frac{M_1\left(a_1(s_0 - s) - \frac{\sqrt{t}}{2}\right)}{\left(a_1(s_0 - s) - \sqrt{t}\right)^2}$$

whenever $0 < s < s_0$ and $0 \leq t \leq a_1^2(s_0 - s)^2$, and if

$$\epsilon_1 \geq \frac{a_1^2(X_0 + Y_0 + Z_0)}{S}, \tag{3.34}$$

then

$$\|w_1'(t) - w_0'\|_r \leq \epsilon_1 S. \tag{3.35}$$

Suppose inequalities (3.31) and (3.30) are valid for $j = k$; if $k \geq 1$, then equation (3.20) gives

$$\begin{aligned}
\|w_{k+1}'(t) - w_k'(t)\|_s = \Bigg\| \int_0^t &\Big[\Phi_t(w_k(\tau)) - \Phi_t(w_{k-1}(\tau)) + \mathrm{X}_t(Dw_k(\tau)) \\
- \quad &\mathrm{X}_t(Dw_{k-1}(\tau)) + \Psi_t(w_k'(\tau)) - \Psi_t(w_{k-1}'(\tau))\Big] d\tau \Bigg\|_s . \qquad (3.36)
\end{aligned}$$

Nagumo's lemma and Inequality (3.16) imply

$$\|w'_{k+1}(t) - w'_k(t)\|_s \leq \left[KT + \frac{LT}{\bar{s} - s} + M\right] \int_0^t \|w'_k(\tau) - w'_{k-1}(\tau)\|_r d\tau \tag{3.37}$$

for $r \in \left(s, s_0 - \frac{\sqrt{t}}{a_k}\right)$. Letting $N = (KT + M)s_0 + LT$ gives the simpler bound

$$\|w'_{k+1}(t) - w'_k(t)\|_s \leq \frac{N}{r - s} \int_0^t \|w'_k(\tau) - w'_{k-1}(\tau)\|_r d\tau. \tag{3.38}$$

For definiteness, let

$$r = \left(s_0 - \frac{\sqrt{t}}{a_k} + s\right) /2. \tag{3.39}$$

Inequality (3.31) implies

$$\begin{aligned} \int_0^t \|w'_k(\tau) - w'_{k-1}(\tau)\|_r d\tau &\leq \int_0^t \frac{M_k \left(a_k(s_0 - r) - \frac{\sqrt{\tau}}{2}\right)}{\left(a_k(s_0 - r) - \sqrt{\tau}\right)^2} d\tau \\ &= \frac{M_k t}{a_k(s_0 - r) - \sqrt{t}} \\ &= \frac{M_k t}{\frac{a_k}{2}\left(s_0 + \frac{\sqrt{t}}{a_k} - s\right) - \sqrt{t}}, \end{aligned} \tag{3.40}$$

and substituting this into inequality (3.38) gives

$$\begin{aligned} \|w'_{k+1}(t) - w'_k(t)\|_s &\leq \frac{N}{\left(s_0 - \frac{\sqrt{t}}{a_k} - s\right)/2} \frac{M_k t}{\frac{a_k}{2}\left(s_0 + \frac{\sqrt{t}}{a_k} - s\right) - \sqrt{t}} \\ &= \frac{4a_k M_k N t}{\left(a_k(s_0 - s) - \sqrt{t}\right)^2}. \end{aligned} \tag{3.41}$$

For each fixed s, the last expression is maximised when t is greatest, that is, when $t = a_{k+1}^2(s_0 - s)^2$. Thus

$$\|w'_{k+1}(t) - w'_k(t)\|_s \leq \frac{4a_k M_k N a_{k+1}^2}{(a_k - a_{k+1})^2}, \tag{3.42}$$

so that inequality (3.30) will be satisfied for $j = k + 1$ provided

$$4M_k N a_k a_{k+1}^2 \leq S\epsilon_{k+1}(a_k - a_{k+1})^2. \tag{3.43}$$

It remains to establish inequality (3.31) for $j = k + 1$. If

$$\frac{4a_k M_k N t}{\left(a_k(s_0 - s) - \sqrt{t}\right)^2} \leq \frac{M_{k+1}\left(a_{k+1}(s_0 - s) - \frac{\sqrt{t}}{2}\right)}{\left(a_{k+1}(s_0 - s) - \sqrt{t}\right)^2} \tag{3.44}$$

holds for $0 < s < s_0$, $0 \le t < a_{k+1}^2(s_0 - s)^2$, inequality (3.41) implies that inequality (3.31) is satisfied. Inequality (3.44) is satisfied if

$$M_{k+1} \ge 8\frac{a_{k+1}^3}{a_k} s_0 N M_k. \tag{3.45}$$

For simplicity let

$$M_j = (8a_1^2 s_0 N)^{j-1} M_1 \tag{3.46}$$

for general j, so that, since $a_{k+1} < a_k < a_1$, inequality (3.45) holds. Substituting this into inequality (3.43) gives

$$4(8a_1^2 s_0 N)^{k-1} M_1 N a_k a_{k+1}^2 \le S\epsilon_{k+1}(a_k - a_{k+1})^2. \tag{3.47}$$

It remains to prove the existence of sequences $\{\epsilon_k\}$ and $\{a_k\}$ such that inequalities (3.23) and (3.47) hold. To this end let $\epsilon_k = r^{k-1}(1-r)$, for some $r \in (0,1)$, so that the relation (3.23) holds with equality. Now a_1 must be chosen small enough to satisfy inequality (3.34). Inequality (3.47) becomes

$$4(8a_1^2 s_0 N)^{k-1} M_1 N a_k a_{k+1}^2 \le S r^k (1-r)(a_k - a_{k+1})^2. \tag{3.48}$$

Since $a_{k+1} < a_k < a_1$, this inequality is satisfied if

$$\frac{a_k}{a_{k+1}} \ge 1 + \sqrt{\frac{4M_1 a_1 N}{Sr(1-r)}\left(\frac{8a_1^2 s_0 N}{r}\right)^{k-1}}. \tag{3.49}$$

Let $\Gamma = \sqrt{\frac{4M_1 a_1 N}{S(r-r^2)}}$ and $\Delta = \sqrt{\frac{8a_1^2 s_0 N}{r}}$. In terms of these constants, inequality (3.49) is

$$a_{k+1} \le \frac{a_k}{1 + \Gamma\Delta^{k-1}}. \tag{3.50}$$

This relation is satisfied with equality if, for all $k \ge 2$,

$$a_k = \frac{a_1}{(1+\Gamma\Delta^{k-2})(1+\Gamma\Delta^{k-3})\cdots(1+\Gamma)}, \tag{3.51}$$

where a_1 will be defined. Now the existence result requires only that $a_* = \lim_{i\to\infty} a_i$ remain positive, and this is true if

$$a_1 > 0 \tag{3.52}$$

and the series

$$\sum_{i=1}^{\infty} \Gamma\Delta^{i-1}$$

converges, i.e., if $0 < \Delta < 1$. By the definition of Δ, this means that

$$a_1 < \sqrt{\frac{r}{8s_0N}}. \tag{3.53}$$

Fixing r arbitarily in $(0,1)$, we now have only the restrictions (3.24), (3.34), (3.52), and (3.53) on a_1, and it is clear that these may be satisfied simultaneously.

■

3.2 Proof of Uniqueness

In this section it is shown that any two continuously differentiable solutions u_r and v_s to initial value problem (3.2), (3.3), mapping from the common interval $[0,T]$ into spaces B_r and B_s, respectively, where $0 < s < r < s_0$, must coincide where their domains overlap, that is, they must be compatible, satisfying

$$I_{r,s}(u_r(t)) = v_s(t). \tag{3.54}$$

Suppose $u_r : [0,T] \to B_r$ is a continuously differentiable solution to initial value problem (3.2), (3.3), i.e.,

$$u_r'(t) = \int_0^t [\Phi_t(u_r(\tau)) + \mathrm{X}_t((Du_r)(\tau)) + \Psi_t(u_r'(\tau))]\, d\tau \tag{3.55}$$

and

$$u_r(0) = w_0 \tag{3.56}$$

in B_r. The operator $I_{r,s}$ is linear and commutes with $'$, D, Φ_t, X_t and Ψ_t; thus, for $u_s = I_{r,s}u_r$,

$$u_s'(t) = \int_0^t [\Phi_t(u_s(\tau)) + \mathrm{X}_t((Du_s)(\tau)) + \Psi_t(u_s'(\tau))]\, d\tau \tag{3.57}$$

and

$$u_s(0) = w_0 \tag{3.58}$$

in B_s. If $v_s \in B_s$ is also a continuously differentiable solution, then

$$v_s'(t) = \int_0^t [\Phi_t(v_s(\tau)) + \mathrm{X}_t((Dv_s)(\tau)) + \Psi_t(v_s'(\tau))]\, d\tau \tag{3.59}$$

and

$$v_s(0) = w_0 \tag{3.60}$$

in B_s. Subtracting equation (3.59) from (3.57) gives

$$u_s'(t) - v_s'(t) = \int_0^t [\Phi_t(u_s(\tau)) - \Phi_t(v_s(\tau)) + \mathrm{X}_t((Du_s)(\tau)) \\ -\mathrm{X}_t((Dv_s)(\tau)) + \Psi_t(u_s'(\tau)) - \Psi_t(v_s'(\tau))]\, d\tau. \qquad (3.61)$$

Using the bounds (3.16), and letting $h(t) = u_s(t) - v_s(t) \in B_s$ for all $t \in [0, T]$, gives

$$\|h'(t)\|_q \leq \int_0^t \left[K\|h(\tau)\|_q + \frac{L}{p-q}\|h(\tau)\|_p + M\|h'(\tau)\|_q \right] d\tau, \qquad (3.62)$$

provided $0 < q < p \leq s$.

It will firstly be shown that there exists a positive number a such that if $h(\tau) = 0$ for all $\tau \in [0, \bar{T}]$, where $\bar{T}$ is any nonnegative number, and $0 < q < s$, then $\|h'(t)\|_q = 0$ for $t \in [\bar{T}, \bar{T} + a(s-q)]$. This relation and the injective property of $I_{s,q}$ can then be used to show that $\|h'(t)\|_s = 0$: because the holomorphic function represented by the point $h'(t)$ is identically zero on the open set $\{z \in \mathcal{C} : |z| < q\}$, it must be identically zero on the full domain $\{z \in \mathcal{C} : |z| < s\}$, whenever $t \in [\bar{T}, \bar{T} + as)$, and certainly whenever $t \in [\bar{T}, \bar{T} + as/2]$. Since $h'(0) = 0$, this means that $h'(t) = 0$ for all $t \in [0, T)$; moreover, since also $\|h(0)\| = 0$, it also implies that $h(t) = 0$ $\forall t \in [0, T]$. Thus, the two arbitrary solutions u_s and v_s whose difference is given by h are equal.

Let $a \in (0, \min\{1/s, T/s\})$. Fix $\bar{T} \in [0, T)$, and assume that $h(t) = h'(t) = 0$ whenever $0 \leq t \leq \bar{T}$. Since $u_s'(t)$ and $v_s'(t)$ are continuous on $[0, T]$, h' is also continuous on this interval and consequently there is a finite number Ξ such that

$$\Xi = \sup\{\|h'(t)\|_q : 0 < q < s \text{ and } \bar{T} \leq t \leq \min\{\bar{T} + a(s-q), T\}\}. \qquad (3.63)$$

It will be shown that $\Xi \leq \rho\Xi$, for some $\rho \in [0, 1)$, and, thus, that Ξ must be zero.

Let q and t satisfy $0 < q < s$ and $\bar{T} \leq t \leq \min\{\bar{T} + a(s-q), T\}$. From the bound (3.62), replacing $\|h'(t)\|$ and $\|h(t)\|$ on the right side both by Ξ for $\tau > \bar{T}$ and by 0 for $\tau \leq \bar{T}$, we obtain

$$\|h'(t)\|_q \leq \Xi \int_{\bar{T}}^t \left[K + \frac{L}{s-q} + M \right] d\tau = \Xi \left[(t - \bar{T})(K + M) + \frac{t - \bar{T}}{s - q} L \right]. \qquad (3.64)$$

Since $t - \bar{T} \leq a(s-q) < as$, and $\frac{t-\bar{T}}{s-q} \leq a$, this becomes

$$\|h'(t)\|_q \leq \Xi a[s(K + M) + L]. \qquad (3.65)$$

Clearly we can now choose some positive number a so small that the factor

$$a[s(K + M) + L]$$

is less than unity; consequently, $\Xi = 0$. Note that a is independent of $\bar{T}$, and thus, the desired uniqueness result is shown.
■

Chapter 4

A More General Retardation Operator

The differential equation (3.2) considered in the last chapter is more general than equation (2.50) considered by Sekine and Yamanaka (*op. cit.*), in that it expresses the current derivative in terms not only of t and retarded values of the solution, but of retarded values of the derivative as well. However, in equation (2.50), the retardation is more general than in equation (3.2). Leaving aside for the moment the dependence on retarded values of the derivative, the right-hand side of equation (3.2) becomes

$$\int_0^t \left[\Phi_t(u(\tau)) + \mathrm{X}_t(D(u(\tau)))\right] d\tau,$$

which is a special case of the right-hand side

$$F\left(t, \overleftarrow{u}(t)\right)$$

of equation (2.50). Therefore, in this chapter, an equation is considered which combines the dependence on retarded values of the time-like derivative appearing in the last chapter with the more general form of retardation considered by Sekine and Yamanaka. The required notation will be explained first.

The solution u will map into the scale of Banach spaces

$$\{H_r : 0 < r < s_0\}$$

with scale operators

$$\{J_{r,s} : 0 < s < r < s_0\},$$

not necessarily the scale

$$\{B_r : 0 < r < s_0\}$$

of holomorphic function spaces. In the initial-value problem (2.50), $u'(t)$ is given in terms of t and values of $u(x)$ for $x \in [t-d, t]$, an interval of fixed size. A point in the Banach space D_r is used to describe the behaviour of u over this interval. Here, instead, $u'(t)$ will be given in terms of t and values of $u(x)$ and $u'(x)$ for $x \in [0, t]$, an interval of varying length. If t is thought of as time, this differential equation gives the rate of change of a system in terms of the current time together with the history of the system over the whole duration from 0 to t. Consequently, a continuum of Banach spaces D_r^t will be required in place of the single space D_r. For each $r \in (0, s_0)$ and $t \geq 0$, let D_r^t be the space of all differentiable mappings from the interval $[0, t]$ into H_r. If $\eta \in D_r$, then define

$$\|\eta\|_r^t = \sup\{\|\eta(x)\|_r : x \in [0, t]\}.$$

Under the norm $\|\cdot\|_r^t$, D_r^t is a Banach space, and, for each $t \geq 0$, the collection

$$\{D_r^t : 0 < r < s_0\}$$

is a scale of Banach spaces. If $0 < s < r < s_0$, the scale operator $Q_{r,s}^t$ is defined by

$$(Q_{r,s}^t \eta)(x) = J_{r,s}(\eta(x))$$

whenever $t \geq 0$, $\eta \in D_r^t$, and $x \in [0, t]$.

If $u : [0, T] \to H_r$, where T is some positive constant, then, for each $t \in [0, T]$, define $\overleftarrow{u}(t)$ to be the point in D_r^t satisfying

$$\left(\overleftarrow{u}(t)\right)(x) = u(x)$$

for each $x \in [0, t]$. Now $\overleftarrow{u}(t)$ describes the behaviour of u over the whole interval $[0, t]$.

The initial-value problem considered in this chapter is

$$u'(t) = F\left(t, \overleftarrow{u}(t), \overleftarrow{u'}(t)\right) \tag{4.1}$$

$$u(0) = u_0. \tag{4.2}$$

The initial value u_0 must belong to every H_r with $0 < r < s_0$. The function F must satisfy certain conditions:

(i) There exists a unique $w_0 \in H_r$ such that

$$F(0, u_0, w_0) = w_0. \tag{4.3}$$

(ii) There are positive constants R_1, R_2 and T such that F maps continuously into H_s at the point (t, v, ξ) wherever $0 < s < r < s_0$, $0 \le t \le T$, $\|v - u_0\|_r^t \le R_1$ and $\|\xi - w_0\|_r^t \le R_2$.

(iii) F is a generalised Cauchy-Riemann operator with respect to its second argument, that is, there is a positive constant C such that if $0 < s < r < s_0$, $0 \le t \le T$, $\|v - u_0\|_r^t \le R_1$, $\|\phi - u_0\|_r^t \le R_1$ and $\|\xi - w_0\|_r^t \le R_2$, then

$$\|F(t, \phi, \xi) - F(t, v, \xi)\|_s \le \frac{C\,\|\phi - v\|_r^t}{r - s}.$$

(iv) F is a bounded operator, with norm $D < 1$ uniform in the scale parameter r, with respect to its third argument, that is, if $0 < r < s_0$, $0 \le t \le T$, $\|v - u_0\|_r^t \le R_1$, $\|\xi - w_0\|_r^t \le R_2$ and $\|\nu - w_0\|_r^t \le R_2$, then

$$\|F(t, v, \nu) - F(t, v, \xi)\|_r \le D\,\|\nu - \xi\|_r^t\,.$$

(v) As $a \to 0$, the norm

$$\|F(a^2(s_0 - r)^2, u_0, w_0) - w_0\|_r \to 0$$

uniformly over $r \in (0, s_0)$, that is,

$$\sup\{\|F(a^2(s_0 - r)^2, u_0, w_0) - w_0\|_r : 0 < r < s_0\} \to 0.$$

Of course, $\overleftarrow{u'}(t)$ can be obtained from $\overleftarrow{u}(t)$ by differentiation, since if $0 \le x \le t$, then

$$\begin{aligned}
\left(\overleftarrow{u'}(t)\right)(x) &= u'(x)\,. \\
&= \frac{d}{dx}(u(x)) \\
&= \frac{d}{dx}\left(\left(\overleftarrow{u}(t)\right)(x)\right).
\end{aligned}$$

An explicit formal dependence of F on $\overleftarrow{u'}(t)$ is maintained, however, so that conditions (iii) and (iv) above are more easily expressible and applicable.

Condition (i) is an algebraic compatibility condition, arising directly from the initial-value problem. The substitution $t = 0$ into equation (4.1) gives

$$u'(0) = F\left(0, \overleftarrow{u}(0), \overleftarrow{u'}(0)\right).$$

The space D_r^0, to which $\overleftarrow{u}(0)$ and $\overleftarrow{u'}(0)$ belong, is isomorphic to H_r. The point $\overleftarrow{u}(0)$ is the function defined on only the singleton $\{0\}$, mapping 0 to $u(0)$, and thus no distinction is made between $\overleftarrow{u}(0)$ and $u(0)$. Similarly $\overleftarrow{u'}(0)$ and $u'(0)$ are equivalent, so that

$$u'(0) = F(0, u(0), u'(0)).$$

Letting $w_0 = u'(0)$ yields equation (4.3); if this cannot be satisfied, then initial-value problem (4.1), (4.2) cannot have a solution.

Condition (ii) describes the domain of the right-hand side operator F. Its first argument must belong to the interval $[0, T]$, and its second and third must belong to the balls in D_r^t of radius R_1 and R_2 centred on u_0 and w_0, respectively. In the proof, the constants R_1 and R_2 will be replaced by

$$S = \min\left\{\frac{R_1}{T'} - \|w_0\|_{s_0}, R_2\right\}, \tag{4.4}$$

where T' is some positive constant not exceeding T, such that

$$\frac{R_1}{T'} - \|w_0\|_{s_0} > 0. \tag{4.5}$$

(This is a generalisation of equation (3.21) in the proof of Theorem 3.) Suppose, for example, that u is a mapping from the interval $[0, T']$ into the scale $\{H_r\}$, satisfying

$$\|u'(t) - w_0\|_r \leq S.$$

Then

$$\|u'(t) - w_0\|_r \leq R_2,$$

so that (ii) can be used to bound $u'(t)$. Further, if $u(0) = u_0$, then

$$\begin{aligned}
\|u(t) - u_0\|_r &\leq \int_0^t \|u'(\tau)\|_r d\tau \\
&\leq \int_0^t (\|u'(\tau) - w_0\|_r + \|w_0\|_r) d\tau \\
&\leq tS + \|w_0\|_r \\
&\leq T'S + \|w_0\|_{s_0} \\
&\leq R_1,
\end{aligned}$$

so that the bound (iii) will hold for $u(t)$. Recall that although the initial value u_0

need not belong to H_{s_0}, but only to every H_r with $0 < r < s_0$, the initial derivative w_0 must belong to H_{s_0}; this paragraph indicates the reason.

Condition (*iii*) states that F is a Cauchy-Riemann operator with respect to its second argument, representing retarded values of the solution. It has been seen that this can be used to model a first order complex differentiation operator, when the scale of Banach spaces under consideration is the scale $\{B_r : 0 < r < s_0\}$ of holomorphic function spaces.

Condition (*iv*) requires F to be bounded, uniformly in the scale parameter r, in its third argument, representing retarded values of the derivative of the solution. Again referring to the case of the scale $\{B_r : 0 < r < s_0\}$ of holomorphic function spaces, the right-hand side of differential equation (4.1) may involve retarded values of quantities such as u and $\frac{\partial u}{\partial z}$, through the second argument, and of derivatives such as $\frac{\partial u}{\partial t}$, through the third, but not of $\frac{\partial^2 u}{\partial t \partial z}$.

Condition (v) allows the first approximation to the solution to be formed in the proof of existence, and is analogous to the bound (2.9) in Theorem (2).

Now it will be proven, using methods similar to those in previous chapters, that the initial-value problem under consideration has a unique local solution.

Theorem 4 *Under conditions (i) to (v) above, the initial-value problem (4.1), (4.2) has a unique solution u mapping into the scale*

$$\{H_r : 0 < r < s_0\}.$$

The domains $[0, T_r)$ of the functions

$$u_r : [0, T_r) \to H_r$$

constituting the mapping u can be chosen to satisfy

$$T_r = a^2 s_0^2$$

for some positive a.

4.1 Proof of Existence

The method of successive approximations will again be used to establish existence. The integral equation

$$u(t) = u_0 + \int_0^t F\left(\tau, \overleftarrow{u}(\tau), \overleftarrow{u'}(\tau)\right) d\tau \tag{4.6}$$

is equivalent to initial-value problem (4.1), (4.2). The first approximation u_1 is defined by

$$u_1(t) = u_0 + \int_0^t F\left(\tau, \overleftarrow{u_0}, \overleftarrow{w_0}\right) d\tau. \tag{4.7}$$

For each $\tau \geq 0$, the second and third arguments in the ordered triple

$$\left(\tau, \overleftarrow{u_0}, \overleftarrow{w_0}\right)$$

are to be interpreted as those elements in D_τ^τ which map every point in the interval $[0, \tau]$ to the points u_0 and w_0 respectively, that is, $\overleftarrow{u_0}$ represents the 'history' from 0 to τ of the constant function u_0, and similarly for $\overleftarrow{w_0}$. The subsequent approximations are defined recursively by

$$u_{k+1}(t) = u_0 + \int_0^t F\left(\tau, \overleftarrow{u_k}(\tau), \overleftarrow{u_k'}(\tau)\right) d\tau, \tag{4.8}$$

for each $k \geq 1$.

Differentiating equation (4.7) with respect to t and evaluating at $t = 0$ gives

$$\begin{aligned} u_1'(0) &= F\left(0, \overleftarrow{u_0}, \overleftarrow{w_0}\right) \\ &= F(0, u_0, w_0) \\ &= w_0. \end{aligned}$$

Suppose $u_k'(0) = w_0$ for some $k \in N$. Then

$$\begin{aligned} u_{k+1}'(0) &= F\left(0, \overleftarrow{u_k}(0), \overleftarrow{u_k'}(0)\right) \\ &= F(0, u_k(0), u_k'(0)) \\ &= F(0, u_0, w_0) \\ &= w_0. \end{aligned}$$

Thus, by induction, for each $k \in N$,

$$u_k'(0) = w_0. \tag{4.9}$$

As in the proof of Theorem 3, it will first be shown that if the derivatives $\{u_k'\}$ of the approximations converge absolutely and uniformly whenever $0 < r < s_0$ and $0 \leq t < a^2(s_0 - r)^2$, for some positive a, then their limit u' defines a solution u to initial-value problem (4.1), (4.2) by

$$u(t) = u_0 + \int_0^t u'(\tau) d\tau.$$

The convergence of $\{u'_k\}$ will then be established. Fix $a > 0$, and suppose there exists a sequence $\{\epsilon_k\}$ of positive numbers satisfying inequality (2.24) such that

$$\|u'_1(t) - w_0\|_r \leq \epsilon_1 S \tag{4.10}$$

and, for each $k \in N$,

$$\|u'_{k+1}(t) - u'_k(t)\|_r \leq \epsilon_{k+1} S \tag{4.11}$$

whenever $0 < r < s_0$ and $0 \leq t < a^2(s_0 - r)^2$. Fix such r and t. The sequence $\{u'_k(t)\}$ is Cauchy in H_r, and therefore converges to some limit $u'(t) \in H_r$. Further, for each $\tau \in [0, t]$, τ also satisfies the inequality $0 \leq \tau < a^2(s_0 - r)^2$, so that the sequence $\{u'_k(\tau)\}$ is Cauchy in H_r, and converges to a limit $u'(\tau)$. This convergence is uniform in τ, so that in fact the sequence $\{\overleftarrow{u'_k}(t)\}$ converges to a limit $\overleftarrow{u'}(t)$ in D^t_r. Choose $q > r$ also satisfying $0 < q < s_0$ and $0 \leq t < a^2(s_0 - q)^2$, for instance,

$$q = \frac{1}{2}\left(s_0 - \frac{\sqrt{t}}{a} + r\right).$$

Then the above holds for q also, that is, $\overleftarrow{u'_k}(t)$ converges to $\overleftarrow{u'}(t)$ in the space D^t_q.

Having defined the derivative u', integrate it to define the mapping u,

$$u(t) = u_0 + \int_0^t u'(\tau)d\tau.$$

Now $u(0) = u_0$, and since each of the approximations u_k also has initial value u_0,

$$\begin{aligned}
\|u(t) - u_k(t)\|_r &= \int_0^t \|u'(\tau) - u'_k(\tau)\|_r d\tau \\
&\leq \int_0^t \sum_{j=k}^{\infty} \|u'_{j+1}(\tau) - u'_j(\tau)\|_r d\tau \\
&\leq \int_0^t \sum_{j=k}^{\infty} \epsilon_{j+1} d\tau \\
&\leq a^2 s_0^2 S \left(\sum_{j=1}^{\infty} \epsilon_j - \sum_{j=1}^{k} \epsilon_j\right) \\
&\to 0
\end{aligned}$$

uniformly in $t \in [0, a^2(s_0 - q)^2)$, so that

$$\left\|\overleftarrow{u(t)} - \overleftarrow{u_k(t)}\right\|_r^t \to 0.$$

Thus

$$\begin{aligned}
&\left\| u(t) - \int_0^t F\left(\tau, \overleftarrow{u}(\tau), \overleftarrow{u'}(\tau)\right) d\tau \right\|_r \\
\leq\ & \|u(t) - u_{k+1}(t)\|_r + \left\| u_{k+1}(t) - \int_0^t F\left(\tau, \overleftarrow{u_k}(\tau), \overleftarrow{u_k'}(\tau)\right) d\tau \right\|_r \\
+\ & \left\| \int_0^t F\left(\tau, \overleftarrow{u_k}(\tau), \overleftarrow{u_k'}(\tau)\right) d\tau - \int_0^t F\left(\tau, \overleftarrow{u}(\tau), \overleftarrow{u'}(\tau)\right) d\tau \right\|_r \\
\leq\ & \|u(t) - u_{k+1}(t)\|_r \\
+\ & \int_0^t \left(\frac{C}{q-r} \left\| \overleftarrow{u_k}(\tau) - \overleftarrow{u}(\tau) \right\|_q^\tau + D \left\| \overleftarrow{u_k'}(\tau) - \overleftarrow{u'}(\tau) \right\|_r^t \right) d\tau.
\end{aligned}$$

Each of the three above norms tends to zero, so that the first expression, having no dependence on k, is equal to zero. This implies that u satisfies integral equation (4.6). Since also

$$u'(0) = \lim u_k'(0) = \lim w_0 = w_0,$$

u satisfies initial-value problem (4.1), (4.2).

As in the previous proofs, a strictly decreasing sequence $\{a_k\}$ of positive numbers will be employed, whose limit a will ultimately be made positive. As in the proof of Theorem 3, it will be shown that, for each $j \in N$, if $0 < r < s_0$ and $0 \leq t < a_j^2(s_0 - r)^2$, then

$$\|u_j'(t) - u_{j-1}'(t)\|_r \leq \epsilon_j S. \tag{4.12}$$

If $t \in [0, a_j^2(s_0 - r)^2)$, then $[0, t] \subset [0, a_j^2(s_0 - r)^2)$, and inequality (4.12) implies

$$\left\| \overleftarrow{u_j'}(t) - \overleftarrow{u_{j-1}'}(t) \right\|_r^t \leq \epsilon_j S. \tag{4.13}$$

The 'catalyst' inequality used in this proof analogous to (3.31) is

$$\|u_j'(t) - u_{j-1}'(t)\|_r \leq M_j \frac{a_j(s_0 - r) - \frac{\sqrt{t}}{2}}{(a_j(s_0 - r) - \sqrt{t})^2}. \tag{4.14}$$

Here $u_{j-1}'(t)$ should be interpreted as w_0 in the case $j = 1$, and $\{M_j\}$ is some sequence of positive numbers to be specified. Note that the right-hand side of inequality (4.14) is increasing with increasing t, so that if inequality (4.14) holds for $0 < r < s_0$, $0 \leq t < a_j^2(s_0 - r)^2$, then also

$$\left\| \overleftarrow{u_j'}(t) - \overleftarrow{u_{j-1}'}(t) \right\|_r^t \leq M_j \frac{a_j(s_0 - r) - \frac{\sqrt{t}}{2}}{(a_j(s_0 - r) - \sqrt{t})^2}. \tag{4.15}$$

An induction will determine what relationships the sequences $\{\epsilon_k\}$, $\{a_k\}$ and $\{M_k\}$ need to satisfy to ensure that inequalities (4.12) and (4.14) hold. Fix T' positive but sufficiently small that inequality (4.5) holds. Each a_k must be chosen small enough that, whenever $0 < r < s_0$ and $0 \le t < a_k^2(s_0 - r)^2$,

$$t \le T'.$$

This is implied by

$$a_k \le \frac{\sqrt{T'}}{s_0},$$

or, since the sequence $\{a_k\}$ is decreasing,

$$a_1 \le \frac{\sqrt{T'}}{s_0}. \tag{4.16}$$

Differentiating equation (4.7),

$$u_1'(t) = F\left(t, \overleftarrow{u_0}, \overleftarrow{w_0}\right).$$

Condition (v) implies that there is some number $\bar{a}$ depending on ϵ_1 such that, if

$$a_1 \le \bar{a}, \tag{4.17}$$

then whenever $0 < r < s_0$ and $0 \le t < a_1^2(s_0 - r)^2$,

$$\|u_1'(t) - w_0\|_r \le \epsilon_1 S, \tag{4.18}$$

so that inequality (4.12) holds in the case $j = 1$.

The expression

$$M_1 \frac{a_1(s_0 - r) - \frac{\sqrt{t}}{2}}{(a_1(s_0 - r) - \sqrt{t})^2}$$

is bounded below, for fixed M_1 and a_1, on the region $0 < r < s_0$, $0 \le t < a_1^2(s_0 - r)^2$, by the expression

$$\frac{M_1}{a_1 s_0},$$

so that if

$$\epsilon_1 S \le \frac{M_1}{a_1 s_0},$$

then, by inequality (4.18), inequality (4.14) holds in the case $j = 1$. Therefore let

$$M_1 = \epsilon_1 a_1 s_0 S. \tag{4.19}$$

Now let $k \in N$ and suppose that inequalities (4.12) and (4.14) hold whenever $0 < r < s_0$ and $0 \le t < a_j^2(s_0 - r)^2$, for $j \in \{1, 2, \ldots, k\}$. Since the region $0 < r < s_0$, $0 \le t < a_k^2(s_0 - r)^2$ is contained in each of the regions $0 < r < s_0$, $0 \le t < a_j^2(s_0 - r)^2$ for $j \in \{1, 2, \ldots, k\}$, inequality (4.12) holds for each such j whenever $0 < r < s_0$ and $0 \le t < a_k^2(s_0 - r)^2$. Thus

$$\begin{aligned} \|u_k'(t) - w_0\|_r &\le \sum_{j=1}^{k} \|u_j'(t) - u_{j-1}'(t)\|_r \\ &\le \sum_{j=1}^{k} \epsilon_j S \le S, \end{aligned}$$

where $u_{j-1}(t)$ is to be interpreted as w_0 in the case $j = 1$. Thus, as explained on page 52, the ordered triple

$$\left(t, \overleftarrow{u_k}(t), \overleftarrow{u_k'}(t)\right)$$

now belongs to the domain of the right-hand side operator F_r (that is, the facet of F which maps from $R \times D_r^t \times D_r^t$ to H_s for each $s \in (0, r)$) at each point in the region, and conditions (*iii*) and (*iv*) are applicable.

Now choose r, s and t satisying

$$0 < s < r < s_0 \tag{4.20}$$

and

$$0 \le t < a_{k+1}^2(s_0 - r)^2 < a_k^2(s_0 - r)^2 < a_k^2(s_0 - s)^2. \tag{4.21}$$

In view of definition (4.8),

$$\|u_{k+1}'(t) - u_k'(t)\|_s = \left\| F\left(t, \overleftarrow{u_k}(t), \overleftarrow{u_k'}(t)\right) - F\left(t, \overleftarrow{u_{k-1}}(t), \overleftarrow{u_{k-1}'}(t)\right) \right\|_s. \tag{4.22}$$

Now conditions (*iii*) and (*iv*) give

$$\|u_{k+1}'(t) - u_k'(t)\|_s \le \frac{C}{r - s} \left\| \overleftarrow{u_k}(t) - \overleftarrow{u_{k-1}}(t) \right\|_r^t + D \left\| \overleftarrow{u_k'}(t) - \overleftarrow{u_{k-1}'}(t) \right\|_s^t. \tag{4.23}$$

If s and t are chosen to satisfy $0 < s < s_0$ and $0 \le t < a_k^2(s_0 - s)^2$, and r is given by

$$r = \frac{1}{2}\left(s_0 - \frac{\sqrt{t}}{a_k} + s\right),$$

then inequalities(4.20) and (4.21) above are satisfied, and inequality (4.23) becomes

$$\|u_{k+1}'(t) - u_k'(t)\|_s \le \frac{2C}{s_0 - \frac{\sqrt{t}}{a_k} - s} \left\| \overleftarrow{u_k}(t) - \overleftarrow{u_{k-1}}(t) \right\|_r^t + D \left\| \overleftarrow{u_k'}(t) - \overleftarrow{u_{k-1}'}(t) \right\|_s^t. \tag{4.24}$$

Because the right-hand side of inequality (4.14) is increasing with t, for each $p \in (0, s_0)$,

$$\left\| \overleftarrow{u'_k}(t) - \overleftarrow{u'_{k-1}}(t) \right\|_p^t \leq M_j \frac{a_j(s_0 - p) - \frac{\sqrt{t}}{2}}{(a_j(s_0 - p) - \sqrt{t})^2}. \tag{4.25}$$

That is,

$$\begin{aligned} \left\| \overleftarrow{u'_k}(t) - \overleftarrow{u'_{k-1}}(t) \right\|_p^t &= \sup \left\{ \|u'_k(\tau) - u'_{k-1}(\tau)\|_p : 0 \leq t \leq \tau \right\} \\ &\leq \sup \left\{ M_k \frac{a_k(s_0 - p) - \frac{\sqrt{\tau}}{2}}{(a_k(s_0 - p) - \sqrt{\tau})^2} : 0 \leq t \leq \tau \right\} \\ &\leq M_k \frac{a_k(s_0 - p) - \frac{\sqrt{t}}{2}}{(a_k(s_0 - p) - \sqrt{t})^2}. \end{aligned}$$

Further, since $u_k(0) = u_{k-1}(0) = u_0$, the bound (4.14) can be integrated to give

$$\|u_k(t) - u_{k-1}(t)\|_p \leq \frac{M_k t}{a_k(s_0 - p) - \sqrt{t}},$$

and the right-hand side of this inequality is increasing with t, so that

$$\left\| \overleftarrow{u_k}(t) - \overleftarrow{u_{k-1}}(t) \right\|_p^t \leq \frac{M_k t}{a_k(s_0 - p) - \sqrt{t}}. \tag{4.26}$$

Putting $p = r$ in inequality (4.26) and $p = s$ in inequality (4.25), inequality (4.24) becomes

$$\begin{aligned} \|u'_{k+1}(t) - u'_k(t)\|_s &\leq \frac{C}{\frac{1}{2}\left(s_0 - \frac{\sqrt{t}}{a_k} - s\right)} \cdot \frac{M_k t}{\frac{1}{2} a_k \left(s_0 + \frac{\sqrt{t}}{a_k} - s\right) - \sqrt{t}} \\ &+ D \frac{M_k \left(a_k(s_0 - s) - \frac{\sqrt{t}}{2}\right)}{(a_k(s_0 - s) - \sqrt{t})^2}, \end{aligned} \tag{4.27}$$

whence judicious use of the inequalities $0 < s < s_0$, $0 \leq t < a_k^2(s_0 - s)^2$ will reveal that

$$\|u'_{k+1}(t) - u'_k(t)\|_s \leq M_{k+1} \frac{a_{k+1}(s_0 - s) - \frac{\sqrt{t}}{2}}{(a_{k+1}(s_0 - s) - \sqrt{t})^2},$$

where

$$M_{k+1} = (16 C a_1^2 s_0 + D) M_k,$$

so that the bound (4.14) holds in the case $j = k + 1$. Hence, in view of equation (4.19), the sequence $\{M_k\}$ is defined by

$$M_k = \epsilon_1 a_1 s_0 S (16 C a_1^2 s_0 + D)^{k-1}. \tag{4.28}$$

In a similar fashion to the derivation of the inequality (4.25), the inequality

$$\left\| \overleftarrow{u'_k}(t) - \overleftarrow{u'_{k-1}}(t) \right\|_p^t \leq \epsilon_k S. \tag{4.29}$$

is implied by the bound (4.12). Inequality (4.27) above was obtained by replacing the norms on the right-hand side of inequality (4.24) with their bounds derived from inequality (4.14) with $j = k$. Estimating the first of these norms in the same way, but instead using inequality (4.29) with $j = k$ for the second, yields

$$\|u'_{k+1}(t) - u'_k(t)\|_s \leq \frac{4CM_k t a_k}{(a_k(s_0 - s) - \sqrt{t})^2} + D\epsilon_k S.$$

With equation (4.28), this becomes

$$\|u'_{k+1}(t) - u'_k(t)\|_s \leq \left(\frac{4Ca_1 a_k s_0 \epsilon_1 t}{(a_k(s_0 - s) - \sqrt{t})^2} (16Ca_1^2 s_0 + D)^{k-1} + D\epsilon_k \right) S.$$

If t is restricted to the interval $[0, a_{k+1}^2(s_0 - s)^2)$, it can be seen that

$$\|u'_{k+1}(t) - u'_k(t)\|_s \leq \left(4Ca_1^2 s_0 \epsilon_1 \left(\frac{a_k}{a_{k+1}} - 1 \right) (16Ca_1^2 s_0 + D)^{k-1} + D\epsilon_k \right) S.$$

Here, one occurance of a_k has been replaced by a_1, since $a_k < a_1$, to simplify the manipulations. Thus, if

$$4Ca_1^2 s_0 \epsilon_1 \left(\frac{a_k}{a_{k+1}} - 1 \right) (16Ca_1^2 s_0 + D)^{k-1} + D\epsilon_k \leq \epsilon_{k+1}, \tag{4.30}$$

then the bound (4.12) holds in the case $j = k + 1$, and the induction step is complete.

This means that if sequences $\{\epsilon_k\}$ and $\{a_k\}$ of positive reals can be found, the latter strictly decreasing, which satisfy the conditions (2.24), (4.16), and (4.30), then the inequalities (4.12) and (4.14) hold for each $j \in N$ whenever $0 < r < s_0$ and $0 \leq t < a_k^2(s_0 - r)^2$. While the catalyst inequality (4.14) is of no further interest here, inequality (4.12) implies, as seen earlier, that the successive approximations u_k converge to a solution u on the region $0 < r < s_0$, $0 \leq t < a^2(s_0 - r)^2$, which will be of positive height if

$$a = \lim_{k \to \infty} a_k > 0.$$

Condition (2.24) is satisfied by the assignment

$$\epsilon_k = (1 - r)r^{k-1}$$

for each k, where r is any ratio strictly between 0 and 1. Put

$$E = 16Ca_1^2 s_0 + D.$$

Require

$$a_1 < \frac{\sqrt{1-D}}{4\sqrt{Cs_0}}, \tag{4.31}$$

so that

$$0 < E < 1.$$

Inequality (4.30) is now

$$4Ca_1^2 s_0(1-r)\left(\frac{a_k}{a_{k+1}} - 1\right)E^{k-1} + D(1-r)r^{k-1} \leq (1-r)r^k, \tag{4.32}$$

or

$$\frac{4Ca_1^2 s_0}{r^{k-1}}\left(\frac{a_k}{a_{k+1}} - 1\right)E^{k-1} \leq r - D. \tag{4.33}$$

Clearly this is possible only if $r > D$, but since $D < 1$, this is allowable.

If condition (4.33) holds with equality, then

$$a_{k+1} = \frac{a_k}{1 + \frac{r-D}{4Ca_1^2 s_0}\left(\frac{r}{E}\right)^{k-1}}.$$

If this is used together with some assignment for a_1 to define recursively the sequence $\{a_k\}$, then inequality (4.30) holds. Now

$$a_k = a_1 \prod_{j=1}^{k-1} \frac{1}{1 + \frac{r-D}{4Ca_1^2 s_0}\left(\frac{r}{E}\right)^{j-1}},$$

so that

$$a = \lim_{k\to\infty} a_k = a_1 \prod_{j=1}^{\infty} \frac{1}{1 + \frac{r-D}{4Ca_1^2 s_0}\left(\frac{r}{E}\right)^{j-1}}.$$

If the series

$$\sum_{k=1}^{\infty} \frac{r-D}{4Ca_1^2 s_0}\left(\frac{r}{E}\right)^{k-1} \tag{4.34}$$

converges, then the infinite product

$$\prod_{k=1}^{\infty}\left(1 + \frac{r-D}{4Ca_1^2 s_0}\left(\frac{r}{E}\right)^{j-1}\right)$$

converges to a finite number (c.f. Titchmarsh [6]), and thus a is positive. The series (4.34) clearly converges if

$$r < E.$$

Therefore fix $E > D$ by some choice of a_1 satisfying inequalities (4.16) and (4.31), and then fix r arbitrarily, such that

$$D < r < E.$$

The conditions (2.24), (4.16), and (4.30) now all hold, and additionally $a > 0$, so that the existence result is established.

■

4.2 Proof of Uniqueness

It will be shown that two solutions u and v to initial-value problem (4.1), (4.2) mapping the interval $[0, \Delta]$ into the space H_q, where $\Delta \in (0, T]$ and $0 < q < s_0$, must be equal. From this it will follow that two mappings into the scale $\{H_r\}$ which solve initial-value problem (4.1), (4.2) must be equal on the intersection of their domains.

Suppose that

$$u : [0, \Delta] \to H_q$$

and

$$v : [0, \Delta] \to H_q$$

are two solutions to initial-value problem (4.1), (4.2). Their difference is a continuous function, and is therefore bounded on the compact interval $[0, \Delta]$. Fix $b \in \left(0, \frac{\sqrt{\Delta}}{q}\right]$, so that whenever $0 < r < q$ and $0 \leq t < b^2(q-r)^2$, $u(t)$ and $v(t)$ both belong to H_r. The expression

$$\frac{(b(s_0 - r) - \sqrt{t})^2}{b(s_0 - r) - \frac{\sqrt{t}}{2}}$$

is bounded above by bs_0 when $0 < r < q$ and $0 \leq t < b^2(q-r)^2$, so that the supremum

$$M = \sup \left\{ \|u'(t) - v'(t)\|_r \frac{(b(q-r) - \sqrt{t})^2}{b(q-r) - \frac{\sqrt{t}}{2}} : 0 < r < s_0 \text{ and } 0 \leq t < b^2(q-r)^2 \right\} \tag{4.35}$$

is some finite nonnegative number. It will be shown that if b is sufficiently small then M must be zero, implying that

$$\|u'(t) - v'(t)\|_r = 0$$

on the region $0 < r < s_0$, $0 \leq t \leq b^2(s_0 - r)^2$. By the injective property of the scale operator, this implies that

$$\|u'(t) - v'(t)\|_q = 0$$

whenever $0 \leq t \leq b^2q^2$, that is, that u' and v' are equal on the interval $[0, b^2q^2]$. Since

$$u(0) = v(0) = u_0,$$

this means also that u and v are equal on the same interval.

Let $0 < s < r < q$ and $0 \leq t < b^2(q-r)^2$. It can be seen from differential equation (4.1) that

$$\begin{aligned} \|u'(t) - v'(t)\|_s &= \left\| F\left(t, \overleftarrow{u}(t), \overleftarrow{u'}(t)\right) - F\left(t, \overleftarrow{v}(t), \overleftarrow{v'}(t)\right) \right\|_s \\ &\leq \frac{C}{r-s} \left\| \overleftarrow{u}(t) - \overleftarrow{v}(t) \right\|_r^t + D \left\| \overleftarrow{u'}(t) - \overleftarrow{v'}(t) \right\|_s^t, \end{aligned} \tag{4.36}$$

using the condtions (*iii*) and (*iv*). Integrating the bound

$$\|u'(t) - v'(t)\|_s \leq M \frac{b(q-s) - \frac{\sqrt{t}}{2}}{(b(q-s) - \sqrt{t})^2}, \tag{4.37}$$

noting that $u(0) = v(0) = u_0$, yields

$$\|u(t) - v(t)\|_s \leq \frac{Mt}{b(q-s) - \sqrt{t}}. \tag{4.38}$$

Because the right-hand sides of inequalities (4.37) and (4.38) are increasing with t, as in the existence proof, these bounds apply also to the retarded expressions, that is,

$$\left\| \overleftarrow{u'}(t) - \overleftarrow{v'}(t) \right\|_s^t \leq M \frac{b(q-s) - \frac{\sqrt{t}}{2}}{(b(q-s) - \sqrt{t})^2}$$

and

$$\left\| \overleftarrow{u}(t) - \overleftarrow{v}(t) \right\|_s^t \leq \frac{Mt}{b(q-s) - \sqrt{t}}.$$

Substituting these bounds into inequality (4.36) gives

$$\|u'(t) - v'(t)\|_s \leq \frac{C}{r-s} \frac{Mt}{b(q-r) - \sqrt{t}} + DM \frac{b(q-s) - \frac{\sqrt{t}}{2}}{(b(q-s) - \sqrt{t})^2},$$

and on letting

$$r = \frac{1}{2}\left(q - \frac{\sqrt{t}}{b} + s\right),$$

this becomes

$$\|u'(t) - v'(t)\|_s \leq \frac{2C}{q - \frac{t}{b} - s} \frac{2Mt}{b(q-s) - \sqrt{t}} + DM \frac{b(q-s) - \frac{\sqrt{t}}{2}}{(b(q-s) - \sqrt{t})^2}.$$

Substituting this into definition (4.35) gives

$$\begin{aligned} M &\leq \sup \left\{ \left(\frac{2C}{q - \frac{t}{b} - s} \frac{2Mt}{b(q-s) - \sqrt{t}} + DM \frac{b(q-s) - \frac{\sqrt{t}}{2}}{(b(q-s) - \sqrt{t})^2} \right) \right. \\ &\qquad \left. \cdot \frac{(b(q-r) - \sqrt{t})^2}{b(q-r) - \frac{\sqrt{t}}{2}} : 0 < r < q \text{ and } 0 \leq t < b^2(q-r)^2 \right\} \\ &\leq \sup \left\{ \left(\frac{4CbMt}{(b(q-s) - \sqrt{t})^2} + DM \frac{b(q-s) - \frac{\sqrt{t}}{2}}{(b(q-s) - \sqrt{t})^2} \right) \frac{(b(q-r) - \sqrt{t})^2}{b(q-r) - \frac{\sqrt{t}}{2}} \right. \\ &\qquad \left. : 0 < r < q \text{ and } 0 \leq t < b^2(q-r)^2 \right\}. \end{aligned} \tag{4.39}$$

Whenever $0 < s < q$ and $0 \leq t < b^2(q-s)^2$,

$$b(q-s) - \frac{\sqrt{t}}{2} > b(q-s) - \frac{b(q-s)}{2} = \frac{b(q-s)}{2},$$

so that

$$t \leq 2b(q-s)\left(b(q-s) - \frac{\sqrt{t}}{2}\right).$$

Thus inequality (4.39) implies

$$\begin{aligned} M &\leq \sup \left\{ \left(\frac{8Cb^2(q-s)M\left(b(q-s) - \frac{\sqrt{t}}{2}\right)}{(b(q-s) - \sqrt{t})^2} + DM \frac{b(q-s) - \frac{\sqrt{t}}{2}}{(b(q-s) - \sqrt{t})^2} \right) \right. \\ &\qquad \left. \cdot \frac{(b(q-r) - \sqrt{t})^2}{b(q-r) - \frac{\sqrt{t}}{2}} : 0 < r < q \text{ and } 0 \leq t < b^2(q-r)^2 \right\} \\ &= M \sup \left\{ 8Cb^2(q-s) + D : 0 < r < q \text{ and } 0 \leq t < b^2(q-r)^2 \right\} \\ &= M(8Cb^2q + D). \end{aligned}$$

Since $D < 1$, b can be chosen sufficiently small that the factor $8Cb^2q + D < 1$, whence, since M is nonnegative and finite, is implied $M = 0$.

As remarked above, the vanishing of M implies that u and v are equal on the interval $[0, b^2q^2]$. If $b^2q^2 < \Delta$ then an induction can be used to show that u and v are equal on the interval $[0, \Delta]$; this is not essentially different from the inductive approach described in the uniqueness proof of Theorem 2, and is omitted.

Let u and v be two mappings into the scale $\{H_r\}$ which solve the initial-value problem (4.1), (4.2). Then, for each $q \in (0, s_0)$, u and v map intervals $[0, T_u)$ and $[0, T_v)$, respectively, into H_q. If t is a nonnegative number belonging to both of these intervals, then u and v each map the interval $[0, t]$ into H_q, so that, as above, u and v are equal on the interval $[0, t]$, and hence $u(t) = v(t)$. Therefore u and v are equal on the intersection of their domains, that is, the solution found in the existence proof is unique.

■

The requirement that the solution w_0 of equation (4.3) be unique can be relaxed. It does not appear at all in the existence proof, and only arises in the uniqueness proof when it is assumed that u and v have the same initial derivative. Thus, if equation (4.3) has more than one solution, then initial-value problem (4.1), (4.2) has a unique solution u for each solution w_0 of the algebraic compatibility condition (4.3), satisfying $u'(0) = w_0$.

Theorem 3 is a corollary of Theorem 4. Let $\{H_r\}$ be the scale $\{B_r\}$ of holomorphic function spaces. Let the operators Φ_t, X_t and Ψ_t satisfy the conditions on Theorem 3, and let

$$F\left(t, \overleftarrow{u}(t), \overleftarrow{u'}(t)\right) = \int_0^t \left(\Phi_t(u(\tau)) + \mathrm{X}_t(Du(\tau)) + \Psi_t(u'(\tau))\right) d\tau.$$

Now conditions (i) to (v) on Theorem 4 hold. The constants R_1, R_2 and T are equal in both theorems, and condition (i) holds. The initial derivative w_0 is equal to zero since

$$u'(0) = \int_0^0 \left(\Phi_0(u(\tau)) + \mathrm{X}_0(Du(\tau)) + \Psi_0(u'(\tau))\right) d\tau = 0,$$

and thus condition (ii) holds. Conditions (iii) and (iv) are easily verified by varying the arguments first of Φ_t and X_t, and then of Ψ_t. Condition (v) is satisfied, since

$$\begin{aligned}
&\left\| \int_0^{a^2(s_0-r)^2} \left(\Phi_{a^2(s_0-r)^2}(u_0) + \mathrm{X}_{a^2(s_0-r)^2}(Du_0) + \Psi_{a^2(s_0-r)^2}(0)\right) d\tau \right\|_r \\
&= a^2(s_0-r)^2 \left(\Phi_{a^2(s_0-r)^2}(u_0) + \mathrm{X}_{a^2(s_0-r)^2}(Du_0) + \Psi_{a^2(s_0-r)^2}(0)\right) \\
&\leq a^2(s_0-r)^2 \frac{X_0 + Y_0 + Z_0}{(s_0-r)^2} \\
&= a^2(X_0 + Y_0 + Z_0) \longrightarrow 0
\end{aligned}$$

uniformly in r as $a \longrightarrow 0$. Now Theorem 4 concludes existence and uniqueness of a solution to initial-value problem (3.2), (3.3).

Chapter 5

Conclusion

5.1 Summary

This thesis has shown how Scales of Banach spaces arise in the treatment of certain partial differential equations. It t is real-valued and z is complex-valued, a scale of complex function spaces can be used to reformulate a partial differential equation such as

$$\frac{\partial u}{\partial t}(t,z) = F(t,z,u(t,z),\frac{\partial u}{\partial z}(t,z)),$$

where u is to be complex valued, as an ordinary differential equation

$$\frac{dU}{dt}(t) = F(t,U(t)),$$

where U is to be valued in the scale of function spaces, and represents the originally sought solution u. Now that only one independent variable is formally present, the technique of successive integral approximations from the standard proof of the Picard-Lindelöf theorem is applicable to an initial-value problem based on this differential equation. This method was illustrated with the inclusion of Theorem 2 and its proof, taken from Tutschke(*op. cit.*).

Initial-value problems in scales of Banach spaces involving retardations in the real independent variable t have been considered by Sekine and Yamanaka (*op. cit.*). The two original theorems presented in this thesis in Chapters 3 and 4 incorporate such retardations, and dependence on retarded values of the timelike derivative was introduced. Without retardation, right-hand side dependence on the timelike derivative in a first order differential equation is meaningless, for instance, the differential equation

$$\frac{dU}{dt}(t) = F(t, U(t), \frac{dU}{dt}(t))$$

can be solved for $\frac{dU}{dt}(t)$ to give

$$\frac{dU}{dt}(t) = G(t, U(t)).$$

However no such simplification is possible in the case of equations (3.2) or (4.1). For each of these equations, existence of a solution to an initial-value problem was shown under certain conditions on the the right-hand side operator F.

5.2 Further Work

One avenue for generalisation from here would be into generalised Cauchy-Riemann operators of the n^{th} order. As in Section 1.3, let f be some holomorphic function on the disc (1.3), and let z_0 be some point on the interior of the disc. The Cauchy integral formula gives a bound not only on $\frac{df}{dz}(z_0)$, but also on $\frac{d^n f}{dz^n}(z_0)$, where n is any positive integer. Integrating around the same circle C of radius ρ, (see Figure (1.1)),

$$\begin{aligned} \left|\frac{d^n f}{dz^n}(z_0)\right| &= \left|\frac{n!}{2\pi i}\oint_C \frac{f(z)dz}{(z-z_0)^{n+1}}\right| \\ &\leq \frac{n!}{2\pi}2\pi\rho\frac{\|f\|}{\rho^{n+1}} \\ &= \frac{n!\|f\|}{\rho^n}. \end{aligned}$$

As $\rho \to s_0 - |z_0|$, the bound

$$\left|\frac{d^n f}{dz^n}(z_0)\right| \leq \frac{n!\|f\|}{s_0 - |z_0|}$$

is obtained. In terms of the scale $\{B_r\}$ of complex function spaces, this means that if the operator

$$D^n = \frac{d^n}{dz^n}$$

is viewed as mapping from B_r to B_s, where $0 \leq s < r \leq s_0$, then

$$\|D^n\| \leq \frac{n!}{(r-s)^n}.$$

Therefore $F : B_r \to B_s$ is a *generalised Cauchy-Riemann operator of order* n if, whenever $u, v \in B_r$,

$$\|F(u) - F(v)\|_s \leq K \frac{\|u - v\|_r}{(r - s)^n}$$

for some positive constant K. It is probable that similar techniques to those used in this thesis could be used to prove existence theorems for initial-value problems in scales of Banach spaces for differential equations whose right-hand sides are generalised Cauchy-Riemann operators of order n, for $n > 1$. In the case of the scale $\{B_r\}$ of complex function spaces, a simple example of a partial differential equation which could be modeled in this way is

$$\frac{\partial u}{\partial t}(t, z) = \frac{\partial^n u}{\partial z^n}(t, z).$$

Another possible generalisation would be to consider systems of simultaneous first order differential equations in scales of Banach spaces. A coupled pair of equations of the class (2.4), for instance, might be considered.

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