A Mathematical Model of Volcanic Plumes

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Joshua Manfred Duley

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Abstract

Volcanic plumes and the resultant tephra fallout are of significant concern to nations the world over. Several recent large-scale eruptions have caused such disruption to air traffic that huge proportions of European commerce have been severely compromised. The plumes of such eruptions exist beyond any human recourse and must simply be left to extinguish themselves in time.

Currently, separate models do exist for plume dynamics and the atmospheric transport of particles, with a mixture of qualitative and quantitative results. In this thesis we develop a mathematical model with some similarities and some differences to those already in use.

The model has its core in the conservation equations of mass, momentum and energy for the plume's driving gases and suspended particles. While these equations are non-linear and difficult (if not impossible) to solve analytically, we can solve the equations numerically using a discretisation along the central vertical axis.

Initially these equations are provided with full time-dependency, with a view to pursuing such results in the future. However, the numerical results contained here are limited to a steady-flow model of an established and sustained, buoyant plume.

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Cover Image: photograph taken from the International Space Station of the Mt Cleveland Eruption, Alaska (May 2006). Image courtesy of the Earth Science and Remote Sensing Unit, NASA Johnson Space Center.

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Notation

Due to the large volume of notation needed in this work, ambiguity exists in the meaning of some terms. However, the context in which the symbology is used is sufficient for a distinction to be made, and throughout this thesis notation is clarified as it is introduced.

Lower case

a	m s^{-2}	Acceleration
c_V	J kg^{-1} K^{-1}	Specific Heat at constant volume
c_P	J $kg^{-1}K^{-1}$	Specific Heat at constant pressure
dZ	m	Height of a small horizontal section of plume
h	J kg^{-1}	Specific Enthalpy
k	$-$	Plume-atmosphere interface coefficients
m	kg m^{-1}	Mass per unit height
n	mol m^{-1}	Number of moles per unit height
p	Pa	Pressure
p	$-$	Proportion of particles in the sloped region of a section of plume
$q_\diamond$	$\diamond$ s^{-1}	Flux of subscript quantity
r, ϕ, z	m,rad,m	Cylindrical co-ordinates
t	s	Time
u	J kg^{-1}	Specific Internal Energy
u	m s^{-1}	Radial velocity component
v	rad s^{-1}	Rotational velocity component
w	ms^{-1}	Vertical velocity component
y	m	Difference in radius across plume section
x, y, z	m	Cartesian co-ordinates

Upper case

A	m^2	Cross sectional area
A	$-$	Suzuki constant
B	m	Radial thickness of a plumes boundary layer
C_D	$-$	Drag coefficient
E	J	Energy
$E_\diamond$	ms^{-1}	Exchange speed at the plume boundary.
		The entrainment speed is given no subscript.
F	N	Force
H	m	Maximum plume height
L	K m^{-1}	Temperature lapse rate
L	m	Length scale
J	kg s^{-1}	Dispersive flux
M	kg mol^{-1}	Molar mass
N	s^{-1}	Brunt-Viäsälä frequency
N	#	Number of particle cohorts
P	m	Perimeter
R	m	Plume radius
R_e	m	Plume radius at the vent
Re	$-$	Reynolds number
RH	$-$	Relative Humidity
R_0	J $K^{-1}mol^{-1}$	Universal Gas Constant $:= 8.314$
S	$-$	Functional particle distribution given by Sparks *et al.* (1997)
T	K	Thermodynamic (absolute) temperatue
V	m^3	Volume
$\overline{W}$	$-$	Dimensionless velocity
Z	m	A given, fixed altitude

Greek

Δ	$-$	Numerical step
ϕ	$^\circ$	Angle about vertical axis
ψ	m	Radius of boundary layer
θ	$^\circ$C	Temperature (Celcius)
$\rho_\diamond$	kg m^{-3}	Concentration or bulk density of subscript component
ν	m^{2}s^{-1}	Kinematic viscosity
μ_p	kg	Mass of a single particle
μ_ϕ	kg	Mean
σ	kg m^{-3}	Intrinsic density
σ_ϕ	$-$	Variance
ω	$-$	Specific humidity
ζ	m^{-1}	Suzuki release distribution

Bold Font

$\mathbf{a}$	m s^{-2}	Mean acceleration
$\mathbf{u}$	m s^{-1}	Mean velocity
$\mathbf{x}$	m	Spatial location
$\mathbf{F}$	N	Force vector

Sub/Sup-Scripts, Prefaces

a	Dry air
c	Volcanic gases
e	Denotes value at vent
f	Frozen water - ice
g	Plume gases
j	Particle cohort index
l	Liquid water
$m\diamond$	Mass of component $\diamond$
m	Total plume mass
p	Particle
s	Pertaining to the particle latency
v	Water vapour
w	Total water
E	Energy
E	Extrainment
L	Denotes value at the plume apex
R	Re-entrainment
T	Terminal velocity
H_2O	Regarding a water molecule
CO_2	Regarding a CO_2 molecule
X	Extrainment
atm	Atmospheric
env	Envelope - plume's immediate boundary
max	Maximum value attainable over domain
$3w$	Triple point of water
0	Boundary condition
$\diamond$	Denotes an unknown variable to be referenced