Some Shock Models in Reliability Theory.

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ABSTRACT

This thesis is concerned with the lifetime distribution of a device subject to environmental shocks. The terms "device" and "shock" are used here in an abstract sense and although industrial interpretations are the most obvious, the models described in this thesis can also be applied in other fields, for example, in biology and finance.

Several models are presented and in each case the main question of interest is to determine the class of distributions to which the lifetime distribution associated with the model belongs. This approach to the study of shock models is taken since it is often difficult to derive an explicit expression for the lifetime distribution of a device, but if the class to which the distribution belongs can be identified it is usually possible to obtain a bound on the distribution.

Since classes of lifetime distribution have an important role to play in the study of shock models and in reliability theory generally, the first part of this thesis is devoted to a review of the classes which have proved useful in these areas. The classes are defined and some justification for their use in reliability theory is provided. In addition, alternative characterisations of the classes are given and it is shown that a function which is closely related to the Laplace transform can be used to characterise all the classes.

The discussion of shock models commences, in chapter two, with a survey of results pertaining to the standard shock model :-

$$\bar{H}(t) = \sum_{k=0}^{\infty} P(N(t)=k)\bar{P}_k,$$

where $\bar{H}(t) = 1 - H(t)$ is the lifetime distribution of a device subject to shocks whose arrival is governed by the stochastic counting process $\{N(t)\}$. The probability that the device survives K shocks is given by $\bar{P}_K$ where $k=0,1,2,\ldots\ldots$ This model has received a good deal of attention in the literature

(see, for example, Esary, Marshall and Proschan (1973), A-Hameed and Proschan (1975), Klefsjo (1981, 1985). The most striking feature of the model is that under appropriate conditions on {N(t)} the lifetime distribution inherits its class from the discrete class of the survival probabilities $(\bar{P}_k)_{k=0}^{\infty}$. Results are presented under a variety of assumptions on {N(t)} ranging from the assumption of a homogeneous Poisson process to the assumption that {N(t)} is a generalised renewal process. In addition, a model where {N(t)} is assumed to be a doubly stochastic Poisson process is introduced. For the more general models it is often the case that the life distribution H inherits its class not only from the survival probabilities but also from the class of the shock interarrival time distribution.

The final part of this thesis is concerned with shock models in which failure occurs according to some specified mechanism. In particular, two methods of failure are considered. Firstly, the case where failure occurs on the occurrence of a shock which exceeds some critical threshold is studied and, secondly, the case where failure occurs when the total accumulated damage due to shocks exceeds some critical threshold is considered. In both cases, the initial approach is to use the standard model with an appropriate structure imposed on the survival probabilities $(\bar{P}_k)_{k=0}^{\infty}$. A more general approach which allows for some dependency between the shock magnitudes and shock interarrival times and, in the case of the cumulative damage model, for wear or recovery between shocks, is then adopted. Such models have been studied by Shanthikumar and Sunmita (1983, 1984) and by Shanthikumar (1984). Their results are summarised and the importance of the class of the shock inter-arrival time distributions in determining the class to which the lifetime distribution of the model belongs is noted. In addition, some minor extensions to Shanthikumar's (1984) work on the cumulative damage model are made.

INTRODUCTION

This thesis is concerned with the lifetime distribution of a device subject to shocks. The terms device and shock are used here in an abstract sense so, depending on the context, a device may be interpreted as: for example, a biological organism, a financial account, or a piece of industrial machinery. Similarly, a "shock" may be interpreted as a myocardial infarction, a demand or withdrawal, or, in the industrial context, as a blow which causes damage to the device or even as a repair causing negative damage to the device.

The study of shock models is approached from the point of view of establishing the class of distributions to which the lifetime distribution of the model belongs. The justification for taking this approach is that :-

a) in practice it is often difficult to obtain an explicit expression for the lifetime distribution of a device subject to shocks, and,

b) if the class to which a distribution belongs can be determined, it is usually possible to obtain bounds on the distribution.

Since the study of Shock Models is approached in this way, Chapter One of this thesis is devoted to a summary of the classes of life distribution useful in studying Shock Models and in Reliability Theory, generally. The classes are defined and some justifications for their use in Reliability Theory is presented. Alternative characterisations of the classes are provided using a function closely related to the Laplace Transorm and, where appropriate, the Total Time on Test transform. The Chapter concludes with a summary of the closure properties of the classes under the reliability operations of formation of coherent systems, mixture and convolution. Some bounds for distributions belonging to the classes are presented and the relationship between the classes are summarised.

Chapter Two includes a survey of the literature on the standard Shock Model,

$$\bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k)\ \bar{P}_k$$

proposed by Esary, Marshall and Proschan (1973). Here, it is assumed that shocks arrive at a device according to the Stochastic counting process $\{N(t)\}$ and that the probability of surviving k shocks is given by $\bar{P}_k$. $\bar{H}(t) = 1 - H(t)$ is the survivor function for the device in question, i.e. if T is a random variable denoting the lifetime of the device $\bar{H}(t) = P(T>t)$.

The main aim of the Chapter is to determine conditions on the survival probabilities $(\bar{P}_k)_{k=0}^{\infty}$ and on the process $\{N(t)\}$ which are sufficient for the lifetime distribution H(.) to belong to one of the classes of distribution introduced in Chapter one.

In Chapter three, some account is made of the actual mechanism by which devices fail. In particular, two modes of failure are considered :-

a) the maximum shock model where failure occurs when the magnitude of a shock exceeds some critical level, and
b) the cumulative damage model where shocks all cause damage which accumulates until the total accumulated damage exceeds some critical threshold and failure occurs.

Initially these models are studied by imposing an appropriate structure on the survival probabilities $(\bar{P}_k)_{k=0}^{\infty}$ of the standard model discussed in Chapter two. A more general approach which allows the possibility of correlation between the shock magnitudes and the intervals between shocks and, in the case of the cumulative damage model, the possibility of wear or recovery between shocks, is then considered.

Some remarks on the layout of this thesis are in order. Theorems, Lemmas and corallaries are numbered within subsections so that Theorem (3.1A.1) is the first theorem presented in subsection 1A of Chapter 3. Equations are numbered sequentially throughout sections, thus equation (2.1.24) is the 24th equation of section 1 of Chapter 2, regardless of whether it is located in subsection 2.1A or 2.1F. This should not cause too much inconvenience since, with few exceptions, numbered equations are only referred to locally.

Where a lemma is specific to a particular Theorem, it has usually been included within the proof of that Theorem so as not to interrupt the flow of the text.

Owing to some technical problems in the typing of this thesis some of the notation is not quite standard, for example 's' has been used where the Greek letter 'lambda' would normally be appropriate. Thus, the birth coefficients of a pure birth process are referred to as s_1, s_2,... and Poisson probabilities are given by $e^{-st}(st)^k/k!$

In section 2.1E a capital letter, B, has been used to denote a probability density function rather than a lower-case character as is more common.

In Chapter three the symbol $d^2/dxdy$ is used to denote partial derivative but it is clear from the context what is meant.

Finally, I would like to thank Dr C. D. Lai for his encouragement in the researching and writing of this Thesis, and Olaf Skarsholt et.al. for their efforts in making it legible for you to read.

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CHAPTER 1 :- CLASSES OF LIFE DISTRIBUTION USEFUL IN THE STUDY OF SHOCK MODELS

~1.0 Preliminaries

The study of shock models in the context of reliability theory is primarily concerned with the lifetime distribution of a machine or device subject to random environmental shocks. In general, an explicit form for the lifetime distribution is difficult to obtain but, provided it can be established that the distribution belongs to a sufficiently well-known (non-parametric) class of life distributions, bounds on the distribution can often be obtained. Consequently, the study of classes of life distribution is of great importance in the analysis of shock models and in reliability theory generally.

In this Chapter we present a summary of the main classes of lifetime distribution which have proved useful in reliability theory and, in particular, in the study of shock models. For each class, we attempt to provide an indication of the reasons for the class's usefulness and some alternative characterisations of the class based on the Laplace transform and the less well-known Total Time on Test transform (TTT transform) which is discussed briefly below.

Total Time on Test

Suppose n components are placed on test for a period of time, t, then the total time on test statistic is defined to be the sum of the failure times for each component. If no failures occur, the total time on test during the interval [0,t] is nt.

The total time on test (TTT) statistic can be defined more formally as follows :- Suppose that the n items placed on test are assumed to have a common life distribution F, and that successive failures are observed at times $Y_{[1]}$, $Y_{[2]}$, ... $Y_{[n]}$. $Y_{[i]}$ can be regarded as the i^{th} order statistic from a sample of size n from the distribution F.

The TTT statistic T(t), is given by :-

$$\tau(t) = nY_{[1]} + (n-1)(Y_{[2]} - Y_{[1]}) + n-2(Y_{[3]} - Y_{[2]}) + \ldots.$$
$$\ldots. + (n-i+1)(Y_{[i]} - Y_{[i-1]}) + (n-i)(t - Y_{[i]})$$

where $i \in (1...n)$ and $Y_{[i]} =< t =< Y_{[i+1]}$

i.e. If we define $Y_{[0]} =^{df} 0$ then,

(1.0.1) $\tau(t) = \Sigma_{j=1}^{i}(n-j+1)(Y_{[j]} - (Y_{[j-1]}) + (n-1)(t-Y_{[i]})$ .....

where i belongs to (1...n) and $Y_{[i]} =< t =< Y_{[i+1]}$

The TTT concept was introduced by Epstein and Sobel (1953) who used the TTT statistic to make inferences about the exponential distribution. Beginning with a paper by Barlow and Campo (1975), the TTT concept has been generalised and found to be of use in a variety of areas of reliability theory, e.g., hypothesis testing, model identification, determination of optimal age replacement intervals and characterisation of classes of life distribution. Bergman and Klefsjo (1983) provide a useful summary of the use of the TTT concept in reliability theory. The generalisation of the TTT concept most useful for characterizing classes of life distribution is the TTT transform which arises as follows :

The TTT at the time of the i^{th} failure is given by :-

$$\tau(Y_{[i]}) = \Sigma_{j=1}^{i}(n-j+1)(Y_{[j]} - Y_{[j-1]})$$

Hence, $\tau(Y_{[i]}) = \int_0^{F_n^{-1}(i/n)} \bar{F}_n(x)dx,$

Where F_n is the empirical distribution function

and $F_n^{-1}(t) =^{df} \inf\{x:F_n(x)>=t\}$

Now, by the Glivenko-Cantelle Lemma and the strong law of large numbers it follows that if F is the underlying distribution then with probability one (W.P.1)

$$\tau(Y_{[i]}) \rightarrow \int_0^{F^{-1}(t)} \bar{F}(x)dx$$

uniformly in (0,1) as $i/n \rightarrow t$ and $n \rightarrow \infty$

i.e. As $i/n \to t$ and $n \to \infty$

then $P(T(Y_{[i]}) \to \int_0^{F^{-1}(t)} \bar{F}(x)dx$ uniformly in $[0,1]) = 1$

(c.f. Barlow, Bartholomew, Bremner and Bunk (1972) p.37).
Consequently, we call the quantity

(1.0.2) $$H_F^{-1}(t) = \int_0^{F^{-1}(t)} \bar{F}(x)dx$$

the TTT transform of the distribution F.

If F has mean, μ, then :-

(1.0.3) $$\phi_F(t) = \frac{H_F^{-1}(t)}{H_F^{-1}(1)} = \frac{H_F^{-1}(t)}{\mu}$$

is called the scaled TTT transform of the distribution F.

An interesting property of the TTT transform is that if

$F(x) = 1 - e^{-sx}$ where $s > 0$, $x > 0$ then $\phi_F(t) = t$, where $0 < t < 1$

i.e. the scaled TTT transform carries the exponential distribution into the uniform distribution on [0,1] Other useful properties of the TTT transform include :-

(i) there is a one to one correspondence between life distributions and their TTT transform

(ii) $H_F^{-1}(t)$ is continuous in t iff F is strictly increasing

(iii) $H_F^{-1}(t)$ is strictly increasing iff F is continuous.

(iv) If F is absolutely continuous, with density f, and strictly increasing, then,

$$d/dt(H_F^{-1}(t)) = 1 / R^{-1}(F(t))$$

for almost all t belongs to [0,1],
where $R(x) = f(x) / F(x)$

(v) $H_F^{-1}(0) = \mu_F$, the mean of F.

We are now in a position to systematically study some classes of life distribution. The classes to be introduced here are all based on notions of ageing, the different notions of ageing giving rise to different classes of distribution.

One of the most straightforward notions of ageing is that of increasing failure rate, which can be characterised by the instantaneous probability of failure at time t given survival to time t, increasing as the device ages. The concept of failure rate is fundamental to much of reliability theory.

Consequently, it is essential to begin our survey of classes of life distribution with the increasing failure rate and decreasing failure rate classes.

~1.1 The Increasing Failure Rate and Decreasing Failure Rate Classes of Lifetime Distribution.

Suppose a device has a life distribution, F, then the conditional probability of failure during the next period of duration x given that the device has survived until time t is given by :-

$$F(x|t) = (F(t+x) - F(t)) / \bar{F}(t)$$

and the conditional failure rate R(t) can be defined by :-

(1.1.1) $$r(t) \stackrel{df}{=} \lim_{x \to 0} 1/x \quad (F(t+x) - F(t) / \bar{F}(t))$$

$$= f(t) / \bar{F}(t)$$

whenever the limit exists and $F'(t) = f(t)$

Thus, the failure rate can be thought of as the instantaneous rate of change of the conditional probability of failure given that the device has survived until the present. Alternatively, r(t)dt represents the conditional probability of failure in the small time interval (t, t+dt) given that the device has survived until time t.

In the interests of generality, we use the conditional life (distribution F(x|t) rather than the failure rate, r(t), to define the IFR and DFR classes as follows:

Definition (1.1.1)

A life distribution F (i.e. a distribution F with $F(0^-) = 0$) belongs to the increasing failure rate (IFR) class of distributions if :-

(1.1.2) $$F(x|t) = (F(x+t) - F(t)) / \bar{F}(t)$$
is increasing in t

If F(x|t) is decreasing in t then F belongs to the decreasing failure rate (DFR) class.

Clearly, if F has a density f then $F(x|t)$ increasing (decreasing) in t is equivalent to $r(t)$ increasing (decreasing) in t .

For convenience, we will often abbreviate the statement "F is a member of the IFR class" to F is IFR or $F \in \{IFR\}$

In the case of a discrete sequence of probabilities, it is possible to define classes analogous to the IFR and DFR classes defined above. Thus we have :-

<u>Definition (1.1.2)</u>

Let $Q_0 = 1$; Q_1, Q_2, Q_3, ... be a sequence of probabilities, then $(Q_n)_{n=0}$ is discrete IFR (DFR) iff :-

$$(Q_n)^2 \geq (\leq) \; Q_{n-1} Q_{n+1} \quad \text{where } n = 1,2,\ldots$$

i.e. $(Q_n)_{n=1}$ is log concave (convex).

In this thesis we will, in the main,be dealing with decreasing sequences of probabilities $(Q_k)_{k=0}^{\infty}$ satisfying $Q_0 = 1$. We will term such sequences as sequences of survival probabilities. The reason for this is as follows : Suppose $(p_k)_{k=0}^{\infty}$ is a discrete probability distribution satisfying $p_0 = 0$. It is convenient to interpret p_k as the P (a device fails on the k^{th} shock it receives). Now, define

$$Q_k =^{df} \Sigma_{j=k+1}^{\infty} p_j \qquad \text{where } k = 0,1,2,\ldots$$

then $Q_0 = 1$ and $(Q_k)_k$ is a decreasing sequence. Further, the obvious interpretation of :-

$$Q_k = \Sigma_{j=k+1}^{\infty} p_j$$

is that Q_k = P(a device <u>survives</u> the first k shocks it receives) where n = 0,1,2,...

Before proceeding with some examples of IFR and DFR distributions, it is worthwhile to consider the distinction between repairable and non-repairable devices. The discussion so far has implicitly assumed that the device in question is discarded upon failure. Often, however, devices are repaired on failure and placed back in service until the next failure. This distinction is of some importance when one considers the notion of failure rate of a repairable device.

The sequence of failures of a repairable device can be represented by a stochastic point process $\{N(t)\}$ where $N(t)$ equals the number of failures up to time t.
If $M = E(N(t))$ it seems entirely reasonable and is certainly intuitively appealing to define the failure rate to be $\mu(t) = M'(t)$ whenever the derivative of M exists. According to Thompson (1981) such a definition is in accord with well established scientific usage of the word "rate". The problem is that this latter definition of the term failure rate is not equivalent to our earlier definition as the following example demonstrates.

The lifetime of a non-repairable device can be represented by a simple stochastic process $\{N(t)\}$ which takes values 0 or 1; so that $E(N(t)) = F(t) = M(t)$; where F is the lifetime distribution of the device. Now, if the derivative exists, then $\mu(t) = M'(t) = f(t) <> r(t)$

Essentially, the difference arises because the failure rate of a repairable device is the unconditional failure rate of the stochastic process governing the sequence of failure, while in the non-repairable case the failure rate is the conditional failure rate of the lifetime distribution of the device.

In this thesis we are primarily concerned with non-repairable devices and consequently it is the failure rate in the sense of distributions that will be of interest to us.

A link between the two notions of failure rate can in fact be drawn provided one introduces a type of conditional failure rate for stochastic processes. Thompson (1981) illustrates this as follows :-

Consider a non-repairable device of which n copies are sampled and let N(t) be the number of failures up to time t. Then the number of copies surviving after time t, S(t) = n - N(t), is a pure death process with transition probabilities given by :-

$$P(S(t) = j \mid S(\tau) = i) \quad \text{where } \tau < t$$

$$= \binom{i}{j} \frac{(\bar{F}(t))^{j} \quad (F(t) - F(\tau))^{i-j}}{\bar{F}(\tau)}$$

$$= \binom{i}{j} \left[\frac{(\bar{F}(t))^{j}}{(\bar{F}(\tau))^{j}} \quad \frac{(F(t) - F(\tau))^{i-j}}{(\bar{F}(\tau))^{i-j}} \right]$$

i.e. the conditional distribution of the number of survivors given the survivors to an earlier time is binomial.

Consequently, $E\,[N(t) - N(\tau) \mid S(\tau)]$

$$= E\,[S(\tau) - S(t) \mid S(\tau)]$$

$$= S(\tau) - S(\tau) \frac{\bar{F}(t)}{\bar{F}(\tau)}$$

$$= S(\tau) \; 1 - \frac{\bar{F}(t)}{\bar{F}(\tau)}$$

$$= S(\tau) \frac{F(t) - F(\tau)}{\bar{F}(\tau)}$$

and hence :-

$$r_{P}(t) = \lim_{t \to \tau^{+}} E\left[\frac{N(t) - N(\tau)}{(t-\tau) S(\tau)} \right] S(\tau)$$

$$= \lim_{t \to \tau^{+}} \left[\frac{F(t) - F(\tau)}{(t-\tau) \bar{F}(\tau)} \right]$$

$$= \frac{f(\tau)}{\bar{F}(\tau)}$$

$$= r(\tau),$$

the failure rate of the distribution, F.

i.e., if n copies of a non-repairable device are sampled, the failure rate r(t) of the life distribution of the device is equivalent to the instantaneous rate of change of conditional expected number of failures (given the number of survivors) with respect to time and per individual at risk.

The function $r_p(t)$ can be interpreted as the conditional failure rate of the stochastic process {N(t)}.

We now present some examples of IFR and DFR distributions:-

1) The exponential distribution is both IFR and DFR since its failure rate is constant. The exponential distribution is the only distribution with this property.

2) The Weibull distribution has distribution functions $F_a(t) = 1 - e^{-(st)^a}$ where s, a > 0 and a polynomial failure rate $r(t) = as(st)^{a-1}$

Consequently, the Weibull distribution is IFR for a >= 1 and DFR for 0 < a =< 1.

3) The Gamma distribution :-

The density of the Gamma distribution is given by :-

$$g_{s,a}(t) = \frac{s^a t^{a-1} e^{-st}}{\Gamma(a)} \quad \text{for } t \geq 0 \text{ and } s,a > 0$$

and the failure rate of the Gamma distribution is given by:

$$[r(t)]^{-1} = \int_0^\infty (1+u/t)^{a-1} e^{-su}\, du$$

hence the Gamma distribution is IFR for a =< 1 and DFR for a >= 1.

4) The IDB distribution recently introduced by Hjorth (1980) differs from the previous three distributions in that its failure rate is not necessarily monotone. Consequently, this distribution is useful for modelling situations involving the so-called bathtub shaped failure rate. The IDB (increasing, decreasing, bathtub) distribution is defined by :-

$$\bar{F}(t) = 1 - F(t) = \frac{\exp(-(st^2)/2)}{(1+Bt)^{\theta/B}}$$

where $s > 0$, $\theta > 0$, $B > 0$

and has failure rate given by :-

$$r(t) = st + \theta/(1 + Bt)$$

Now if $st = 0$ then F is DFR, and,
if $s >= \theta B$ then F is IFR, and,
if $0 < s < \theta B$ then r(t) has the bathtub shape.

The failure rate of a distribution may not always be easy to obtain explicitly and consequently it is of some use to have alternative methods of identifying IFR and DFR distribution.

We will present two alternative characterisations of the IFR and DFR classes, the first of these is based on a function related to the Laplace transform, $F^*(s) = \int_0^\infty e^{-sx} dF(x)$

Let $$A_n^F(s) = \frac{(-1)^n}{n!} \frac{d^n}{ds^n} \left[\frac{1-F^*(s)}{s}\right]$$

(1.1.3) $$= \int_0^\infty x^n/n! \; e^{-sx} \; \bar{F}(x)dx$$

and define :-

(1.1.4) $$a_0^F(s) = 1, \quad a_{n+1}^F(s) =^{df} S^{n+1} A_n^F(s)$$

Vinogradov (1973) established the following result :-

Theorem (1.1.1)

Let F be a life distribution (i.e. $F(0^-) = 0$) Then, F is IFR (DFR) iff $(a_n^F(s))_{n=1}$ is discrete IFR (DFR) for every $s > 0$

i.e. for all $s > 0$, $a_n^{F2}(s) >= (=<) a_{n-1}^F(s) a_{n+1}^F(s)$ where $n = 1,2,\ldots$

PROOF

We present the proof only in the IFR case. The DFR case follows similarly. First, suppose that F is IFR and note that in Barlow and Proschan (1965) it was shown that if F is IFR then :-

(1.1.5) $A_n^{F2}(0) >= A^F_{n-1}(0) A^F_{n+1}(0)$ where $n=1,2,\ldots$

Now if F is IFR so is $G(t) = 1 -(1-F(t)e^{-st})$, for every $s > 0$ and replacing F by G in (1.1.4) above yields

(1.1.6) For all $s > 0$, $A_n^{F2}(s) >= A^F_{n-1}(s) A^F_{n+1}(s)$ where $n = 1,2,\ldots$

and hence $a_n^{F2}(s) >= a^F_{n-1}(s) a^F_{n+1}(s)$ as required.

The proof of sufficiency requires the following two lemmas; details of their proofs are omitted here but can be found in Vingradov (1973). The first lemma establishes the desired result for all life distributions with bounded, continuous densities while the second lemma is used in the extension of this result to more general distributions.

Lemma (1.1.1)

Let F be any life distribution with a continuous bounded density, f, and suppose $A^{F2}_n(s) >= A^F_{n-1}(s) A^F_{n+1}(s)$ holds for all $s >= 0$ where $n=1,2,\ldots$, then F is IFR.

Proof

The proof relies on an inversion formula for Laplace transforms and the fact that the failure rate $r(t) = f(t) / \bar{F}(t)$ can be written in terms of the $A^F{}_n(s)$ and as a consequence of the condition of the lemma can be shown to be increasing.

Lemma (1.1.2)

Let $F_1(t)$ be any life distribution and $F_2(t)$ defined by $F_2(t) = 1 - e^{-st}$ where $t >= 0$.

Suppose that for all $s >= 0$, $(A^F{}_n(s))^2 >= A^F{}_{n-1}(s)A^F{}_{n+1}(s)$ where n=1,2,...

then,

$$(B_n(s))^2 >= B_{n-1}(s)B_{n+1}(s)$$

where

$$B_n(s) = \int_0^\infty \frac{e^{-st}\ t^n}{n!} \bar{\Phi}(t)dt$$

$$\bar{\Phi}(t) = 1 - \Phi(t) = 1 - \int_0^t F_2(t-y)dF_1(y).$$

Now, continuing with the proof of sufficiency of the theorem, let $F_2{}^*(t) = 1 - (1+bt)e^{-bt}$ then it can be shown that Lemma 2 applies with $F_2{}^*$ in place of F_2. This follows from the fact that

$F_2{}^*(t) = \int_0^t F_2(t-y)dF_2(y)$ and by repeated application of Lemma 2. Now, if F_1 is any arbitrary life distribution then Feller vol.2 (1966) has shown that the distribution $\Phi(t) = \int_0^t F_2{}^*(t-y)dF_1(y)$ has a continuous, bounded density and by Lemma (1.1.2)

$(B_n(s))^2 >= B_{n-1}(s)B_{n+1}(s)$ and hence by lemma (1.1.1) Φ is IFR but Φ -> F_1 weakly as $s \to \infty$ and since the class of IFR distributions is closed under weak convergence (c.f. Basu and Bhattacharjee (1984)) F_1 is IFR. Since F_1 was arbitrary the theorem is proved.

The $(A_n(s))_{n=1}^{\infty}$ of the previous theorem can be given a shock-model interpretation as follows :-

Suppose a device is subject to shocks which each cause a random amount of damage and let E_i denote the amount of damage caused by the i^{th} shock. If the E_i are i.i.d. with a common exponential distribution of mean 1/s where s > 0 then $X = \Sigma_{i=1} E_i$ is a gamma random variable with density function :-

$$f_n(s,x) = \frac{S^n X^{n-1} e^{-sx}}{(n-1)!} \qquad \text{where } s > 0,\ n >= 0,\ x >= 0$$

Now, suppose that damage is caused only by shocks and that the device fails when the total accumulated damage exceeds some critical threshold, C. Further suppose that C is itself a random variable as would be the case if we were considering a population of identical devices with significant variation in individual device's ability to withstand damage.

Now the lifetime distribution of the device is given by

$\bar{H}(t) = P(\Sigma_{i=1}^{N(t)} E_i < C)$ where $\{N(t)\}$ is the stochastic Point Process governing the arrival of the shocks.

Hence $\bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k) P(\Sigma_{i=1}^{k} E_i < C)$ where $P(\Sigma_{i=1}^{k} E_i < C)$ is the probability of the device surviving k shocks.

Now, $P(\Sigma_{i=1}^{k} E_i < C)$

$$= \int_0^{\infty} P(C > \Sigma_{i=1}^{k} E_i \mid \Sigma_{i=1}^{k} E_i = x)\, f_k(s,x)\,dx$$

$$= \int_0^{\infty} \frac{s^k x^{k-1}}{(k-1)!} e^{-sx}\, \bar{F}(x)\,dx$$

$$= a_k(S)$$

where F is the distribution of the threshold, C.

So the $(a_k(-))_{k=1}$ defined by (1.1.3) are the survival probabilities in a cumulative damage random threshold model. We will meet this model again in Chapter three.

The Total Time on Test transform can also be used to obtain an alternative characterisation of the IFR and DFR classes. The relevant result is contained in the following Theorem due to Barlow and Campo (1975).

Theorem (1.1.2)

A life distribution F with finite mean, μ, is IFR (DFR) iff the scaled TTT transform :-

$$H_F^{-1}(t)/\mu = \left(\int_0^{F^{-1}(t)} \bar{F}(x)dx \right)/\mu$$

is concave (convex).

Proof

Only the case where F is absolutely continuous (a.c.) and strictly increasing is considered here. Barlow and Campo (1975) extend the result to more general F via limiting arguments, while Langberg et al (1980) present an alternative proof which avoids some of the technical difficulties inherent in the limiting arguments of Barlow and Campo.

Let F be an a.c. and strictly increasing life distribution. Now :-

> $H_F^{-1\prime}(t) = 1 / r(F^{-1}(t)$ (by property (iv) of TTT transforms mentioned in section 1.0); where r(.) is the failure rate of F.

So, if F is IFR (DFR) then $H_F^{-1\prime}(t)$ is decreasing (increasing) in the interval $(F^{-1}(0), F^{-1}(1))$, hence $(H_F^{-1}(t)) / \mu$ is concave (convex). Conversely, assume $(H_F^{-1}(t)) / \mu$ is concave (convex) in t then $H_F'(t) = 1 / r(F^{-1}(t)$ is decreasing (increasing) in t

i.e. r(.) is decreasing (increasing). F is IFR (DFR) as required.

Other characterisations of the IFR class are possible, e.g. Langberg et al (1980) established a characterisation based on weighted spacings between order statistics.

We conclude this section with a brief discussion of a sub-class of the IFR class.

Recall that a life distribution F was defined to be IFR iff, $[F(x+t) - F(t)] / \bar{F}(t)$ was increasing in t for all $x >= 0$. This definition is equivalent to F is IFR iff

(1.1.7) $\bar{F}(x|t) = \bar{F}(t+x) / \bar{F}(t)$

is decreasing in t for each x. The sub-class we are interested in consists of those IFR distributions with densities which satisfy (1.1.7). Any non-negative function, h, satisfying (1.1.7) for all x in its' support is known as a 'Polya Frequency function of Order 2 (PF_2)'

Consequently, if a life distribution F has a density f for which (1.1.7) holds, i.e. :-

(1.1.8) $f(t+x) / f(t)$ decreasing in t for all $x > 0$ we say f is a PF_2 density.

Many standard distributions in statistics, for example, the uniform and the gamma, have PF_2 densities.

It is easy to show that if f is a PF_2 density, then the corresponding distribution, F, is IFR (see Barlow and Proschan (1975) p.77). Thus the class of life distributions with PF_2 densities is indeed a sub-class of the IFR class.

It should also be noted that the statement "F is IFR" is equivalent to "$\bar{F}$ is PF_2". The condition (1.1.8) is equivalent to :-

(1.1.8)' $$\begin{vmatrix} f(x_1-y_1) & f(x_1-y_2) \\ f(x_2-y_1) & f(x_2-y_2) \end{vmatrix} >= 0$$

for all $-\infty < x_1 < x_2 < \infty$ and,

$-\infty < y_1 < y_2 < \infty$ and to :-

(1.1.8)" $\log f(x)$ is concave on $(-\infty,\infty)$

We could define an analogous sub-class of the DFR class either by reversing the inequality in (1.1.8)' or imposing the condition :-

(1.1.9) $\log f(x)$ is convex.

~1.2 The Increasing Failure Rate Average and Decreasing Failure Rate Average Classes of Life Distribution

The IDB distribution introduced in the previous section is an example of a distribution with a non-monotone failure rate. Such distributions arise quite frequently in reliability problems, e.g., the lifetime distribution of a device consisting of two independent but non identically exponentially distributed components in parallel has a failure rate which is initially increasing but eventually decreases (c.f. Barlow and Proschan (1975) p.83).

It may be the case, however, that a failure rate which is not monotone increases or decreases "on the average". The average failure rate of a distribution F with density f is given by :-

$$G(t) = \left(\int_0^t r(x)dx \right) / t \quad \text{where} \quad r(x) = f(x) / \bar{F}(x)$$

Since $\quad - \log \bar{F}(t) = \int_0^t r(x)dx$

it is clear that, $\quad G(t) = (- \log \bar{F}(t)) / t$

hence we have,

Definition (1.2.1)

A life distribution, F belongs to the Increasing Failure Rate Average (IFRA) Class iff

(1.2.1) $\quad (- \log \bar{F}(t)) / t \quad$ is increasing in t.

Definition (1.2.2)

A life distribution F belongs to the Decreasing Failure Rate Average (DFRA) Class iff

(1.2.2) $(-\log \bar{F}(t)) / t$ is decreasing in t.

The IFRA and DFRA classes are characterised by the behaviour of $[\bar{F}(t)]^{1/t}$ i.e.,

(1.2.3) F is IFRA iff $[\bar{F}(t)]^{1/t}$ is decreasing in $t > 0$

(1.2.4) F is DFRA iff $[\bar{F}(t)]^{1/t}$ is increasing in $t > 0$

If $0 < a < 1$ then $[\bar{F}(t)]^{1/t}$ decreasing implies
$[\bar{F}(t)]^{1/t} \leq [\bar{F}(at)]^{1/at}$
i.e. $[\bar{F}(t)]^{a} \leq \bar{F}(at)$.

Hence the IFRA and DFRA classes can be defined by :-

(1.2.5) F is IFRA (DFRA) iff
$[\bar{F}(t)]^{a} \leq (\geq) \bar{F}(at)$ where $0 < a < 1$

In the discrete case definitions analagous to (1.2.3) and (1.2.4) can be used to define discrete IFRA or DFRA sequences of probabilities as follows :-

Definition (1.2.3)

Let $(Q_n)_{n=0}^{\infty}$ be a discrete sequence of probabilities with $Q_0 = 1$ then then $(Q_n)_{n=0}^{\infty}$ is a discrete IFRA (DFRA) sequence iff $(Q_n)^{1/n}$ is decreasing (increasing) in n.

The IFRA and DFRA classes arise naturally in a number of reliability situations, e.g., the distribution, mentioned earlier, of a device consisting of two independent exponentially distributed components can be shown to be IFRA. We will see, in our consideration of cumulative damage Shock Models in Chapter three that the IFRA class has an important role to play in this area too.

Barlow and Proschan (1975) provide a good summary of the properties of the IFRA and DFRA classes including the following result which will prove useful in the study of some shock models considered later.

Lemma (1.2.1)

A life distribution F is IFRA (DFRA) iff for each $s > 0$ $\bar{F}(t) - e^{-st}$ has at most one change of sign and if one change of sign does occur then it occurs from + to - (from - to +).

It is well known that the IFRA class contains the IFR class and that the DFRA class contains the DFR class. This follows from the fact that F IFR is equivalent to $\bar{F}$ PF_2 i.e. log $\bar{F}$ is concave.

As with the IFR and DFR classes, it is possible to obtain an alternative characterisation of both the IFRA and DFRA classes, via the Laplace transform. The following Lemma due to Block and Savitts (1980) will be of use in establishing the characterisation as well as alternative characterisation classes to be considered subsequently.

Lemma(1.2.2)

Let F be a life distribution with $(A^F_n(s))_{n=0}^{\infty}$ and $(A^F_n(s))_{n=0}^{\infty}$ defined as in (1.1.3) and (1.1.4) respectively. If $u > 0$ is a continuity point of F, $S = S(n,u)$ and $\lim_{n\to\infty} n/s = u$ then

(1.2.6) $\lim_{n\to\infty} a_{n+1}(s) = \bar{F}(u)$

Proof

$$a^F_{n+1}(s) = \int_0^\infty \frac{s^{n+1} x^n}{n!} \bar{F}(x) e^{-sx}\, dx$$

$$= \int_0^\infty \bar{F}(x)\, dG_n(x)$$

(1.2.7) where $$dG_n(x) = \frac{s^{n+1} e^{-sx} x^n\, dx}{n!}$$

i.e. $G_n(x)$ is a gamma distribution and consequently has characteristic function :-

$$h_n(t) = E(e^{ixt})$$
$$= [s / s-it]^n$$

So,
$$\lim_{n\to\infty} h_n(t) = \lim_{n\to\infty} [(1-it) / n/u]^{-n}$$
$$= e^{iut}$$

which is the characteristic function of the one point distribution :-

$$G(x) = \begin{cases} 0 & \text{for } x < u \\ 1 & \text{for } x \geq u \end{cases}$$

hence $G_n(x) \to G(x)$ weakly since convergence of characteristic functions implies weak convergence of the corresponding distribution (c.f. Bilingsley (1968).

So
$$\lim_{n\to\infty} a^F_{n+1}(s) = \lim_{n\to\infty} \int \bar{F}(x)\, dG_n(x)$$
$$= \int \bar{F}(x)\, dG(x)$$
$$= \bar{F}(u) \qquad \text{as required.}$$

Klefsjo (1982) has shown, by a similar argument, that :-

$$(1.2.8) \qquad \lim_{n\to\infty} \int_0^\infty G_n(x)\bar{F}(x)dx = \int \bar{F}(x)dx$$

and by (1.1.3) and (1.1.4),

$$1/s \quad \Sigma_{k=n+1} a_k^F(s) = \int_0^\infty G_n(x)\bar{F}(x)dx$$

$$(1.2.9) \qquad \text{So} \quad \lim_{n\to\infty} [1/s \quad \Sigma_{k=n+1} a_k^F(s)] = \int \bar{F}(x)dx$$

whenever $\lim_{n\to\infty} n/s=u$. (1.2.8) and (1.2.9) will prove useful in establishing alternative characterisation of classes of distribution to be discussed later.

We can now establish the following result also due to Block and Savits (1980).

<u>Theorem (1.2.1)</u>

A life distribution F is IFRA (DFRA) iff $(a_n^F(s))^{1/n}$ is discrete IFRA (DFRA) in n for every $s > 0$

<u>Proof</u>

Only the IFRA case is considered here. The DFRA case is very similar. First suppose F is IFRA and let $s > 0$ be fixed but arbitrary. Then $A_n^F(s) = \int_0^\infty (x^n/n!)\ \bar{J}(x)d(x)$ where $\bar{J}(x) = e^{-sx}\bar{F}(x)$

Clearly from the definition of the IFRA class J(x) is IFRA iff F(x) is IFRA.

Now, $A_n^F(s) = \int_0^\infty (x^n/n!)\ \bar{J}(x)d(x)$

and integration by parts yields :-

$$A_n^F(s) = 1/(n+1)!\ \int_0^\infty x^{n+1} dG(x)$$
$$= E(x^{n+1})\ /\ (n+1)!$$
$$= V_{n+1}$$

where $(V_n)^{1/n}$ is decreasing in n=1,2,... by corrollary 6.5 p.112 of Barlow and Proschan (1975).

i.e. $(A_n^F(s))^{1/n}$ is decreasing in n and consequently so is $(a_n^F(s))^{1/n}$

i.e. $(a_n^F(s))^{1/n}$ is discrete IFRA and since s was arbitrary the desired result is established for all s > 0.

Now suppose $(a_n^F(s))_n$ is decreasing in n. This condition can be rewritten as :-

(1.2.10) (for all s > 0) $a_{n+1}^F(s) \geq [a_{n+k+1}^F(s)]^{n+1/n+k+1}$
where n,k=1,2,...

Now let X = p/q where p and q are integers such that 0 < p < q and let X > 0 and aX be continuity points of F (1.2.10) holds for all s > 0, n,k, > 0 so in particular it holds for n = mp, k = m(q-p) where m is a positive integer, and s = n / aX

As m -> ∞; n/s -> aX, (n+k)/s -> a and (n+1)/(n+k+1) -> a

So by taking limits on both sides of (1.2.10) we have by Lemma (1.2.2). :-

$$\bar{F}(aX) \geq (\bar{F}(X))^a \quad \text{where } 0 < a \leq 1$$

i.e. $(\bar{F}(X))^{1/X}$ is decreasing in X hence since the set of points {X} of the type considered is dense F is IFRA.

The TTT transform is not as useful in the identification of IFRA distribution as it is in the IFR case. Barlow (1979), however, has shown that if F is IFRA (DFRA) then the inverse of its TTT transform $H_F(x)$ satisfies.

(1.2.11) $H_F(x)/x$ is increasing (decreasing) in x where $0 < x < \mu$ where $\mu = \int x dF(x)$

The converse is unfortunately false. A function which satisfies (1.2.11) on its support is sometimes referred to as a starshaped (anti-starshaped) function. Thus the IFRA property can be written as :-

(1.2.12) F is IFRA iff $-\log \bar{F}(t)$ is starshaped.

We conclude this section with two further characterisations of the IFRA and DFRA Classes, both due to Langberg et al (1980)

<u>Theorem (1.2.2)</u>

Let F be a life distribution and let $X_{i:n}$ be the i^{th} order statistic from a sample of size n, from F, then F is IFRA (DFRA) iff $P(X_{1:n} > x/n)$ is increasing (decreasing) in $n >= N$ for some N.

<u>Proof</u>

Details of the proof are omitted here but we note that the proof is based on the fact that :-

$$P(X_{1:n} > x/n) = (\bar{F}(x/n))^n$$

Langberg et al (1980) have also established the following result :-

<u>Theorem (1.2.3)</u>

Let F be a continuous life distribution with finite mean. If the support of F is an interval and F(0) = 0 then F is IFRA (DFRA) iff

$$E(X_{i:n}) / [\Sigma_{k=1} 1/(n-k+1)]$$

is decreasing (increasing) in i where i = 1,2,...,n for infinitely many n.

1.3 The Decreasing Mean Residual Life and Increasing Mean Residual Life Classes of Life Distribution.

An alternative to the failure rate as a means of describing the ageing of a device is the mean residual life. The mean residual life corresponding to a life distribution F is given by :-

(1.3.1) $e_F(t) = E(T-t \mid T>=t)$

where T is the random lifetime of the device in question.
Now,

(1.3.2) $E(T-t \mid T>=t) = (\int_0^\infty \bar{F}(t+x)) / \bar{F}(t) dx$

hence

(1.3.3) $e_F(t) = (\int_t^\infty \bar{F}(x)dx) / \bar{F}(t)$; when $\bar{F}(t) <> 0$

or $e_F(t) = 0$; when $\bar{F}(t) = 0$

$e_F(t)$ can be used to define two classes of life distribution in an obvious way.

Definition (1.3.1)

Let $e_F(t)$ be defined as in (1.3.1) or (1.3.2) then if $e_F(t)$ is decreasing we say F belongs to the Decreasing Mean Residual Life Class (DMRL) and if $e_F(t)$ is increasing we say F belongs to the Increasing Mean Residual Life Class (IMRL).

The IMRL Class has proved useful in a social science context e.g., empirical labour turnover studies have indicated that the length of tenure in a job can be well fitted by an IMRL distribution, i.e., the longer one has spent in a job the more likely one is to stay in the job (see e.g.,Bartholomew (1982) and Gerchak (1984)). Now, since the IFR class is characterised by :-

$$1-F(x|t) = \bar{F}(x|t) = \bar{F}(t+x) / \bar{F}(t)$$

decreasing in t for all $x > 0$ it follows from (1.3.2) that the DMRL Class contains the IFR class and similarly the IMRL Class contains the

DFR Class. No such relationship has been found to exist between the IFRA and DMRL Classes or between the DFRA and IMRL Classes.

The DMRL and IMRL Classes are related to the IFR and DFR Classes in another way, as is shown below.

Suppose F is a life distribution with mean μ_F and denote by :-

(1.3.4) $$\hat{F}(t) = 1/\mu_F \int_0^{\infty} \bar{F}(x)dx$$

the distribution of the first interval, X_1 in the equilibrium renewal process with common interval distribution, F for intervals X_i where $i=2,3,\ldots$ Then we have :-

<u>Theorem (1.3.1)</u>

A life distribution F is DMRL (IMRL) iff $\hat{F}$ is IFR (DFR)

<u>Proof</u>

Recall that for any IFR (DFR) distribution G, $\log \bar{G}$ is concave (convex) i.e., $-\log \bar{G}$ is convex (concave) and conversely.

Hence $\hat{F}$ is IFR (DFR)

iff $\hat{R}(t) = -\log(1-\hat{F}(t))$ is convex (concave).

Now $\hat{R}(t) = -\log(1/\mu_F \int_t^{\infty} \bar{F}(x)dx)$

and $$\hat{R}'(t) = \frac{\mu_F \bar{F}(t)}{\int_t^{\infty} \bar{F}(x)dx} = \frac{\mu_F}{e_F(t)}$$

hence $\hat{R}(t)$ is convex (concave) iff $e_F(t)$ is decreasing (increasing) hence the desired result follows. This result is due to Rolski (1975), although he stated the result only in the DMRL case.

the context of renewal theory the result states that if the stribution of the first interval in the equilibrium renewal process IFR (DFR) then the common distribution of all succeeding intervals at least DMRL (IMRL) (and conversely).

The DMRL (IMRL) class has the following property which will prove useful both in establishing an alternative characterisation of the class and later in establishing the DMRL property of shock models under certain conditions.

F is DMRL (IMRL) iff :-

(1.3.5) $h(u) = \int_u^\infty \bar{F}(y)dy - c\bar{F}(u)$ where $c >= 0$

changes sign at most once and if once from + to - (- to +). Discrete DMRL and IMRL classes can be defined as follows :-

Definition (1.3.2)

A sequence of probabilities Q_0, $Q_1,\ldots$ is discrete DMRL (IMRL) iff

$(\Sigma_{k=1}^{n} Q_k) / Q_n$ is decreasing (increasing) in n = 1,2,...

We can now present a characterisation of the DMRL and IMRL classes, based on the Laplace transform.

Theorem (1.3.2) (Block and Savitts (1980)).

Let F be a life distribution, then F is DMRL (IMRL) iff $(a_k)_{k=0}^{\infty}$ is discrete DMRL (IMRL) where a_k is defined as in (1.1.4).

Proof

Only the DMRL case is considered here, the IMRL case follows by an exactly similar argument.

We first prove necessity :-

$$a^F_{k+1}(s) = s \int_0^\infty ((sx)^k/k!) \; e^{-sx} \; \bar{F}(x)dx$$

Note that the kernel :-

$$g_x(k,s) = ((sx)^k / k!) e^{-sx},$$

is PF_2. (cf.1.1) Now, PF_2 functions possess the variation diminishing property which is discussed in Chapter 2 (see Theorem (2.0.1)) and can be summarised here as follows :-

If $g(k,s)$ is PF_2 then :-

$$h(k) = \int_0^\infty g(k,s)f(s)ds$$

follows the same sequence of sign changes as f(s) i.e., if f(.) has one sign change from + to - then so does h(.).

We note also that :-

$$s \int_0^\infty e^{-sx} ((sx)^k / k!) (\int_x^\infty \bar{F}(y)dy - c\bar{F}(x))dx$$

$$= 1/s \; \Sigma_{i=k+2} \; a_i(s) - ca_{k+1}(s) \quad \text{where} \quad c >= 0$$

So, by the sign change property of DMRL distributions described by (1.3.5) and the variation diminishing property of the PF_2 kernel :-

$$g_x(k,s) = ((sx)^k e^{-sx}) / k!$$

it follows that :-

$1/s \; \Sigma_{i=k+2}^\infty \; a^F_i(s) - ca^F_{k+1}(s)$ changes sign at most once and if once from + to -

i.e. $$\frac{\Sigma_{i=k+2} \; a^F_i(s)}{a^F_{k+1}(s)}$$

is decreasing in n for all $s > 0$.

Hence $$\frac{\Sigma_{i=k+1}^{\infty} a_i^F(s)}{a_k^F(s)}$$

is decreasing in n for all $s > 0$, as is,

$$\frac{\Sigma_{i=k+1}^{\infty} a_i^F(s)}{a_k^F(s)} + \frac{a_k^F(s)}{a_k^F(s)}$$

$$= \frac{\Sigma_{i=k}^{\infty} a_i^F(s)}{a_k^F(s)} \qquad \text{for all } s > 0$$

i.e. $(a_k^F(s))_{k=0}^{\infty}$ is discrete DMRL.

Now, to prove sufficiency, suppose :-

$(a_k^F(s))_{k=0}^{\infty}$ is discrete DMRL

i.e. $$\frac{\Sigma_{i=k}^{\infty} a_i^F(s)}{a_k^F(s)}$$

is decreasing in k for each $s > 0$; hence :-

$$a_{k+n+1}^F(s)\ \Sigma_{i=k+1}^{\infty} a_i^F(s) >= a_{k+1}^F \Sigma_{i=k+n+1}^{\infty} a_i(s)$$

for n=1,2,... and k=1,2,... and $s > 0$ therefore :-

(1.3.6) $$\lim_{k\to\infty} a_{k+n+1}^F(s)\ 1/s\ \Sigma_{i=k+1}^{\infty} a_i^F(s) >=$$

$$\lim_{k\to\infty} a_{k+1}^F(s)\ 1/s\ \Sigma_{i=k+n+1}^{\infty} a_i^F(s)$$

where n=1,2,... and $s > 0$

In particular, (1.3.6) holds for :-

$n = [Kv / u]$; $s = k / u$ where u and v are continuity points of F. Thus :-

$$\lim_{n\to\infty} (n+k / s) = (u + v) \text{ and } u = (k / s)$$

so by lemma 1.2.2 :-

(1.3.7) $$\lim_{k\to\infty} a_{k+1}^F(s) = \bar{F}(u) \quad \text{and}$$

$$\lim_{k\to\infty} a_{k+n+1}^F(s) = \bar{F}(u+v)$$

Now, recall that by (1.2.9)

(1.3.8) $\lim_{k \to \infty} 1/s \ \Sigma_{i=k+1}^{\infty} a_i(s) = \int_u^{\infty} \bar{F}(x)dx$

whenever $\lim_{k \to \infty} (k / s) = u$

hence using (1.3.7) and (1.3.8) in (1.3.6) we have

(1.3.9) $\bar{F}(u+v) \int_u^{\infty} \bar{F}(x)dx \ >= \ \bar{F}(u) \int_{u+v}^{\infty} \bar{F}(x)dx$

for all continuity points u, v.

i.e. $e_F(u) = (\int_u^{\infty} \bar{F}(x)dx) / \bar{F}(u)$

is decreasing in u, and since the set of continuity points of a DMRL distribution is dense, hence the desired result is established.

The DMRL and IMRL Classes can also be characterised by the behaviour of a function related to the scaled TTT transform. Klefsjo (1982) established the following result :-

<u>Theorem (1.3.3)</u>

Let $Q(t) = (1-\phi_F(t)) / 1-t$
where $\phi_F(t)$ is the scaled TTT transform of the Life Distribution F, then F is DMRL (IMRL) iff Q(t) is decreasing (increasing) in t.

Graphically, this result can be interpreted as in the Figure (1.3.1) following :-

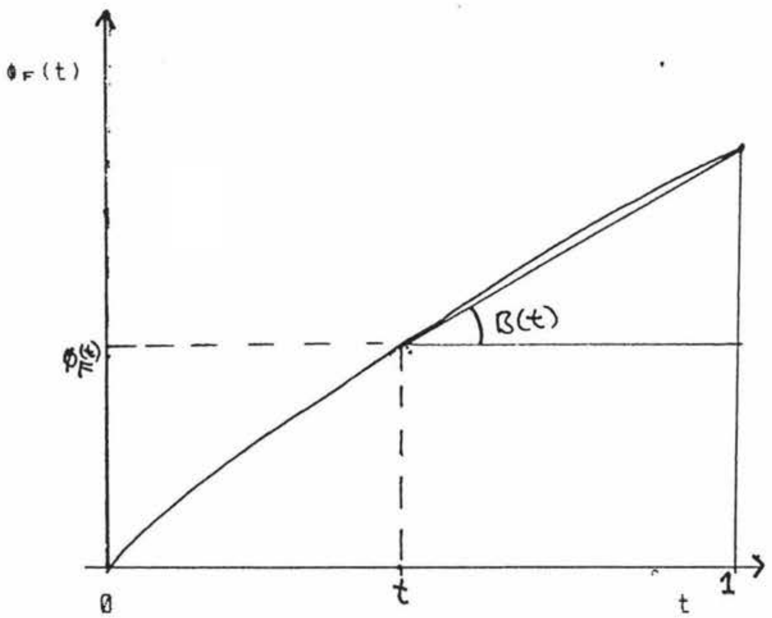

Figure (1.3.1) F is DMRL iff B(t) is decreasing (increasing) as a function of t.

~1.4 The New Better than Used (NBU) and New Worse than Used (NWU) Classes of Life Distribution

These classes of life distribution arise most naturally in the study of maintenance and replacement policies but the mathematical tractability of their definition together with the fact that the classes encompass a wide range of distribution has ensured their importance in the study of shock models. In particular, the NBU and NWU Classes have been used to describe some of the more complex models considered in Chapter three where, for example, recovery between shocks takes place.

Definition (1.4.1)

A Life distribution, F, is NBU (NWU) iff

(1.4.1) $\bar{F}(x+t)/\bar{F}(t) \leq (\geq) \bar{F}(x)$ for all $t \geq 0$

i.e. F is NBU (NWU) iff

(1.4.2) $\bar{F}(x+t) \leq (\geq) \bar{F}(x)\bar{F}(t)$

The expression on the left in inequality (1.4.1) is the conditional reliability of a used device of age t, while the expression on the right is the reliability of a new unit, hence the terms 'New Better than Used' and 'New Worse than Used'.

In the discrete case an analagous definition is as follows :-

Definition (1.4.2)

A sequence of probabilities $(Q_k)_{k=0}^{\infty}$ is discrete NBU (NWU) iff

$$Q_{k+l} \leq (\geq) Q_k Q_l$$

where $k = 0,1,2,...$ and $l = 0,1,2,...$

The NBU and NWU Classes contain the IFRA and DFRA Classes respectively (and hence the IFR and DFR Classes) and the same relationship exists between the discrete NBU (NWU) and the discrete IFRA (DFRA) Classes. No such relationship has been found to exist between the NBU and DMRL or NWU and IMRL Classes.

As with the other classes of life distribution studied so far, Block and Savits (1980) have presented a characterisation of the NBU and NWU Classes based on the Laplace transform. This result is contained in the following theorem :-

Theorem (1.4.1)

A life distribution F is NBU (NWU) iff

$(a_k^F(s))_{k=0}^{\infty}$ is discrete NBU (NWU)

for all $s > 0$ where :-

$(a_k^F(s))_{k=0}^{\infty}$ is as defined in (1.1.4).

Proof

The proof of necessity is very similar to the proof of necessity in Theorem (1.2.1) (the corresponding characterisation of the IFRA (DFRA) Classes) except that the condition $(V_n)^{1/n}$ decreasing is replaced by the result, also due to Barlow and Proschan (1975), that if $\bar{J}(x) = e^{-sx}\bar{F}(x)$ and $c_n = (\mu_n/n!)$ where μ_n is the n^{th} moment of $J(.)$ then $J(.)$ NBU implies (c_n) is discrete NBU.

To prove sufficiency the condition of the Theorem can be re-written as :-

$$a_{n+1}^F(s)\ a_{m+1}^F(s) >= a_{n+m+2}^F(s)$$

for $m,n=1,2,...$ and for all $s > 0$.

Now arguing as in the proof of sufficiency in Theorem (1.3.2) (the DMRL case) and again using Lemma 1.2.2. it follows that :-

$$\bar{F}(u+v) \leq \bar{F}(u)\bar{F}(v) \quad \text{for all } u, v > 0$$

No relationship has been found to exist between the TTT transform and the NBU and NWU Classes.

Langberg (et al) give two further characterisations of the NBU Class but for our purposes the following result presented by Block and Savits (1978) will prove more useful.

<u>Theorem (1.4.2)</u>

A life distribution F is NBU (NWU) iff

$$\int_x^\infty g(t-x)dF(t) \leq (\geq) \quad \bar{F}(x) \int_x^\infty g(t)dF(t)$$

for all $x \geq 0$, $g \geq 0$ and increasing on $(0,\infty)$.

<u>Proof</u>

Sufficiency follows by considering :-

$$g(t) = I_{(u,\infty)}(t) \quad ; \quad u \geq 0$$

where I_A is the indicator function of the set A. To prove necessity, first note that the result certainly holds for functions of the form $g(t) = I_A(t)$ where $A = [t,\infty)$ or (t,∞). The desired result follows from the fact that any non-negative increasing function can be written as limit of linear combinations of such functions (i.e. as the limit of simple functions). We will have cause to use the following corrollary to the above theorem in a later Chapter (2).

Corrollary (1.4.1) (Block and Savits (1978)).

Let X and Y be non-negative random variables and suppose X has a NBU distribution, F, then for all x, y >= 0.

$$P(X+Y \geq x+y \mid X>x) \leq P(X+Y>y)$$

Proof

$$P(X+Y > x+y \mid X>x) = \int_0^\infty \bar{G}(x+y-z)\,dF(z)$$

(where G is the distribution of Y)

$$\leq \bar{F}(x) \int_0^\infty \bar{G}(y-z)\,dF(z)$$

(since F is NBU and $\bar{G}$ is non negative and increasing in z).

$$= \bar{F}(u)\; P(X+Y>y)$$

$$\leq P(X+Y>y) \quad \text{as required.}$$

Another useful characterisation of the NBU and NWU Classes is :-

(1.4.3) F is NBU (NWU) iff

$- \log \bar{F}(t)$ is superadditive (sub-additive).

i.e. $- \log \bar{F}(t+s) \geq (\leq) - \log \bar{F}(t) - \log \bar{F}(s)$,

for all $t, s > 0$

This result follows almost directly from the definition of NBU (NWU) Class.

We will have cause to use (1.4.3) in Chapter 2.

~1.5 The New Better than Used in Expectation (NBUE) and New Worse than Used in Expectation (NWUE) Classes of Life Distribution

These classes of life distribution are defined by comparing the conditional expected remaining life of a used device with the expected lifetime of a new device. Consequently, they are closely related to the DMRL and IMRL Classes.

Formally, the NBUE and NWUE Classes are defined as follows:

Definition 1.5.1

A life distribution, F, is NBUE (NWUE) iff

a) F has a finite (finite or infinite) mean, μ
   and

b) (1.5.1) $e_F(t) = (\int_t^\infty \bar{F}(x)dx) / F(t) \ =< \ (>=) \ \mu$

i.e.

(1.5.2) $\int_t^\infty \bar{F}(x)dx \ =< \ \mu \bar{F}(t)$

Now suppose F is DMRL then,

$$e_F(t) = (\int_t^\infty \bar{F}(x)dx) / F(t)$$

is decreasing in t and hence,

(1.5.3) $(\int_t^\infty \bar{F}(x)dx) / \bar{F}(t) \ =< \ (\int_0^\infty \bar{F}(x)dx) \bar{F}(0) - \mu_F / \bar{F}(0)$

Where μ_F is the mean of F.

Now, the only DMRL distribution with mass at 0 is the distribution degenerate at 0, so by 1.5.3 it follows that F DMRL => F NBUE. Similarly, F IMRL => F NWUE.

As would be expected, it is also easy to show that the NBUE (NWUE) Class contains the NBU (NWU) Class. In the discrete case, NBUE and NWUE Classes can be defined as follows :-

Definition (1.5.2)

A sequence of probabilities $(Q_k)_{k=0}^{\infty}$ is discrete NBUE (NWUE) iff

a) $\Sigma_{k=0}^{\infty} Q_k$ is finite (finite or infinite)

and

b) $\Sigma_{k=n}^{\infty} Q_k =< (>=) Q_n \Sigma_{k=0}^{\infty} Q_k$

The relationships between the discrete NBUE (NWUE) and the discrete DMRL (IMRL) and between the discrete NBUE (NWUE) and discrete NBU (NWU) Classes are the same as in the general case.

We now present three alternative characterisations of the NBUE and NWUE Classes of life distribution beginning as usual with a characterisation based on the Laplace transform and due to Block and Savits (1980).

Theorem (1.5.1)

Define the sequence $(a_k^F(s))_{k=0}^{\infty}$ as in (1.1.3) and (1.1.4) there is a life distribution F is NBUE (NWUE) iff

$(a_k^F(s))_{k=0}^{\infty}$ is discrete NBUE (NWUE) for all $s > 0$

Proof

We prove necessity first and consider the NBUE case only since the NWUE case is very similar. We must show that,

$$\Sigma_{k=n}^{\infty} a_k^F(s) \;=<\; a_n^F(s)\, \Sigma_{k=0}^{\infty} a_k^F(s) \quad \text{for all } s > 0$$

Now $\quad A_n^F(s) \;=\; \int_0^{\infty}(u^n/n!)e^{-su}\,\bar{F}(u)du$

(as defined in (1.1.2))

$$>= \;1/\mu \int_0^{\infty}(u^n/n!)e^{-su} \int_u^{\infty}\bar{F}(w)dwdu$$

(where μ is the mean of F) Since F is NBUE; and swapping the order of integration gives :-

$$A_n^F(s) \;>=\; 1/\mu \int_0^{\infty}\bar{F}(w) \int_0^{w}(u^n/n!)e^{-su}dudw$$

Now, repeated integration of the inner integral by parts with summation of the resultant series, gives :-

$$A_n(s) \;>=\; \Sigma_{k=n+1}(s^k w^k/k!)e^{-sw}dw$$

Reversing the order of integration and summation gives :-

$$(1.5.1) \qquad A_n^F(s) \;>=\; 1/\mu s^{n+1} \;\Sigma_{k=n+1}(s^k A_k^F(s)$$

i.e.

$$(1.5.2) \qquad \mu s\, a_{n+1}^F(s) \;>=\; \Sigma_{k=n+2}\, a_k^F(s)$$

Now, $\quad \Sigma_{k=1}\, a_k^F(s) \;=\; \Sigma_{k=1}\, s^k A_{k-1}^F(s)$

$$= \;\Sigma_{k=1} s^k \int_0^{\infty}(\, u^{k-1}/(k-1)!\,)\, e^{-su}\bar{F}(u)du$$

$$= \;s\int_0^{\infty}\Sigma_{k=1}((su)^{k-1}) \;/\; (k-1)!\; e^{-su}\bar{F}(u)du$$

$$= \;s \int_0^{\infty}\bar{F}(u)du$$

$$= \;s\mu$$

and substituting this result in (1.5.2) we have :-

$$(1.5.4) \qquad a_{n+1}^F(s)\;\Sigma_{k=1}^{\infty}a_k^F(s) \;>=\; \Sigma_{k=n+2}^{\infty}a_k^F(s)$$

Now, by adding $a^F_{n+1}(s)$ to both sides of (1.5.4) and noting that $a^F_0(s) = 1$ we have,

$$a^F_{n+1}(s)\ \Sigma^{\infty}_{k=0}\ a^F_k(s) \quad >= \quad \Sigma_{k=n+1}\ a^F_k(s)$$

where n = 1,2,...

and hence,

$$a^F_n(s)\ \Sigma^{\infty}_{k=0}\ a^F_k(s) \quad >= \quad \Sigma^{\infty}_{k=n}\ a^F_k(s)$$

where n = 1,2,...

i.e. $(a^F_k(s))^{\infty}_{k=0}$ is discrete NBUE. Now suppose $(a^F_k(s))^{\infty}_{k=0}$ is discrete NBUE, i.e.,

(1.5.5) for all $s > 0$,

$$a^F_n(s)\ \Sigma^{\infty}_{k=0} a^F_k(s) \quad >= \quad \Sigma^{\infty}_{k=n} a^F_k(s)$$

where n = 1,2,...

In the proof of necessity it was shown that :-

$$\Sigma^{\infty}_{k=1}\ a^F_k(s) \quad = \quad s\mu \quad \text{and consequently,}$$

$$\Sigma^{\infty}_{k=0}\ a^F_k(s) \quad = \quad 1 + s\mu$$

Thus (1.5.5) can be rewritten as :-

for all $s > 0$,

$$a^F_n(s)(1 + s \int_0^{\infty} \bar{F}(u)du) \quad >= \quad \Sigma_{k=n}\ a^F_k(s)$$

where n = 1,2,...

i.e. for all $s > 0$

$$a^F_n(s)s \int_0^{\infty} F(\omega)du \quad >= \quad \Sigma_{k=n+1} a^F_k(s)$$

where n = 1,2,...

and hence,

(1.5.6) $s a_{n+1}(s) \int_0^\infty \bar{F}(u)du \geq \Sigma_{k=n+2}^{\infty} a_k^F(s)$

where n = 1,2,...

Now, define $s =^{df} n/a$ for some continuity point a of F, then :-

$\lim_{n\to\infty} n/s = \lim_{n\to\infty} ((n+1)/s) = a$

Hence by lemma (1.2.2) and equation (1.2.9) letting $n \to \infty$ on both sides of (1.5.6) yields :-

$\bar{F}(x) s\mu \geq s \int_x^\infty \bar{F}(u)du$

i.e. $\bar{F}(x)\mu \geq \int_x^\infty \bar{F}(u)du$

and arguing as in the proof of sufficiency in Theorem (1.3.2) (the DMRL case) it follows that F is NBUE.

The following integral inequality can also be used to characterise the NBUE and NWUE Classes.

<u>Theorem (1.5.2)</u> (Block and Savits (1978))

A life distribution, F, is NBUE, (NWUE) iff

a) F has a finite (finite or infinite) mean, $\mu = \int_0^\infty \bar{F}(x)dx$ and

b) $\int_0^\infty g(z)\bar{F}(z)dz \leq (\geq) \mu \int_0^\infty g(z)dF(z)$

for all non-negative g increasing in $(0,\infty)$.

Proof

Exactly as for Theorem (1.4.2) (the corresponding result in the NBU (NWU) case).

The scaled TTT transform is quite useful in identifying NBUE and NWUE distribution as the following result shows :-

Theorem (1.5.3)

A life distribution, F, is NBUE (NWUE) iff

$$\phi_F(t) \geq (\leq) \ t \quad ; \quad 0 \leq t \leq 1$$

where $\phi_F(t)$ is the scaled TTT transform of F, as defined by (1.0.3).

Proof

For strictly increasing and continuous distributions F the result follows directly from the definition of the scaled TTT transform and the NBUE (NWUE) class, once one makes the substitution $t = F(x)$. Klefjo (1982b), however, has pointed out that care must be taken in making this substitution in the more general case since not all NBUE (NWUE) distributions are strictly increasing and continuous.

The above theorem was first presented by Bergman (1979) in the NBUE case and later extended to the NWUE case by Klefjo (1982b).

The NBUE and NWUE Classes are probably the classes of life distribution for which the TTT transform is most useful, since if the scaled TTT transform $\phi_F(t)$ can be obtained in a manageable form, a quick graphical comparison with the line $\phi_F(t) = t$ can indicate whether or not the distribution of interest is NBUE or NWUE.

~1.6 The Harmonic New Better than Used in Expectation (HNBUE) and Harmonic New Worse than Used in Expectation (HNWUE) Class of Life Distribution.

The role of the exponential distribution is central to much of reliability theory and application. It seems reasonable therefore to classify distributions by in some sense comparing them with the exponential distribution. The classes considered in this section are defined via such a comparison.

In reliability and queueing theory, the following method of ordering distribution is sometimes used (c.f. Rolski (1975)): If F and G are two distributions, we say F $<_c$ G iff

$$\int_0^\infty x\,dG(x) < \infty \quad \text{and,}$$

(1.6.1) $$\int_t^\infty \bar{F}(x)\,dx \leq \int_t^\infty \bar{G}(x)\,dx \quad , \quad t \geq 0$$

It can be shown that (1.6.1) holds iff for every increasing convex function f,

$$\int_0^\infty f(x)\,dF(x) \leq \int_0^\infty f(x)\,dG(x)$$

If in (1.6.1) G denotes the expondential distribution with mean, μ, then we have :-

(1.6.2) $$\int_t^\infty \bar{F}(x) \leq \mu e^{-t/\mu} \quad , \quad t \geq 0$$

and it is this inequality which is used to define the HNBUE and HNWUE Classes as follows :-

Definition (1.6.1)

A Life distribution F is HNBUE (HNWUE) iff

(1.6.3) $\int_t^{\infty}\bar{F}(x) \;=<\; (>=)\; \mu_F e^{-t/\mu_F}$

where $\mu_F = \int_0^{\infty}\bar{F}(x)dx$; the mean of F.

Thus, a distribution F is HNBUE iff

$$\int_t^{\infty}\bar{F}(x) \;=<\; \int_t^{\infty}\bar{G}(x)$$

where G is the expondential distribution with the same mean as F.

The HNBUE Classes were introduced by Rolski (1975) and further studied by Klefsjo (1982a).

The name HNBUE arises from the fact that it can be shown that (1.6.3) holds iff

(1.6.4) $t \,/ \int_0^{t} e_F(x))^{-1}dx\;) \;=<\; \mu \;, \quad t >= 0$

where, as before, the mean residual life :-

$$e_F(x) = (\int_x^{\infty}\bar{F}(t)dt\,)\;/\;\bar{F}(x)$$

(1.6.4) says that the integral harmonic mean of the mean residual life of a used unit is less than or equal to the mean life of a new unit.

Of course, since if G is the exponential distribution of mean μ then :-

$$t \,/ \int_0^{t} e_G(x))^{-1}dx\;) = \mu$$

equation (1.6.4) can also be interpreted as a comparison between the integral harmonic means of $e_F(x)$ and $e_G(x)$.

Since the geometric distribution is the discrete analog of the exponential, it seems reasonable to define discrete HNBUE and HNWUE Classes, as follows :-

Definition (1.6.2)

A sequence of survival probabilities $(Q_k)_{k=0}$ is discrete HNBUE (HNWUE) iff

$$\Sigma_{k=n}^{\infty} Q_k \ =< \ (>=) \ \ \mu[1-1/\mu]^n$$

$$\text{where } \mu = \ \Sigma_{k=0}^{\infty} Q_k.$$

Notice that if $(\bar{P}_k)_{k=0}^{\infty}$ are the survival probabilities corresponding to the geometric distribution and $\mu = \Sigma_{k=0}^{\infty}\bar{P}_k$ then $\bar{P}_k = (1-1/\mu)^k$ and $\Sigma_{k=n}^{\infty}\bar{P}_k = \mu[1-1/\mu]^n$

We turn now to the question of providing additional characterisations for the HNBUE and HNWUE Classes.

As in the DMRL (IMRL) case, it is possible to characterise the HNBUE (HNWUE) class via a condition on the first interval in the equilibrium renewal process

If $F\hat{}(t) = 1/\mu_F \int \bar{F}(x)dx$ is the distribution function of the first interval in the equilibrium process with underlying distribution F then we have :-

Theorem (1.6.1)

F is HNBUE (HNWUE) iff

$$[1-F\hat{}(t)] \ =< \ (>=) \ \ \exp[-t/\mu_F] \qquad , \ t >= 0$$

where μ_F is the mean of F (= $\int_0^{\infty}\bar{F}(x)dx$)

Proof

F is HNBUE iff

$$\int_t^{\infty}\bar{F}(x)dx \leq \mu_F \exp[-t/\mu_F] \quad , t \geq 0$$

i.e. iff $\mu_F - \int_0^t \bar{F}(x)dx \leq \mu_F \exp[-t/u_F]$

$$\iff \quad [1-F\hat{}(t)] \leq \exp[-t/\mu_F]$$

The HNWUE case follows, similarly, by reversing the inequalities.

The above result is due to Klefsjo (1982a) as is the following characterisation based on the Laplace transform.

Theorem (1.6.2)

A life distribution F is HNBUE (HNWUE) iff $(a_k^F(s))_k$ is discrete HNBUE (HNWUE) for every $s > 0$; where the $a_k^F(s)$ are defined as in (1.1.3) and (1.1.4).

Proof

We prove necessity first.
Assume F is HNBUE, then we must show that :-

for all $s > 0$,

$$\Sigma_{k=n}^{\infty} a_k^F(s) \leq \mu(s)(1-1/\mu(s))^n$$

where $\mu(s) = \Sigma_{k=0} a_k^F(s)$ and $n=1,2,...$

Now, $\Sigma_{k=n}^{\infty} a_k^F(s) = \Sigma_{k=n}^{\infty} s \int_0^{\infty} \frac{(sx)^{k-1}e^{-sx}}{(k-1)!}\bar{F}(x)dx$

$$= \int_0^{\infty} s\bar{F}(x) \, \Sigma_{k=n}^{\infty}\left(\frac{(sx)^{k-1}e^{-sx}}{(k-1)!}\right)dx$$

$$= \int_0^{\infty} s\bar{F}(x) \int_0^x \frac{s^{n-1}u^{n-2}e^{-su}}{(n-2)!} \, dudx$$

and reversing the order of integration gives :-

$$\Sigma_{k=n}^{\infty} a_k^F(s) = \int_0^{\infty} \frac{s^n u^{n-2} e^{-su}}{(n-2)!} \int_u^{\infty} \bar{F}(x)\,dx\,du$$

$$=< \int_0^{\infty} \frac{s^n u^{n-2} e^{-su}}{(n-2)!} \mu_F \exp[-u/\mu_F]\,du$$

since F is HNBUE and making the substitution $\mu t = x(s\mu_F + 1)$ yields :-

$$\sum_{k=n}^{\infty} a_k(s) =< \frac{(s\mu_F)^n}{(1+s\mu_F)^{n-1}} \int_0^{\infty} \frac{t^{n-2}e^{-t}}{(n-2)!} dt$$

$$= \frac{(s\mu_F)^n}{(1+s\mu_F)^{n-1}}$$

but $\mu(s) = (1+s\mu_F)$ (c.f. proof of Theorem 1.5.1)

hence,

$$\Sigma_{k=n}^{\infty} a_k^F(s) =< \frac{(\mu(s)-1)^n}{(\mu(s))^{n-1}}$$

i.e. $$\Sigma_{k=n}^{\infty} a_k^F(s) =< \mu(s)(1 - 1/\mu(s))^n$$

as required. Now to prove sufficiency, suppose that the condition of the theorem holds,

i.e., $$\Sigma_{k=n}^{\infty} a_k^F(s) =< \mu(s)(1 - 1/\mu(s))^n$$

where $s > 0$, $n = 1,2,...$

So, $$\Sigma_{k=n+1}^{\infty} a_k^F(s) = \mu(s)(1 - 1/\mu(s))^n - a_n^F(s)$$

$$= (\mu(s) - 1)(1 - 1/\mu(s))^n$$

Since $$\Sigma_{k=0}^{\infty} a_k^F(s) = \mu(s)$$

(1.6.5) $$1/s \quad \Sigma_{k=n+1}^{\infty} a_k^F(s) = \mu_F(1 - 1/(1+s\mu_F))^n$$

where for all $s > 0$, $n = 1,2,...$

Now let x be a continuity point of F and choose $s = n/x$ then letting $n \to \infty$ on both sides of (1.6.5) and using equation (1.2.9) and the fact that :-

$$\lim_{n\to\infty} [\ 1 - (x/\mu_F)\ /\ (n+x/\mu_F)\ ]^n$$

$$= \exp[-x/\mu_F]$$

we have,

$$\int_x^\infty \bar{F}(u)du \leq \mu_F \exp[-x/\mu_F]$$

i.e. F is HNBUE.

The final characterisation of the HNBUE class that we shall present here is based on the TTT transform.

<u>Theorem (1.6.3)</u>

A strictly increasing life distribution, F, of finite mean, u, is HNBUE (HNWUE) iff

$$\phi_F(t) \geq (\leq)\ 1 - \exp[-F^{-1}(t)/\mu] \qquad 0 \leq t \leq 1$$

where $\phi_F(t)$ is the scaled TTT transform of F.

Proof

F is HNBUE iff

$$\int_x^{\infty} \bar{F}(u)du \leq \mu e^{-x/\mu} \quad , \quad x > 0$$

and since F is strictly increasing it follows that F is HNBUE iff

$$\int_{F^{-1}(t)}^{\infty} \bar{F}(u)du \leq \mu \exp[-x/F^{-1}(t)] \quad 0 \leq t \leq 1$$

i.e. $$1 - 1/\mu \int_0^{F^{-1}(t)} \bar{F}(u)du \leq \exp[-x/F^{-1}(t)]$$

$$\iff \quad \phi_F(t) \geq 1 - \exp[-x/F^{-1}(t)]$$

$$0 \leq t \leq 1$$

It should be noted that the HNBUE (HNWUE) Class contains the NBUE (NWUE) Class, since,

F is NBUE (NWUE) with mean μ =>

$$e_F(x) \leq (\geq) \mu \quad ; \quad x \geq 0$$

i.e. $$t \leq (\geq) \int_0^t \mu/e_F(x)\, dx$$

i.e. $$t \,/ \int_0^t (e_F(x))^{-1}\, dx \leq (\geq) \mu$$

i.e. F NBUE (NWUE) => F HNBUE (HNWUE).

To conclude this section we briefly mention a generalisation of the HNBUE (HNWUE) Class which has been introduced recently by Basu and Ebrahimi (1984).

<u>Definition (1.6.3)</u>

A life distribution F, of finite mean μ is K-order HNBUE (K-order HNWUE) iff

$$t \,/\, (e_F^{-k}(x)dx) \;=< \;(>=)\; \mu^k$$

where $k = 1,2,...$

Definition (1.6.3) says that a distribution is K-order HNBUE if the K order integral harmonic mean of the mean residual life of a used unit is less than the integral harmonic mean of the mean life of a new unit or, alternatively, of the mean residual life of a used unit with an exponential life distribution.

We will often write K-HNBUE (K-HNWUE) for K-order HNBUE (K-order HNWUE).

Of course, for K=1 the K-HNBUE (K-HNWUE) Class is just the HNBUE (HNWUE) Class. (c.f. (1.6.4)).

Basu and Ebrahimi state that their motivation for introducing these classes of distribution was two-fold: firstly, to provide a larger class of alternatives to the exponential distribution in hypothesis testing situations and, secondly, to establish analytical results for as large a class of life distributions as possible.

The HNBUE Class is contained by the K-HNBUE Class K >= 2 and, in fact, the K-HNBUE Class is contained by the K+1-HNBUE Class, K >= 1 so that as K increases the K-HNBUE Classes form an increasing sequence of classes.

In the HNWUE case the direction of inclusion is reversed, i.e., the HNWUE Class contains the K-HNWUE Class, K >= 2. This follows from the fact that the K-HNWUE Class contains the K+1-HNWUE Class, K >= 1 i.e. as K increases the K-HNWUE Classes form a decreasing sequence of life distribution.

Since the HNWUE class also contains the NWUE Class, the relationship between the K-HNWUE and NWUE Classes is of interest and it turns out that the K-HNWUE Class contains the NWUE Class for $K \geq 1$.

Basu and Ebrahimi have established that every binary random variable has a distribution which is 2-HNBUE, e.g., If F is defined by :-

$$F(x) = \begin{cases} 1 & 0 \leq x < 1 \\ 1/4 & 1 \leq x < 5 \\ 0 & x \geq 5 \end{cases}$$

Then F is 2-HNBUE. Note, however, that F is not HNBUE and hence the conclusion of the HNBUE Class within the K-HNBUE Class; $K \geq 2$ is strict.

We conclude this discussion by noting that the results for shock models presented in Chapters two and three have not as yet been extended to the K-HNBUE (K-HNWUE) Classes.

~1.7 The L and $\bar{L}$ Classes of Distribution

Recently, Klefsjo (1983) proposed two additional classes of life distribution, defined via a comparison of the Laplace transform of a distribution with the Laplace transform of the exponential distribution of the same mean as the distribution of interest. Thus we have :-

Definition (1.7.1)

A Life distribution F belongs to the Class L ($\bar{L}$) iff

$$(1.7.1) \qquad \int_0^\infty e^{-st}\bar{F}(t)dt \geq (\leq) \; \mu/(1+s\mu)$$

$$\text{where } \mu = \int_0^\infty \bar{F}(t)dt$$

Klefsjo suggests a number of reasons why the L and $\bar{L}$ Classes are of interest in reliability theory. The most obvious of these is the fact that the Laplace transform of a distribution is often easier to obtain in a manageable form than an explicit expression for the distribution function itself. Another interpretation particularly relevant in a shock model context is as follows: Suppose a device with distribution F, as of mean u, is subject to the possibility of a catastrophe (or shock) which is always fatal. Suppose further, that the time to the catastrophe is exponentially distributed with mean 1/s, then $\int_0^\infty e^{-st}\bar{F}(t)dt$ is the probability of a catastrophe in the lifetime of the device.

Hence, (1.7.1) says :-

P(a catastrophe in the component's lifetime)

$>= \mu / (1+s\mu)$

i.e. The probability of a catastrophe during the lifetime of the device is at least as large as the probability of a catastrophe during a the life of a device with an exponential lifetime distribution.

In the discrete case we can define Classes G and $\bar{G}$, analogous to L and $\bar{L}$, by comparing the probability generating function (p.g.f.) of a discrete sequence of survival probabilities with the p.g.f. of the survival probabilities corresponding to the geometric distribution with the same mean as the distribution of interest.

Definition (1.7.2)

Let $(Q_k)_{k=0}^{\infty}$ be a sequence of survival probabilities so that :-

$$Q_0 = 1 \quad \text{and} \quad \mu = \Sigma_{k=0}^{\infty} Q_k$$

then $(Q_k)_{k=0}^{\infty}$ belongs to the Class G ($\bar{G}$) iff

$$(1.7.2) \qquad \Sigma_{k=0}^{\infty} Q_k\, p^k \;>=\; (=<) \;\; \mu/p+(1-p)\mu$$

where $0 =< p =< 1$

It should be noted that the L ($\bar{L}$) Class is strictly larger than the HNBUE (HNWUE) Class. This follows from the fact that :-

$$(1.7.3) \qquad \int_0^\infty e^{-st}F(t)dt =$$

$$-e^{-st}\int_0^\infty F(x)dx \,\Big|_{t=0} - s\int_0^\infty e^{-st} \left(\int_t^\infty \bar{F}(x)dx\right)dt$$

$$= \mu - s\int_0^\infty e^{-st} \left(\int_t^\infty F(x)dx)dt\right)$$

and, consequently, if F is HNBUE. We have :-

$$\int_0^\infty e^{-st}\, \bar{F}(t)dt \;>=\; \mu - s\int_0^\infty \mu e^{-t(s+1/\mu)}\, dt$$

$$= \mu(1 - s\mu/(1+s\mu))$$

$$= \mu(1/(1+s\mu))$$

If F is HNWUE, reversing the inequalities in the above argument shows that F belongs to $\bar{L}$.

That the inclusion in both the L and [cases is strict follows from the following examples due to Klefsjo :-

example a) Let F be defined by :-

$$F(t) = \begin{cases} 1 & 0 =< t < 0.3 \\ 0.7 & 0.3 =< t < 3 \\ 0 & t >= 0 \end{cases}$$

then F belongs to L but F is not HNBUE.

example b) Define F by :-

$$F(t) = \begin{cases} 1/2 & 0 =< t < 1 \\ 0 & t >= 1 \end{cases}$$

then F belongs to L but is not HNWUE.

The relationship between G and the discrete HNBUE Class and between $\bar{G}$ and the discrete HNWUE Class is the same as in the general case discussed above.

Since the L and [Classes are defined via the Laplace transform, it is not surprising that, as with the other classes of distribution we have encountered, it is possible to characterise the L and [Classes in terms of a sequence of functions related to the Laplace transform, namely, the $a_k(s)$ of (1.1.4).

The appropriate result is presented below. It was established by Klefsjo (1983) in a shock model context, i.e., with the $a_k(s)$ interpreted as the survival probabilities in a random threshold cumulative damage shock model (c.f. the interpretation of the $a_k(s)$ given in Section 1.1).

Theorem (1.7.1)

A Life distribution F belongs to L ($\bar{L}$) iff $(a_k^F(s))_{k=0}$ belongs to G ($\bar{G}$) for every s > 0

Proof

Suppose F belongs to L i.e. for all s > 0,

$$\int_0^\infty e^{-sx}\ \bar{F}(x)dx \ \ >= \ \ \mu_F\ /\ (1+s\mu_F) \quad \text{where } \mu_F = \int^\infty \bar{F}(x)dx$$

Now, $\Sigma_{k=0}^{\infty}\ a_k(s)p^k = 1 + \Sigma_{k=0}^{\infty}\ (ps)^{k+1} \int_0^\infty x^k/k!\ e^{-sx}\bar{F}(x)dx$

$$= 1 + sp\int_0^\infty \Sigma_{k=0}^{\infty}\ (psx)^k/k!\ e^{-sx}\ \bar{F}(x)dx$$

(1.7.4) $$= 1 + sp\int_0^\infty e^{-sx(1-p)}\ \bar{F}(x)dx$$

(1.7.5) $$>= 1 + ps \frac{\mu_F}{1+s(1-p)\mu_F}$$

since F belongs to L

Now recall that : $\mu(s) = \Sigma_{k=0}\ a_k(s) = 1 + s\mu_F$

hence, $$1 + \frac{ps\ \mu_F}{1 + s(1-p)\mu_F}$$

$$= \frac{1 + s\mu_F}{1+s\mu_F - ps\mu_F}$$

$$= \frac{\mu(s)}{\mu(s) - ps\mu_F}$$

$$= \frac{\mu(s)}{\mu(s)(1-p)+p}$$

i.e. by (1.7.4),

$$\Sigma_{k=0}^{\infty}\ a_k^F(s)p^k \ \ >= \ \ \frac{\mu(s)}{\mu(s)(1-p)+p}$$

and this holds for all $s > 0$ and for all p in $(0,1)$ so $(a_k^F(s))_{k=0}^{\infty}$ belongs to G

Now assume $(a_k^F(s))_{k=0}^{\infty}$ belongs to G so that by (1.7.5) and the definition of G :-

$$1 + sp \int_0^{\infty} e^{-sx(1-p)} \bar{F}(x)dx \geq \frac{\mu(s)}{p+(1-p)\mu(s)}$$

for all $s > 0$ and $p \in (0,1)$

i.e.
$$\int e^{-sx(1-p)} \bar{F}(x)dx \geq \frac{\mu(s)}{ps(p+(1-p)\mu(s))} - \frac{1}{sp}$$

$$= \frac{(1+s\mu_F)-(p+(1-p))(1+s\mu_F)}{ps(p+(1-p)(1+s\mu_F)}$$

(1.7.7)
$$= \frac{\mu_F}{1+(1-p)s\mu_F}$$

where $s > 0$, p in $(0,1)$

i.e.

(1.7.8)
$$\int_0^{\infty} e^{-s'x} \bar{F}(x)dx \geq \frac{\mu_F}{1+s'\mu_F}$$

where $s' = s(1-p)$

and since (1.7.7) holds for every $s > 0$ and every p belongs to $(0,1)$ so also does (1.7.8) hold for every $s' > 0$
i.e. F belongs to L.

The $\underline{L}$ case follows by reversing inequalities.

1.8 Closure properties

In the previous sections we have studied a number of classes of life distributions. In the first part of this section we turn our attention to the question of whether distributions from a particular class yield distributions from that class under certain operations, e.g., a reliability practitioner may be interested in the convolution of NBU distributions since convolution corresponds to the addition of life lengths of independent components.

Three operations on distribution functions, often referred to in the literature as reliability operations, are of particular interest in reliability theory :-

a) Formation of Coherent Systems

Suppose a system consists of n components, then at any given time each component may be in one of two states, functional (denoted by a '1') or non-functional (denoted by a '0'). Let $\phi(x_1,...x_n)$ be the structure function of the system where $x_i \in \{0,1\}$ and $i=1,2,...n$ and $\phi(\underline{x}) \in \{0,1\}$ then a system is said to be coherent if ϕ is increasing in each argument and for :-

$$\phi(x_1,...x_{i-1},\ 1,\ x_{i+1},...x_n) <> \phi(x_1,...x_{i-1},\ 0,\ x_{i+1},...x_n) \quad \text{where } i = 1,...n$$

Series and Parallel systems are of course coherent.

It is of interest to determine whether the life distributions of coherent systems of components with life distributions in a particular class belong to that class.

b) Convolution

As stated previously, convolution of distributions arise naturally in reliability as the distribution of the sum of independent life lengths.

c) Mixture

Suppose $H_{s(.)}(t)$ is the survivor function of a device subject to shocks whose arrival is governed by an inhomogeneous Poisson process with mean value function,

$$I(t) = \int_0^t s(x)dx \;/\; t$$

Now suppose the rate (or intensity) function is itself random i.e. for every $x > 0$, $S(x)$ is a random variable, so that the process governing the arrival of shocks is in fact a doubly stochastic Poisson process. The survivor function of a device subject to such a shock process is given by :-

$$\bar{H}(t) = E_{S(.)}[\bar{H}_{s(.)}(t)]$$

i.e. a mixture of the survivor functions $\bar{H}_{s(.)}(t)$.

We will study such a model in Chapter two and, consequently, it is of interest to determine whether or not a class of distributions is closed under the mixture operation, i.e., whether a mixture of distributions from a particular class is still a member of that class. The closure properties of the classes considered in this Chapter, under the operations a), b) and c) described above, are summarised in the following tables :-

Table 1 : Closure properties of classes with "positive" ageing under reliability operations.

	Formation of Coherent Systems	Convolution	Mixture
IFR	Not Closed	Closed	Not Closed
IFRA	Closed	Closed	Not Closed
DMRL	Not Closed	Not Closed	Not Closed
NBU	Closed	Closed	Not Closed
NBUE	Not Closed	Closed	Not Closed
HNBUE	Not Closed	Closed	Not Closed
HNBUE	Not Closed	Unknown	Not Closed
L	Not Closed	Closed	Not Closed

Table 2 : Closure Properties of Classes with negative ageing under reliability operations.

	Formation of Coherent System	Convolution	Mixture
DFR	Not Closed	Not Closed	Closed
DFRA	Not Closed	Not Closed	Closed
IMRL	Not Closed	Not Closed	Closed
NWU	Not Closed	Not Closed	Closed
NWUE	Not Closed	Not Closed	Not Closed
K-HNWUE	Not Closed	Not Closed	Closed
NWUE	Not Closed	Not Closed	Closed
$\bar{L}$	Not Closed	Not Closed	Closed

It should be noted that a mixture of HNBUE distributions, all of which have the same mean .s HNBUE and the NWU and NWUE Classes are closed under mixtures of distribution that do not cross.

1.9 Reliability bounds and Relationships between the Classes of Life Distribution.

One of the motivating factors for the study of classes of life distribution is the derivation of bounds on the survivor function for distributions belonging to a particular class. In this section, a few such bounds are presented. The bounds that we will present are based on a known mean, u, and/or a known point of the distribution, i.e., F(t*) is assumed known for some t*.

The bounds based on a known mean and a known point of the distribution are better than those based only on a known mean.

It should also be noted that the smaller the class the better the bound. Now, in the course of sections 1.1 - 1.7, the following relationships between the classes of distribution discussed there, were established.

```
(a)     {IFR}     -->     {IFRA}
          |                  |
          |                  v
          |               {NBU}
          v                  |
         DMRL     -->     {NBUE}
                             |
                             v
                          {HNBUE}    -->    {K-HNBUE}
                             |
                             v
                          { L }

(b)     {DFR}     -->     {DFRA}
          |                  |
          |                  v
          |               {NWU}
          v                  |
        {IMRL}    -->     {NWUE}
                             |
                             v
                          {K - HNWUE}
                             |
                             v
                          {HNWUE}
                             |
                             v
                          { L }
```

Hence the fact that the bounds for smaller classes are better than those for larger ones means that if, for example, a distribution F is IFR, the bound for the IFR class provides a closer approximation to F than does the HNBUE bound.

Many bounds are available in the literature but only a few basic bounds are presented here. Useful references in this context are Barlow and Marshall (I and II, 1964, 1965), Marshall and Proschan (1972) and Haines and Singpurwalla (1974).

We begin with the IFR Class.

If F is IFR with known mean, μ —

$$(1.9.1) \qquad \bar{F}(t) \geq \begin{cases} e^{-t/\mu} & t < \mu \\ 0 & t \geq \mu \end{cases}$$

and,

$$(1.9.2) \qquad \bar{F}(t) \leq \begin{cases} 1 & t \leq \mu \\ w_0 & t > \mu \end{cases}$$

where w_0 uniquely satisfies $\mu = t \int_0^1 w_0^x \, dx$

If F is DFR with mean μ then,

$$(1.9.3) \qquad \bar{F}(t) \leq \begin{cases} e^{-t/\mu} & t \leq \mu \\ \mu e^{-1}/t & t \geq \mu \end{cases}$$

If F is IFRA with known mean μ then,

$$(1.9.4) \qquad \bar{F}(t) \leq \begin{cases} 1 & t \leq \mu \\ e^{-wt} & t > \mu \end{cases}$$

where $w_1 = w_1(t)$ satisfies $1 - w_1\mu = e^{-w_1 t}$

If F is DMRL with known mean μ and $\bar{F}(t^*) = a$ for some fixed t^* and known a, then,

$$(1.9.5) \qquad \bar{F}(t) \geq \max [\bar{F}(t^*),\ (\mu - t)/\mu], \qquad \text{where } 0 \leq t \leq t^* \leq \mu$$

and

$$\bar{F}(t) \geq \bar{F}(t^*) - [F(t^*)]^2 (t - t^*)/(\mu - t^*) \qquad \text{where } 0 \leq t^* \leq t \leq \mu_{t^*}$$

and

$$\bar{F}(t) \geq 0 \qquad \text{where } t \geq \mu_{t^*}$$

and where
$$\mu_{t^*} = \frac{\mu + t^* \bar{F}(t^*) - t^*}{\bar{F}(t^*)}$$

If F is IMRL with known mean μ and $\bar{F}(t^*) = a$ for some fixed t^* and known a, then,

$$(1.9.6) \qquad \bar{F}(t) \leq \mu/(\mu + t) \qquad \text{where } 0 < t \leq t^*$$

$$\bar{F}(t) \leq \frac{F(t^*)[\mu - t^* F(t^*)]}{\mu + tF(t^*) - 2t^* F(t^*)} \qquad \text{where } t \geq T^*$$

If F is NBU with $\bar{F}(t^*) = a$ for some fixed t^* and known a,

$$\bar{F}(t) \geq a^{1/k} \qquad \text{for } t^*/k+1 < t < t^*/k \qquad \text{for } k = 0,1,2,\ldots$$

and

$$\bar{F}(t) \leq a^k \qquad \text{for } kt^* \leq t < (k+1)t^*$$

Similarly, if F is NWU with $F(t^*) = a$ for some fixed t^* and known a, then :-

(1.9.7) $\bar{F}(t) \leq a^{(1/k)+1}$, $t^*/k+1 \leq t < t^*/k$

where $k = 0,1,2...$

$\bar{F}(t) \geq a^{k+1}$, $kt^* \leq t < (k+1)t^*$

If F is NBUE with known mean μ then,

(1.9.8) $\bar{F}(t) \geq (\mu-t)/\mu$ where $t \leq \mu$

If in addition it is known that $F(t^*) = a$ for some t^* such that $0 \leq t^* \leq \mu$, then,

(1.9.9) $\bar{F}(t) \geq \max[a,(\mu-t)/\mu]$, $0 \leq t \leq t^* \leq \mu t^*$

$\bar{F}(t) \geq 1/\mu\ [\mu-t^*-a(t-t^*)]$

and where $t^* \leq t \leq \mu t^*$

where μ_{t*} is as defined in (1.9.5).

If F is NWUE with known mean μ,

(1.9.10) $\bar{F}(t) \leq \mu/(\mu+t)$, $t \geq 0$

and if in addition $F(t^*) = a$ for some t^* then,

$F(t) \leq \mu/(\mu+t)$, $0 \leq t \leq t^* < \infty$

(1.9.11) $\bar{F}(t) \leq \min[a,\mu/(\mu+t)]$, $t^* \leq t < t_\infty$

$$\bar{F}(t) \leq \frac{\mu - t^*a}{\mu+t-t^*} \quad , \quad t_\infty \leq t < \infty$$

If F is HNBUE with known mean μ,

(1.9.12) $\bar{F}(t) \le \exp[(\mu - t)/\mu]$, $t \ge \mu$

$\bar{F}(t) \ge e^{-w/\mu}$, $0 < t < \mu$

where $w = w(t)$ is the largest non-negative number for which

$(w - t + \mu)\exp(-w/\mu) - \mu + t = 0$

If F is HNWUE with known mean μ, then,

(1.9.13) $\bar{F}(t) \le \mu/t\,(1 - e^{-t/\mu})$, $t \ge 0$

If a life distribution F is K-HNBUE with mean μ then,

(1.9.14) $$\bar{F}(t) \ge \frac{\exp(-w_3/\mu^k)}{w_3 - t}(w_3 - t + \mu^k - \mu)$$

for $t < \mu^k$

Here, $w_3 = w_3(t)$ is the largest non-negative solution of :-

$1/\mu^{k-1}\,(w_3 - t + \mu^k)\exp(-w/\mu^k) - \mu + t = 0$

If a life distribution F is K-HNWUE with mean μ then,

(1.9.15) $$\bar{F}(t) \le \frac{\mu t^{1/k}}{t^{(1/k+1)} + \mu}$$ for $t \ge 0$

It appears that no bounds for the L or $\bar{L}$ classes have yet been presented in the literature.

CHAPTER 2 : THE STANDARD SHOCK MODEL

~2.0 INTRODUCTION

Suppose a device is subject to shocks or blows which occur at discrete points in time. Further, suppose that the occurrence of these shocks is governed by a stochastic point process $\{N(t)\}$ where for each $t > 0$, $N(t)$ is the number of shocks suffered by the device up to time t. The origin, time $t_0 = 0$, can be thought of either as the beginning of the device's period of service or as the time at which observation of the device commences.

We are interested in the lifetime distribution, F, or more particularly the survivor function $1-F$ of the device. If we assume that the probability of surviving k shocks is given by P_k where $k = 1,2,\ldots$ and that the device fails only on the occurrence of a shock, so that no failures occur between shocks, we have :-

$$(2.0.1) \qquad P(T>t) = \bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k)\bar{P}_k$$

where T is the lifetime of the device. We will assume throughout that $\bar{P}_0 = 1$

Our aim is to establish sufficient conditions on the $(\bar{P}_k)_{k=0}^{\infty}$ and on the process $\{N(t)\}$ so that $H(t)$ belongs to one of the classes of life distribution discussed in Chapter one. Of particular interest will be the degree to which the continuous life distribution $H(t)$ inherits its class from the analogous discrete class of the survival probabilities $(\bar{P}_k)_{k=0}^{\infty}$.

The justification for approaching the study of Shock Models in this way is that once the class to which the life distribution belongs is known, it is possible to calculate bounds on the distribution, (c.f. 1.9) also knowledge of the class of distribution to which a distribution belongs can often lead to more precise estimation results and a more appropriate choice of maintenance policy.

The model (2.0.1) has been discussed extensively in the literature; Esary, Marshall and Proschan (1973) considered this model in the case that {N(t)} is a homogeneous Poisson process while A-hameed and Proschan (1973) discussed the more general case where {N(t)} is a non-homogeneous Poisson process. A-Hameed and Proschan (1975) and Klefsjo (1980) have considered the model (2.0.1) under the condition that {N(t)} is a pure birth process and recently Thall (1981) has discussed the model (2.0.1) assuming a Poisson cluster process for the process governing the arrival of shocks. Still more general models for {N(t)} have been discussed by Block and Savits (1978) and Klefsjo (1981, 1983).

In this Chapter we will summarise the results of the above papers and, in addition, introduce a model in which {N(t)} is a doubly stochastic Poisson process.

Firstly, however, it is worth making some general comments about the model :-

$$\bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k)\ \bar{P}_k.$$

One notable feature of this model is that the survival probabilities ($\bar{P}_k$) are implicitly assumed to be independent of time. The area of applicability of the model (2.0.1) is wider than this, however, since if $\bar{P}_k(t)$ = P(surviving K shocks in (0,t)) = $\bar{F}(t)\bar{P}_k$ where $\bar{F}(t)$ = P(no failures not due to shocks in (0,t)) then the survivor function for the device in question is given by :-

$$\bar{H}_F(t) = \Sigma_{k=0}^{\infty} P(N(t)=k)\ \bar{F}(t)\bar{P}_k$$

$$= \bar{F}(t)\Sigma_{k=0}^{\infty} P(N(t)=k)\bar{P}_k$$

$$(2.0.3) \qquad = \bar{F}(t)\ \bar{H}(t)$$

where H(t) is defined by (2.0.1).

Now, (2.0.3) is the survivor function of two independent components connected in series hence using the closure properties of the classes discussed in Chapter one under the formation of coherent systems (c.f. 1.9), the class to which $H_F(t)$ belongs can be determined provided the classes to which F(.) and H(.) belong are known and are classes which are closed under the formation of Coherent Systems. Hence, even in this case the survivor function $\bar{H}(t) = \Sigma_{k=0} P(N(t)=k)\bar{P}_k$ is of considerable interest.

The term "shock" has connotations of causing damage. However, in this thesis the term "shock" is an abstraction which can best be defined as "an identifiable event in the lifetime of a device" so that, for example, the shocks of model (2.0.1) may in fact be inspections of the device. Damage may accumulate continuously but because inspection takes place at discrete points in time, the model (2.0.1) applies. In this case the "shocks" themselves cause no damage (assuming inspection does not damage the device). Another possibility is that the shocks of the model are actually repairs so that the shocks cause negative damage. As a consequence of our general definition of the term "shock" the area of applicability of the results which we will present is quite wide. It is also pertinent to note that a shock in the context of reliability may be interpreted as a demand in the theory of inventory control or a claim in risk analysis and, of course, shock models also arise in Biometry.

Several of the results to be presented in this Chapter rely on the notion of a totally positive function and some of the results from the theory of total positivity. For this reason, a brief discussion of total positivity follows. A totally positive function is a generalisation of a PF_2 function (c.f. 1.1) and can be defined as follows :-

Definition (2.0.1)

Let A and B be subsets of the real line. A function $K(x,y)$ on $A \times B$ is said to be totally positive of order n (TP_n) if :-

$$\begin{vmatrix} K(x_1,y_1) & \dots & K(x_1,y_r) \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \vdots & & \vdots \\ K(x_r,y_1) & \dots & K(x_r,y_r) \end{vmatrix} \geq 0$$

for all $x_1 < \dots < x_r$ in A and

$y_1 < \dots < y_r$ in B where $r = 1,2,\dots n$

A function which is totally positive of all finite orders is said to be totally positive. Note that if $h(x)$ is PF_2 then $K(x,y) = h(x-y)$ is TP_2 where x and y range over the real line.

An important property of totally positive function is the variation diminishing property, stated in the following theorem.

Theorem (2.0.1) (c.f.Barlow and Proschan p.93. Theorem 3.5)

Let K (x,y) be TP_r on $A \times B$ and let f be a bounded measurable function on B. Let $g(x)= \int_B K(x,y)f(y)dy$ be finite for each x in A then :-

$S(g) \leq S(f)$ provided $S(f) \leq r-1$ where for any function h

$S(h) = \sup S\,[h(t_1),h(t_2),\dots h(t_n)]$ and $S(x_1,\dots x_m)$ is the number of sign changes of the sequence $(x_1,\dots x_m)$ zero terms being discarded.

Essentially, the above theorem states that if $K(x,y)$ is TP_r then $g(x) = \int_B K(x,y) f(y)dy$ has at most as many sign changes as f. In addition, it can be shown that under the conditions of Theorem (2.0.1) if $S(g) = S(f) \leq r-1$ then f and g exhibit the same sequence of signs.

Another useful result is the Basic composition formula (c.f. Karlin, p.17 or Barlow and Proschan p.100) which is stated below.

<u>Theorem (2.0.2)</u>

Let $W(x,z) = \int U(x,y)V(y,z)d@(y)$ where the integral converges absolutely and $d@(y)$ is a sigma-finite measure. Then,

$$\begin{vmatrix} W(x_1,z_1) & \ldots & W(x_1,z_1) \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \vdots & & \vdots \\ W(x_n,z_1) & \ldots & W(x_n,z_n) \end{vmatrix}$$

$$= \int \cdots \int_{y_1 < \ldots < y_n} \begin{vmatrix} U(x_1,y_1) & \ldots & U(x_1,y_n) \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \vdots & & \vdots \\ U(x_n,y_1) & \ldots & U(x_n,y_n) \end{vmatrix} \quad X$$

$$\begin{vmatrix} V(b_1,z_1) & \ldots & V(y_1,z_n) \\ \vdots & & \vdots \\ \vdots & & \vdots \\ \vdots & & \vdots \\ V(y_n,z_1) & \ldots & V(y_n,z_n) \end{vmatrix} \quad d@(y_1) \ldots d@(y_n)$$

The theory of total positivity has application in many areas, mathematics as well as in mechanics, economics and statistics. The most comprehensive reference is Karlin (1968).

~2.1 <u>Poisson and related models</u>

In this section we consider the model :-

$$H(t) = \Sigma_{k=0}^{\infty} P(N(t)=k)\bar{p}_k$$

under a variety of assumptions on the form of {N(t)} As stated in 2.0 our aim is to establish sufficient conditions on the survival probabilities $(\bar{p}_k)_{k=0}$ and on the process {N(t)} so that H(t) belongs to one of the classes discussed in Chapter one.

One approach would be to begin by assuming {N(t)} to be a homogeneous Poisson process and then to consider progressively more general processes, e.g., inhomogeneous Poisson processes and pure birth processes for {N(t)}. This is in fact the way in which the subject has evolved with the paper by Esary, Marshall and Proschan (1973) serving as a starting point.

Our approach, however, will be slightly different in that we will begin by studying the model (2.0.1) under the condition that {N(t)} is a pure birth process and obtain results for the case where {N(t)} is a homogeneous Poisson process as a special case. This approach has the advantage of simplifying several of the proofs.

~2.1A The Stationary Pure Birth Shock Model

Suppose that a device is subject to shocks whose occurrence is governed by a Markov process {N(t)} with the following transition probabilities :-

(2.1 .1) (i) $P[N(t+d) - N(t) = 1 \mid N(t) = k]$
$= s_k d + o(d)$

(2.1 .2) (ii) $P[N(t+d) - N(t) = 0 \mid N(t) = k]$
$= 1 - s_k d + o(d)$
for small d.

(iii) $P[N(t+d) - N(t) < 0 \mid N(t) = k) = 0$

then {N(t)} is a stationary pure birth process (c.f. Karlin (1966) p. 177) and we will refer to the model :-

$$\bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k)\bar{P}_k$$

as the stationary pure birth shock model.

Clearly if $s_k = s$ where $k = 0,1,2,...$ then {N(t)} is just the familiar homogeneous Poisson process.

We are interested only in sequences $(s_k)_{k=0}^{\infty}$ for which the probability of infinitely many shocks in $(0,t)$ is zero. This is equivalent to the condition $\Sigma_{k=0}^{\infty} P(N(t)=k) = 1$ and by the Feller-Lindberg theorem (c.f. Feller (1968) p. 452) this is equivalent to $\Sigma_{k=0}^{\infty} 1/s_k = \infty$.

One important feature of the stationary pure birth process is that the intervals between the k^{th} and $k+1^{th}$ shock where $k = 0,1,2,...$ are independently and exponentially distributed with mean $1/s_k$.

We note also that if $\{N(t)\}$ is a stationary pure birth process with birth coefficients $(s_k)_{k=0}$ then,

(2.1.3) $$z_k(t) = P(N(t)=k) =$$

$$= s_{k-1} \exp(-s_k t) \int_0^t \exp(s_k x) z_{k-1}(x)dx$$
where $k = 1,2,...$

(2.1.4) $$z_0(t) = \exp(-s_0 t)$$

(2.1.5) and $$z_0'(t) = - s_0 z_0(t)$$

(2.1.6) $$z_n'(t) = - s_n z_n(t) + s_{n-1} z_{n-1}(t)$$
where $n >= 1$

(c.f. Karlin (1966) pp.177-179). Furthermore, if $H(t)$ is defined by,

$$\bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k) \bar{P}_k = \Sigma_{k=0} z_k(t)\bar{P}_k$$

H has a density h given by :-

(2.1.7) $$h(t) = \Sigma_{k=0}^{\infty} z_k(t) s_k p_{k+1}$$

where $p_k = \bar{P}_{k-1} - \bar{P}_k$, the probability of failure on the k_{th} shock. This follows from (2.1.5) and (2.1.6).

Another useful result is that if $\{N(t)\}$ is a stationary pure birth process then $z_k(t) = P(N(t) = k)$ is TP (in k and t). This follows from Theorem 3 of Karlin and Proschan (1960) which states that if $(f_i)_{i=1}$ is a sequence of densities of non-negative random variables and each f_i is PF_k then :-

$$h(n,x) = [F_1 * F_2 * \ldots F_n(x)] - [F_1 * F_2 \ldots F_{n+1}(x)]$$

is TPk.

Here * denotes convolution. Since the shock inter-arrival times in the stationary pure birth process are independently, exponentially distributed and the exponential distribution is PF it follows that $z_k(t)$ is TP.

We can now state the main result for the stationary pure birth shock model.

<u>Theorem (2.1A.1)</u>

Let $\{N(t)\}$ be a stationary pure birth process with birth coefficients $(s_k)_{k=0}^{\infty}$ then if H(t) is defined by :-

$$\bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k)\bar{p}_k$$

a) H is IFR whenever $(s_k)_{k=0}^{\infty}$ is increasing in K and $(\bar{p}_k)_{k=0}^{\infty}$ discrete IFR.

b) H is IFRA whenever $(s_k)_{k=0}^{\infty}$ is increasing and $(\bar{p}_k)_{k=0}^{\infty}$ discrete IFRA

c) H is DMRL wherever $(s_k)_{k=0}^{\infty}$ is increasing and $(\bar{p}_k)_{k=0}^{\infty}$ is discrete DMRL.

d) H is NBU wherever $(s_k)_{k=0}^{\infty}$ is increasing and $(\bar{p}_k)$ discrete NBU.

e) H is NBUE whenever $(s_k)_{k=0}^{\infty}$ is increasing and $(\bar{p}_k)_{k=0}^{\infty}$ is discrete NBUE.

f) H is HNBUE wherever :

$$\Sigma_{j=k}^{\infty} \bar{P}_j \, s_j^{-1} =< a_0 \prod_{j=0} (1-(a_0 s_j)^{-1})$$

for k = 1,2,3,... where $a_0 = \Sigma_{j=0}^{\infty} \bar{p}_j \, s_j$

g) H belongs to L wherever $s_j >= s_0$ for j = 1,2,3..

and $(P_k/s_k)_{k=0}^{\infty}$ is decreasing and in G.

Proof

a) Suppose $(\bar{P}_k)$ discrete IFR and $(s_k)_{k=0}^{\infty}$ increasing in k and consider the determinant :-

$$D = \begin{vmatrix} -h(t_1) & H(t_1) \\ -h(t_2) & H(t_2) \end{vmatrix} \quad \text{where } t_1 < t_2$$

(2.1.7)

$$D = \begin{vmatrix} -\Sigma_{k=0}\, z_k(t_1) s_k p_{k+1} & \Sigma_{k=0}\, z_k(t_1) \bar{p}_k \\ -\Sigma_{k=0}\, z_k(t_2) s_k p_{k+1} & \Sigma_{k=0}\, z_k(t_2) \bar{p}_k \end{vmatrix}$$

and by the basic decomposition formula this is equal to :-

$$\sum_{0<k_1<k_2<\infty} \begin{vmatrix} z_{k_1}(t_1) & z_{k_2}(t_1) \\ z_{k_1}(t_2) & z_{k_2}(t_2) \end{vmatrix} \times \begin{vmatrix} -s_{k_1} p_{k_1+1} & \bar{p}_{k_1} \\ -s_{k_2} p_{k_2+1} & \bar{p}_{k_2} \end{vmatrix}$$

Now, the first determinant is non-negative since $z_k(t)$ is TP (Karlin and Proschan (1960).

The second determinant is non-negative since $s_{k1} < s_{k2}$. As $(s_k)_{k=0}^{\infty}$ is increasing and the IFR property of $(\overline{P}_k)_{k=0}$ implies :

$$\frac{p_{k1+1}}{p_{k1}} < \frac{p_{k2+1}}{p_{k1}}$$

Hence $D = \bar{H}(t_1)h(t_2) - \bar{H}(t_2)h(t_1) > 0$

i.e. $r(t_2) > r(t_1)$ where $r(t)$ is the failure rate.
Consequently H is IFR.

b) To establish the IFRA result a preliminary lemma due to A-Hameed and Proschan (1975) is required.

Lemma 2.1A.1

Let $f(t)$ be an increasing function satisfying $f(0^-)=0$ and for each $0 \le B < \infty$ let there exist a function $g_B(t)$ such that $g_B(t)$ is increasing in t and

$$\begin{aligned} g_B(t) - f(t) &\ge 0 \quad \text{for } 0 \le t \le B \\ &\le 0 \quad \text{for } B \le t \le \infty \end{aligned}$$

Then $f(t)/t$ is increasing in t.

Now, let $H_p(t) = \Sigma_{k=0}^{\infty} z_k(t)p^k$ where $0 \le p \le 1$ then $H_p(t)$ is IFR by part (a) of this theorem since (s_k) is increasing and (p^k) is discrete IFR.

$(P_k)_{k=0}^{\infty}$ is discrete IFRA implies $\bar{P}_k - p^k$ changes sign at most once and if once from + to - for each fixed $0 \le p \le 1$ (c.f Lemma 1.2.1) and using the variation diminishing property of the totally positive kernel $z_k(t) = P(N(t)=k)$ it follows that :-

$$\bar{H}(t) - \bar{H}_p(t) = \Sigma_{k=0}^{\infty} z_k(t)(\bar{P}_k - p^k)$$

changes sign at most once (in t) and if once from + to −. Moreover, since the log function is monotone :-

$$-\log \bar{H}_p(t) - (-\log \bar{H}(t))$$

changes sign at most once and if once from + to −. Note that since H_p is IFR it is also IRFA, i.e. $-\log \bar{H}_p(t) / t$ is increasing in t. Now, given $0 \le a \le \infty$ and letting p vary between 0 and 1.

$$-\log \bar{H}_p(a) - [-\log \bar{H}(a)]$$

varies continuously between

$$(s_0 a - [-\log \bar{H}(a)] \; (\geq 0) \text{ and } -[-\log \bar{H}(a)] \; (\leq 0).$$

Hence there exists a p_* such that :- $-\log \bar{H}_p(a) - [-\log \bar{H}(a)] = 0$

Consequently by the sign change property of :-

$-\log \bar{H}_p(a) - [-\log \bar{H}(a)]$ and Lemma 2.1A.1

$-\log \bar{H}(t) / t$ is increasing in t, i.e., H is IFRA.

c) To establish the desired result in the DMRL case we first show that H(t) is DMRL if $(s_k)_{k=0}^{\infty}$ $(\bar{P}_k)_{k=0}^{\infty}$ satisfy the weaker conditions.

$$\bar{P}_j \Sigma_{k=j}^{\infty} \bar{P}_k / s_k \text{ decreasing in } j.$$

We then show that the conditions of the theorem, $(s_k)_{k=0}^{\infty}$ increasing and $(\bar{P}_k)_{k=0}^{\infty}$ discrete DMRL imply the above condition. To show that H(t) is DMRL if $\bar{P}_j \Sigma_{k=j}^{\infty} \bar{P}_k / s_k$ is decreasing in j we first note that (i) :-

$$(2.1.8) \qquad \int_0^{\infty} z_k(u)du = 1/s_k$$

and

$$(2.1.9) \qquad \int_0^{\infty} z_k(u)du = 1/s_k \Sigma_{j=0}^{\infty} z_j(t)$$

where $z_j(u) = P(N(u)=j)$

(see A-hameed and Proschan (1975) for details of the proof of ii). Now, for $C \geq 0$:-

$$\int_t^{\infty} \bar{H}(u)du - C\bar{H}(t) = \Sigma_{k=0}^{\infty} 1/s_k \, \bar{P}_k \Sigma_{j=0}^{k} z_j(t) - C\Sigma_{j=0}^{\infty} z_j(t) \, \bar{P}_j$$

$$= \Sigma_{j=0}^{\infty} z_j(t) \, [\Sigma_{k=j}^{\infty} \bar{P}_k / s_k - C\bar{P}_j]$$

Now, by assumption $\bar{p}_j^{-1}\Sigma_{k=j}^{\infty} \bar{P}_k/s_k$ is decreasing in j hence $\Sigma_{k=j}^{\infty} \bar{p}_k/s_k - C\bar{p}_j$ changes sign at most once and if once from + to - and by the variation diminishing property of the TP kernal $z_j(t)$ it follows that $\int_t^{\infty}\bar{H}(U)du - C\bar{H}(t)$ has the same sign change property, consequently by (1.3.5) H is DMRL.

To complete the proof we must show that $(s_k)_{k=0}^{\infty}$ increasing and $(\bar{p}_k)_{k=0}^{\infty}$ discrete DMRL implies $\bar{p}_k\Sigma_{k=j}^{\infty} \bar{p}_j/s_j$ decreasing in k.

Now, if $(s_k)_{k=0}^{\infty}$ is increasing then :-

$$\lim_{k\to\infty} s_k = s^{-1} >= 0 \text{ exists.}$$

Consequently,

(2.1.10) $\bar{p}_k^{-1}\Sigma_{j=k}^{\infty} \bar{p}_j s_j^{-1} - \bar{p}_{k+1}^{-1} \Sigma_{j=k+1}^{\infty} \bar{p}_j s_j^{-1}$

$$= [s^{-1}(\bar{p}_k^{-1}\Sigma_{j=k}^{\infty} \bar{p}_j - \bar{p}_{k+1}^{-1} \Sigma_{j=k+1} \bar{p}_j)]$$

$$+ [\bar{p}_k^{-1}\Sigma_{j=k} \bar{p}_j(s_j^{-1}-s^{-1}) - \bar{p}_{k+1}\Sigma_{j=k+1} \bar{p}_j(s_j-s^{-1})]$$

The first term on the right hand side of (2.1.10) is non-negative since $(\bar{p}_k)_{k=0}^{\infty}$ is discrete DMRL

If in the second term on the right hand side of (2.1.10), we write $(s_j-s^{-1}) = \Sigma_{v=j}^{\infty} (s_v-s_{v+1})$ and then change the order of summation we have :-

$$[\bar{p}_k^{-1}\Sigma_{j=k}(s_j-s^{-1}) - \bar{p}_{k+1} \Sigma_{j=k+1} \bar{p}_j(s_j-s^{-1})]$$

$$= \Sigma_{v=k}^{\infty}(s_v^{-1}-s_{v+1}^{-1})\bar{p}_k^{-1} \Sigma_{j=k}^{\infty} \bar{p}_j^{-1}$$
$$- \Sigma_{v=k+1}^{\infty}(s_v^{-1}-s_{v+1}^{-1})\bar{p}_{k+1}^{-1} \Sigma_{j=k+1}^{\infty} \bar{p}_j^{-1}$$

(2.1.11) $= (s_k^{-1}-s_{k+1}^{-1})\bar{p}_k\Sigma_{j=k}^{v} \bar{p}_j + \Sigma_{v=k+1}(s_v^{-1}-s_{v+1}^{-1})$
$\quad X \quad (\bar{p}_n^{-1}\Sigma_{j=k}^{v} \bar{p}_j-\bar{p}_{k+1}^{-1} \Sigma_{j=k+1}^{v} \bar{p}_j)$

Now since $(s_k)_{k=0}^{\infty}$ is increasing in k the right hand side of (2.1.11) is non-negative provided :-

$$\phi_k(v) = (\bar{p}_k^{-1}\Sigma_{j=k}^{v} \bar{p}_j - \bar{p}_{n+1}^{-1}\Sigma_{j=k+1}^{\infty} \bar{p}_j) >= 0$$

for $v = k+1, k+2, ...$

But this follows from the facts that :-

$$\phi_k(v) - \phi_k(v+1) = \bar{p}_{v+1}(\bar{p}_{k+1}-\bar{p}_k) >= 0$$

since $(\bar{p}_n)_{n=0}^{\infty}$ is decreasing and,

$$\lim_{v->\infty} \phi_k(v) = \bar{p}_k^{-1}\Sigma_{j=k} \bar{p}_j - \bar{p}_{k+1}^{-1}\Sigma_{j=k+1}^{\infty} \bar{p}_j >= 0$$

since $(\bar{p}_k)_{k=0}^{\infty}$ is discrete DMRL.

So we have shown that $(s_k)_{k=0}^{\infty}$ is increasing in K and $(\bar{p}_k)_{k=0}^{\infty}$ discrete DMRL => $(\bar{p}_k \ \Sigma_{j=n} \bar{p}_j/s_j)$ is decreasing in k which implies H is DMRL.

d) To establish the desired result in the NBU case, we recall that the shock interarrival times $(Y_k)_{k=0}^{\infty}$ have independent exponential distributions $(F_k(.))_k$ with mean $1/s_k$. Thus the $(F_k(.))$ are NBU where $k = 0,1,2...$ and further, if $(s_k)_{k=0}^{\infty}$ is increasing the $(F_k(.))_{k=0}^{\infty}$ are decreasing in k. Now,

$$H(t_1+t_2) = \Sigma_{k1=0}^{\infty}\Sigma_{k2=0}^{\infty} P(N(t_1)=k_1) P[N(t_1+t_2)-N(t_1)$$

$$= k_2 \mid N(t_1)=k_1] \bar{p}_{k1+k2}$$

$$=< \Sigma_{k1=0}^{\infty}\Sigma_{k2=0}^{\infty} P(N(t_1)=k_1) P[N(t_1+t_2)-N(t_1)$$

$$= k_2 \mid N(t_1) = k_1] \bar{p}_{k1}\bar{p}_{k2}$$

(since $(\bar{p}_k)_k$ is discrete NBU,)

$= \Sigma_{k1=0}^{\infty}\Sigma_{k2=0}^{\infty} P(N(t_1)=k_1)P_{k1}\ P[N(t_1+t_2)-N(t_1)$

$< k_2 \mid N(t_1)=k_1]p_{k2}$

where $p_{k2} = \bar{p}_{k2-1} - \bar{p}_{k2}$

Since, $\Sigma_{k2=0}^{\infty} P(N(t_1+t_2) - N(t_1) = k_2 \mid N(t_1) = k_1)\bar{p}_{k2}$

$=$ P(survival to time $t_1+t_2 \mid N(t_1) = k_1)$

$= \Sigma_{k2} P(N(t_1+t_2) - N(t_1) < k_2 \mid N(t_1) = k_1\ \bar{p}_{k2}$

i.e. $\bar{H}(t_1+t_2) =< \Sigma_{k1=0}^{\infty}\Sigma_{k2=0}^{\infty} P(N_2(t_1)=k_1)\bar{p}_{k1}$ X

X $P(U_{k1}+U_{k2+1} +\ldots U_{k1+k2} > t_2+t_1-Tk_1)\ \bar{p}_{k2}$

where Tk_1 is the time of the $k_1{}^{th}$ shock

$=< \Sigma k_{1=0}^{\infty}\Sigma k_{2=0}^{\infty} P(N_2(t_1)=k_1\ p_{k1}\ P(>_{j=k1} U_j>t_2)$ X

X $P(\Sigma_{j=k1}^{\infty} U_j > t_1-T_{k1})p_{k2}$

$=< \Sigma_{k1=0}^{\infty}\Sigma_{k2=0}^{\infty} P(N(t_1)=k_1)p_{k1}\ P(\Sigma_{j=k1}^{\infty} U_j > t_2)p_{k2}$

$=< \Sigma_{k1=0} P(N(t_1)=k_1)p_{k1}\ \Sigma_{k2=0} P(\Sigma_{j=0}^{\infty} U_j > t_2)p_{k2}$

(since $(F_k(.))_k$ is decreasing in k)

$= \Sigma_{k1=0}^{\infty} P(N(t_1)=k_1)p_{k1}\ \Sigma_{k2=0}^{\infty} P(N(t_2)<k_2)p_{k2}$

$= \Sigma_{k1=0}^{\infty} P(N(t_1)=k_1)p_{k1}\ \Sigma_{k2=0}^{\infty} P(N(t_2)=k_2)p_{k2}$

$= \bar{H}(t_1)\bar{H}(t_2)$ i.e. H is NBU

e) We first note that by an argument very similar to that used in (c) to prove that $(s_k)_k$ increasing and $(\bar{p}_k)$ discrete DMRL implies $(\bar{p}_k \Sigma_{j=k} p_j/s_j)$ decreasing in k, with $(s_k)_k$ increasing and $(\bar{p}_k)$ discrete NBUE implies :-

(2.1.12) $$\bar{p}_k \Sigma_{j=0}^{\infty} \bar{p}_j/s_j >= \Sigma_{j=k}^{\infty} \bar{p}_j/s_j$$

where $k = 0,1,2...$

we now show that condition (2.1.12) implies that H is NBUE

$$\bar{H}(t) \int_0^{\infty} \bar{H}(u)du = \Sigma_{k=0} z_k(t)\bar{p}_k \Sigma_{j=0}^{\infty} \bar{p}_j/s_j$$

(since $z_j(u)du = 1/s_j$),

$$>= \Sigma_{k=0}^{\infty} z_k(t)\Sigma_{j=k}^{\infty} \bar{p}_j/s_j \qquad \text{by (2.1.12)}$$

$$= \Sigma_{j=0}^{\infty} (\bar{p}_j/s_j) \Sigma_{k=0}^{\infty} z_k(t)$$

$$= \Sigma_{j=0}^{\infty} (\bar{p}_j/s_j) s_j \int_t^{\infty} z_k(u)du$$

(since $\int_t^{\infty} z_k(u)du = 1/s_k \Sigma_{j=0} z_j(t)$)

$$= \bar{H}(u)du$$

i.e. $$\bar{H}(t) \int_0^{\infty} \bar{H}(u)du >= \int_t^{\infty} \bar{H}(u)du$$

i.e. $\bar{H}$ is NBUE

f) We have to show that :-

(2.1.13) $$\Sigma_{j=k}^{\infty} p_j s_j =< a_0 \prod_{j=0}^{k-1} (1-(a_0 s_j)^{-1}) \quad \text{where } k = 1,2,...$$

=> H is HNBUE

where $a_0 = \Sigma_{j=0}^{\infty} \bar{p}_j/s_j$

The following Lemma will prove useful.

<u>Lemma</u> (2.1A.2) (c.f Klefsjo (1983)).

Let $\{N(t)\}$ be a stationary pure birth process with birth coefficients $(s_k)_{k=0}^{\infty}$.

Suppose there exists a j_0 such that $0 < 1/\theta \leq s_j$ for $j \geq j_0$, define,
$E_0 =^{df} 1$ and $E_k =^{df} \prod_{j=0}^{k-1} (1-\theta s_j^{-1})$, where k= 1,2,... and
let $J(t) = \Sigma_{k=0}\, z_k(t)E_k$ where $z_k(t) = P(N(t)=k)$ then,

(2.1.14) $\qquad J(t) = \exp(-t/\theta)$ provided $\Sigma_{k=0}\, 1/s_k = \infty$

Now, $\qquad \int_t^\infty A(u)du = \int_t^\infty \Sigma_{j=0}\, z_k(u)\bar{P}_j du$

$$= \Sigma_{j=0}^{\infty}\, 1/s_j\ (\Sigma_{k=0}^{\infty}\, z_k(t))\bar{p}_j$$

by (2.1.9)

$$= \Sigma_{k=0}^{\infty}\, (\Sigma_{j=k}^{\infty}\, \bar{p}_j/s_j)\ z_k(t)$$

(2.1.15) $\qquad \leq \Sigma_{k=0}^{\infty}\, z_k(t)\ a_0 \prod_{j=0}^{k-1} (1-(a_0 s_j)^{-1})$

by (2.1.13).

(2.1.13) also implies that :-

$$1/a_0 \leq s_k \text{ for every } k \text{ such that } \bar{p}_k > 0.$$

So by Lemma (2.1.2) we have, via (2.1.15) :-

$$\int_t^\infty A(u)du \leq a_0 \exp(-t/a_0)$$

and the proof is completed by noting that :-

$$\mu = \int_0^\infty A(u)du = \Sigma_{k=0} \int_0^\infty z_k(u)du\ \bar{p}_k$$

$$= \Sigma_{k=0}\, \bar{p}_k/s_k = a_0$$

(by definition, c.f. (2.1.13)).

So $\qquad \int_t^\infty \bar{A}(u)du \leq \mu\exp(-t/\mu)$

i.e. $\qquad$ H is HNBUE

g) Recall that $a_0 = \Sigma_{k=0}^{\infty} \bar{p}_k / s_k = \int_0^{\infty} \bar{H}(t)dt$

Now we have to show that :-

$$\int_0^{\infty} H(t)e^{-st}dt >= a_0 / (1+sa_0)$$

$$\int_0^{\infty} H(t)e^{-st}dt = \Sigma_{k=0}^{\infty} \bar{p}_k \int z_k(t)e^{-st}dt$$

(2.1.16) $$= \Sigma_{k=0}^{\infty} \bar{p}_k Q_k(s)$$

where $Q_k(s)$ is the Laplace transform of :-

$$z_k(t) = P(N(t)=k)$$

i.e.

$$Q_k(s) = 1 / s_0+s \qquad \text{for } k = 0$$

$$Q_k(s) = \prod_{j=0}^{k-1}(s_j/s_j .\ ) (1/s_k \ .)$$

Now, since the Laplace transform is scale invariant we can w.l.o.g. take $s_0 = 1$. So,

$$Q_k(s) >= \prod_{j=0}^{k-1}(1/(1+s)) (1/s_k(1+s))$$

(2.1.17) $$= (1/1+s)^{n+1} (1/s_k)$$

since $s_1 >= s_0$, $j = 1,2,...$ and $k = 0,1,2,...$

Thus from (2.1.16) and (2.1.17) we have :-

$$\int_0^{\infty} \bar{H}(t)e^{-st}dt >= 1/(1+s)\Sigma_{k=0} (\bar{p}_k/s_k)(1/(1+s))^k$$

and since $(\tilde{p}_k/s_k)$ belongs to G and $a_0 = \Sigma_{k=0} \bar{p}_k/s_k = \int_0^{\infty} \bar{H}(t)dt$ it follows that,

$$\int_0^{\infty} H(t)e^{-st}dt >= a_0/(1+sa_0)$$

i.e. H belongs to L

It should be noted that in the IFRA, DMRL, NBU, NBUE and L cases in Theorem (2.1A.1) above the relevant results have not been stated under the most general possible condition on $(s_k)_{k=0}^{\infty}$ and $(\bar{p}_k)_{k=0}^{\infty}$.

The conditions used are those which best show the relationship between the class to which $(p_k)_k$ belongs (or $(\bar{p}_k/s_k)_k$ in the L case) and the class of life distribution to which $\bar{H}(t)$ belongs. In the DMRL, NBUE and L cases, weaker conditions on $(\bar{p}_k)_k$ were actually established in the course of the respective proofs. In the NBU case it is the condition on $\{N(t)\}$ that can be weakened. i.e.,Theorem (2.1A1(d)) holds for processes more general than stationary pure birth processes. In the IFRA case, Klefsjo (1980) has shown that the result of Theorem (2.1A.1(b)) holds if the conditions $(s_k)_k$ increasing and P_k discrete IFRA are replaced by :-

(i) for all θ such that $0 < \theta < s_0$ the sequence
$\bar{p}_k - \prod_{j=0}^{k-1} (1-\theta/s_j)$ for $k = 1,2,...$ changes sign at most once and if once from + to - and,

(ii) There exists a j_0 such that $s_j >= s_0$ for $j >= j_0$ Klefsjo has also shown that these conditions are weaker than those of Theorem (2.1A.1).

A companion result to Theorem (2.1A.1) can be established for the dual classes DFR, DFRA, IMRL, NWU, NWUE, HNWUE and L

For completeness, we state the appropriate result below and note that the proof follows by reversing inequalities and direction of monotonicity in Theorem (2.1A.1) except that some care is needed in the HNWUE case and consequently an extra condition is included in the HNWUE part of the Theorem.

Theorem (2.1A.2)

Let $\{N(t)\}$ be a stationary pure birth process with birth coefficients $(s_k)_{k=0}$ then if H(t) is defined by :-

$$\bar{H}(t) = \Sigma_{k=0} P(N(t)=k)\bar{p}_k$$

$$= \Sigma_{k=0} z_k(t)\ \bar{p}_k$$

a) H is DFR if $(s_k)_{k=0}^{\infty}$ decreasing and $(\bar{p}_k)_{k=0}^{\infty}$ is discrete DFR.

b) H is DFRA if $(s_k)_{k=0}^{\infty}$ is decreasing and $(\bar{p}_k)_{k=0}^{\infty}$ is discrete DFRA.

c) H is IMRL if $(s_k)_{k=0}^{\infty}$ is decreasing and $(\bar{p}_k)_{k=0}^{\infty}$ is discrete IMRL.

d) H is NWU if $(s_k)_{k=0}^{\infty}$ is decreasing and $(\bar{p}_k)_{k=0}^{\infty}$ is discrete NWU.

e) H is NWUE if $(s_k)_{k=0}^{\infty}$ is decreasing and $(\bar{p}_k)_{k=0}^{\infty}$ is discrete NWUE with $\mu = \Sigma_{k=0} \bar{P}_k < \infty$.

f) H is HNWUE if $\Sigma_{j=k} P_j/s_j >= a_0 \prod_{j=0}^{k-1} (1-(a_0 s_j)^{-1})$ and $\exists\ k_0$ such that $a_0 >= 1/s_k$ for every $k > k_0$ for which $\bar{P}_k > 0$ where $a_0 = \Sigma_{j=0}^{\infty} \bar{p}_j/s_j$.

g) H belongs to Γ if $s_j =< s_0$, j = 1,2,3..., $(\bar{P}_k/s_k)_{k=0}^{\infty}$ is increasing and in $\bar{G}$.

As in Theorem (2.1.1) the conditions in parts (b), (c), (d), (e) and (g) of the above theorem can be weakened.

~2.1B The Homogeneous Poisson Shock Model

If, in the stationary pure birth process {N(t)} the birth coefficients $s_k \equiv s, k = 0,1,2...$ then {N(t)} is the familiar homogeneous Poisson Process of rate s.

Consequently, the survivor function of a device subject to shocks whose arrival is governed by an homogeneous Poisson Process is given by :-

(2.1.18) $$\bar{H}(t) = \Sigma_{k=0}^{\infty} (e^{-st}(st)^k)/k!\ \bar{P}_k$$

Particularly appealing is the case that $\bar{P}_k = @^k$ where $0 =< @ =< 1$ and $k = 0,1,2,...$

$$\bar{H}(t) = e^{-st(1-@)}$$

Note that if $\bar{P}_k = @^k$ where $0 =< @ =< 1$ and $k = 0,1,2,...$ then $\bar{H}(t)$ is the p.g.f. of P(N(t)=k). It can be shown that H is exponential iff $\bar{P}_k = @^k$ where $0 =< @ =< 1$ and $k = 0,1,2,...$

By differentiating (2.1.18) it can be seen that the density function for the life distribution of an homogeneous Poisson Shock Model is given by :-

(2.1.19) $$h(t) = s\ \Sigma_{k=1}^{\infty} \bar{P}_k (e^{-st}(st)^{n-1}) / (k-1)!$$

where $p_k = \bar{P}_{k-1}\ \bar{P}_k$

If $\bar{P}_k = 1$ where $k = 0,1,...n$

$\bar{P}_. = 0$ where $k > n$

then (2.1.19) is the density of a gamma distribution of order n+1.

Our interest in studying the homogeneous Poisson Shock Model is to determine sufficient conditions for the life distribution H to belong to one of the classes discussed in Chapter one. To this end, we obtain the following corollary to Theorems (2.1A.1) and (2.1A.1) simply by setting $s_k = s$ where $k = 0,1,2,...$

<u>Corollary (2.1B.1)</u>

Let {N(t)} be a homogeneous Poisson Process of rate s. Then if H(t) is defined by :-

$$\bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k)\bar{p}_k$$

$$= \Sigma_{k=0}^{\infty} (e^{-st}(st)^k) / k!\ \bar{p}_k$$

a) H is IFR (DFR) if $(\bar{p}_k)_{k=0}^{\infty}$ is discrete IFR (DFR).

b) H is IFRA (DFRA) if $(\bar{p}_k)_{k=0}^{\infty}$ is discrete IFRA (DFRA).

c) H is DMRL (IMRL) if $(\bar{p}_k)_{k=0}^{\infty}$ is discrete NBU (NWU).

d) H is NBU (NWU) if $(\bar{p}_k)_{k=0}^{\infty}$ is discrete NBU (NWU).

e) H is NBUE (NWUE) if $(\bar{p}_k)_{k=0}^{\infty}$ is discrete NBUE (NWUE).

f) H is HNBUE (HNWUE) if $(\bar{p}_k)_{k=0}^{\infty}$ is discrete HNBUE (HNWUE).

g) H belongs to L ($\bar{L}$) if $(\bar{p}_k)_{k=0}^{\infty}$ is in G ($\bar{G}$).

Although we have obtained this result as a corollary of the corresponding results for the stationary pure birth shock model it should be noted that Esary, Marshall and Proschan (1973) established parts (a) to (e) of corollary (2.1B.1) directly using, in the main, the total positivity of the kernel :-

$$z_k(t) = \frac{e^{-st}(st)^k}{k!}$$

and Klefsjo (1981) obtained part (f) also without reference to the stationary pure birth shock model.

One possible application of the homogeneous Poisson Shock Model, proposed by Barlow (1985) is in the area of Software failure in Computer Science. Let N^* be the number of distinct types of input to the software and let N be the number of input types that result in failure. N^* is assumed to be large relative to N.

It is also assumed that processing of an input is instantaneous and that once failure occurs, errors in the logic or coding of the program are corrected with the result that N is reduced by one. If inputs to the software occur according to a homogeneous Poisson process of rate s then the probability that no software failures occur in (0,t) is given by :-

(2.11.20) $$\bar{H}(t) = \Sigma_{k=0}\left((e^{-st}(st)^k) / k!\right)((N^*-N)/N^*)^k$$

which is of the form (2.1.18) with :-

$$\bar{p}_k = ((N^* - N)/N^*)^k$$

Of course, since $\bar{p}_k$ is of the form $\bar{p}_k = @^k$, where $0 =< @ = (N^*-N)/N^* < 1$ the survivor function in (2.1.20) is that of an exponential distribution.

2.1C The Non-Stationary Pure-birth Shock Model

The Non-Stationary Pure birth Shock Model is a generalisation of the stationary pure birth model considered in section 2.1A. In this model shocks are assumed to arrive at a device in accordance with a Markhov Process, {N(t)} with the following transition probabilities :-

(2.1.21) $$P(N(t+d)-N(t) = 1 \mid N(t)=k) = s_k s(t)d + o(d)$$
for sufficiently small d

(2.1.22) $$P(N(t+d)-N(t) > 1 \mid N(t)=k) = o(d)$$
for sufficiently small d

Clearly if $s(t) = 1$ and $t > 0$ then the non-stationary pure birth process defined by (2.1.18) and (2.1.19) reduces to the stationary pure birth process while if $s_k = s$ and $k = 0,1,2,\ldots$ then (2.1.21) and (2.1.22) define an inhomogeneous (or non-stationary) Poisson process.

The non-stationary pure birth shock model was introduced by A-Hameed and Proschan (1975) and further studied by Klefsjo (1981). It should be noted that the non-stationary pure birth process can be obtained from the stationary pure birth process via the transformation $t \to I(t)$ where $I(t) = \int_0^t s(x)dx$ This follows since if $N^*(t)$ is a stationary pure birth process then by (2.1.1) :-

$$P(N^*(I(t)+d) - N^*(I(t)) \mid k \text{ shocks in } (0,I(t))$$

$$= s_k I'(t)d + o(d)$$

$$= s_k s(t)d + o(d)$$

the transition probability for a non-stationary pure birth process.

Now, the survivor function of a device subject to shocks whose arrival is governed by a non-stationary pure-birth process $\{N(t)\}$ is given by :-

$$\text{(2.1.23)} \qquad \bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k)\bar{P}_k$$

$$\text{(2.1.24)} \qquad = \Sigma_{k=0}^{\infty} P(N^*(I(t)=k)\bar{P}_k$$

where $\{N(t)\}$ is a stationary pure birth process. So :-

$$\text{(2.1.25)} \qquad \bar{H}(t) = \bar{H}^*(I(t))$$

where $\bar{H}^*(.)$ is the survivor function of the stationary pure birth model. This relationship (2.1.25) together with the following lemmas provide a straightforward method of establishing a result corresponding to Theorem (2.1.1) for the non-stationary pure birth shock model.

Lemma (2.1C.1)

Let $K(t) = F(G(t))$ then :-

a) if F is increasing and F and G are both convex (concave) then K is convex (concave)

b) if F is increasing and F and G are both starshaped (antistarshaped) i.e. (F(t)/t increasing (decreasing) and G(t)/t increasing (decreasing)).

c) if F is increasing and F and G are both superadditive (subadditive) then K is superadditive (subadditive).

Lemma (2.1C.2)

Let $K(t) = F(G(t))$ where F and K are lifetime distributions then :-

(a) if G is increasing and convex (concave)and F is DMRL (IMRL) then K is DMRL (IMRL).

(b) if G is increasing and starshaped (antistarshaped) and F is NBUE (NWUE) then K is NBUE (NWUE).

(c) if G is starshaped (antistarshaped) and F is HNBUE (HNWUE) then K is HNBUE (HNWUE).

d) if G is starshaped (anti-starshaped) and F belongs to L ($\bar{L}$) then K belongs to L ($\bar{L}$).

Lemma (2.1C.1) and parts (a) and (b) of Lemma (2.1C.2) are due to A-Hameed and Proschan (1975) while parts (c) and (d) of Lemma (2.1C.2) are due to Klefsjo (1981, 1983).

Using the above Lemmas and the relationship (2.1.15) the next result follows almost immediately.

Theorem (2.1C.1)

Let {N(t)} be a non-stationary pure birth process as defined by (2.1.21) and (2.1.22) and let :-

$$\bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k)\bar{p}_k$$

is a decreasing sequence of survival probabilities then :-

a) H is IFR if $(s_k)_{k=0}^{\infty}$ is increasing, s(t) increasing and $(\bar{p}_k)_{k=0}^{\infty}$ is discrete IFR

b) H is IFRA if $I(t) = \int_0^t s(x)dx$ is starshaped (i.e. I(t) increasing in t), $(s_k)_{k=0}^{\infty}$ increasing and $(\bar{p}_k)$ discrete IFRA.

c) H is DMRL if $(s_k)_{k=0}^{\infty}$ is increasing s(t) is increasing and $(\bar{p}_k)_{k=0}^{\infty}$ discrete DMRL.

d) H is NBU if $(s_k)_{k=0}^{\infty}$ is increasing I(t) is superadditive and $(\bar{p}_k)_{k=0}^{\infty}$ is discrete NBU.

e) H is NBUE if $(s_k)_{k=0}^{\infty}$ is increasing I(t) is starshaped and $(\bar{p}_k)_{k=0}^{\infty}$ is discrete NBUE.

f) H is HNBUE if $\Sigma_{j=k}^{\infty} P_j/s_j =< a_0 \prod_{j=0}^{k-1} (1-(a_0 s_j)^{-1})$ where k = 1,2,... and $a_0 = \Sigma_{j=k}^{\infty} P_j/s_j$ and is starshaped.

g) H belongs to L if $s_k >= s_0$ k = 1,2,3,... $(\bar{p}_k/s_k)_{k=0}^{\infty}$ is decreasing and in G and I(t) is starshaped (i.e. I(t)/t increasing)

Proof

For parts (d) - (g) the Theorem follows directly from Theorem (2.1A.1) and Lemma (2.1C.2).

To establish part (a) one must recall that H is IFR <==> - log $\bar{H}(t)$ is convex. The result then follows from Lemma (2.1.2(a)) and Theorem (2.1A.1). In part (b) the result follows from the fact that the IFRA property is characterised by (- log $\bar{H}(t)$)/t increasing in t, and Lemma (2.1C.1(b)) together with Theorem (2.1A.1(b)).

The NBU result follows from the fact that H is NBU <==> - log $\bar{H}(t)$ is superadditive (c.f. (1.4.3) together with Lemma (2.1C.1) and Theorem (2.1A.1(c)).

As in Theorem (2.1.1) it is possible to weaken the condition on $(s_k)_{k=0}^{\infty}$ and on $(\bar{p}_k)_{k=0}^{\infty}$ in the IFRA, NRU, NBUE and L cases. In the NBU case, Theorem (2.1C.1) holds for more general {N(t)} than non-stationary pure-birth processes (see section 2.2).

For the dual classes DFR, DFRA, IMRL, NWU, NWUE, HNWUE and $\bar{L}$ a result exactly analagous to Theorem (2.1C.1) holds and can be established by a straight forward application of Lemmas (2.1C.1) and (2.1C.2) and Theorem (2.1A.2).

~2.1D The Non-Stationary Poisson Shock Model

If in the non-stationary pure birth model just considered we set $s_k \equiv s$ then the stochastic process $\{N(t)\}$ just defined is a non-stationary Poisson process with mean value :-

$$I(t) = s \int_0^t s(x)dx \quad \text{and,}$$

$$P(N(t)=k) = (e^{-I(t)} (I(t))^k) / k!$$

Consequently, the survivor function of a device subject to shocks which arrive in accordance with a non-stationary Poisson process is given by :-

$$(2.1.26) \quad H(t) = \Sigma_{k=0} \left((e^{-I(t)})(I(t))^k) / k! \right) \bar{P}_k$$

$$\text{where } , \bar{P}_0 = 1 >= \bar{P}_1 >= \bar{P}_2$$

We will refer to H(t) as defined by (2.1.26) as the survivor function of the non-stationary Poisson Shock Model. Since the non-stationary Poisson process can be obtained from the non-stationary pure birth process simply by setting $s_k = s > 0$ it is straight forward to obtain sufficient conditions for H(t) as defined by (2.1.23) to belong to one of the classes of life distribution discussed in Chapter one. Consequently, we have the following corollary to Theorem (2.1C.1)

Corollary (2.1D.1)

Let $\{N(t)\}$ be a non-stationary Poisson Process with mean value function I(t) and define H(t) by (2.1.26) then

a) H is IFR (DFR) if $s(t) = I'(t)$ is increasing (decreasing) and $(\bar{P}_k)_{k=0}^{\infty}$ is discrete IFR (DFR).

b) H is IFRA if $I(t)/t$ is increasing (decreasing) and $(\bar{P}_k)_{k=0}^{\infty}$s is discrete IFRA (DFRA).

c) H is DMRL (IMRL) if $s(t) = I'(t)$ is increasing (decreasing) and $(\bar{P}_k)_{k=0}^{\infty}$ is discrete DMRL (IMRL)

d) H is NBU (NWU) if I(t) is superadditive (subadditive) and $(\bar{P}_k)_{k=0}^{\infty}$ is discrete NBU (NWU).

e) H is NBUE (NWUE) if I(t)/t is increasing (decreasing) and $(\bar{P}_k)_{k=0}^{\infty}$ is discrete NBUE (NWUE)

f) H is HNBUE (HNWUE) if I(t)/t is increasing (decreasing) and $(\bar{P}_k)_{k=0}^{\infty}$ discrete HNBUE (HNWUE).

g) H belongs to L ($\bar{L}$) if I(t)/t is increasing (decreasing) and $(\bar{P}_k)_{k=0}^{\infty}$ belongs to G ($\bar{G}$)

It should be noted that in each part of the above corollary the condition required by $(\bar{P}_k)_{k=0}^{\infty}$ is the discrete analog of the condition desired for H and in the IFR, IFRA and NBU cases, the condition imposed on I(t) implies that $e^{-I(t)}$ is respectively IFR, IFRA or NBU. In all the other cases, however, the condition on $e^{-I(t)}$ is stronger than that required for $(\bar{P}_k)_{k=0}^{\infty}$ and concluded to hold for H. Whether it is possible to weaken the condition on $e^{-I(t)}$ appears to be an open question at the moment.

The question is complicated, somewhat, by the fact that if the arrival of shocks is governed by a non-stationary Poisson process the intervals between the shocks are not independent (c.f. Cox and Isham (1950) p. 48). As a consequence of this, it is not possible to answer this question by using the results of the next section where the arrival of shocks is assumed to be governed by a generalised renewal process in which the interarrival times are independent and have distribution belonging to the same class but are not necessarily identical. Of course, a similar question to the one just raised arises in considering the non-stationary pure birth shock model discussed earlier.

We note also that just as theorem (2.1C.1) was obtained from the corresponding result for the stationary pure birth shock model by using the transformation t -> I(t), Corollary (2.1D.1) can be obtained from Corollary (2.1B.1), the corresponding result for the homogeneous Poisson Shock Model, using the same technique. This was the approach adopted by A-hameed and Proschan (1973).

~2.1E Doubly Stochastic Poisson Shock Models

In their discussion of non-stationary Poisson Shock Models, A-hameed and Proschan (1973) note that the rate, s(t), at which shocks arrive at a device may well be subject to random variation. This suggests that in many practical situations it may be appropriate to regard s(t) as a realisation of a Stochastic process, {S(t)} say so that for each t, s(t) is a realisation of a random variable.

A stochastic process with a rate function which is itself a stochastic processis referred to as a doubly stochastic process. In particular, given a realisation of the rate process S(.) the Point process {N(t)} is a non-stationary Poisson process so that :-

(2.1.27) $$P(N(t) = k \mid \{S(x) = s(x)\}_0^t)$$

$$= \exp\left(-\int_0^t s(x)dx\right) \left(\left(\int_0^t s(x)dx\right)^k\right) / k!$$

{N(t)} is said to be a doubly stochastic Poisson process.

Note that we use the notation $\{S(x) = s(x)\}_0^t$ to represent the event that S(x) = s(x) over the interval (0,t) i.e. s(.) is a sample path of S(.) in the interval (0,t).

Of course,our interest centers on the unconditional probability that a device subject to shocks governed by a double stochastic Poisson process, {N(t)} survives for a time t. Let T denote the lifetime of the device, then :-

(2.1.28) $$P(T>t) \mid \{S(x)=s(x)\}_0^t) =$$
$$\Sigma_{k=0}^{\infty} \exp\left(-\int_0^t s(x)dx\right) \left(\left(\int_0^t s(x)dx\right)^k\right) / k!\ \bar{P}_k$$

(2.1.28)

$$= \bar{F}_{s(.)}, \text{ say}$$

and the unconditional survivor function is given by:-

$$H(t) = P(T>t) = E_{S(x)} (P(T>t \mid \{S(x)\}_0^t)$$

$$(2.1.29) \quad = E_{\{S(x)\}} \Sigma_{k=0}^{\infty} \left(\exp\left(-\int_0^t s(x)dx\right) \left(\left(\int_0^t s(x)dx\right)^k\right) / k!\right) \bar{P}_k$$

$$(2.1.30) \quad = E\left(\Sigma_{k=0} \left(\exp\left(-\int_0^t S(x)dx\right) \left(\left(\int_0^t S(x)dx\right)^k\right) / k!\right) P_k\right)$$

where $E_{\{S(x)\}}(Z)$ is the expectation of Z with respect to $\{S(x)\}$ over the range $(0,t)$.

The use of the expectation operator in (2.1.29) and (2.1.30) is equivalent to "integrating" over all possible sample paths of the rate process, $\{S(.)\}$. Hence, (2.1.29) and (2.1.30) are really mixtures of the conditional survivor functions $F_{s(.)}$ defined in (2.1.28).

This fact, together with the closure properties of the DFR, DFRA, IMRL, NWU, NWUE, HNWUE and L classes under mixture c.f. (1.8) leads to the following corollary to Theorem (2.1C.1). (In fact, the result is a further corollary to corollary (2.1D.1)).

<u>Corollary (2.1E.1)</u>

Let a life distribution H be defined by (2.1.29) or (2.1.30) then,

a) H is DFR if $(\bar{P}_k)_{k=0}^{\infty}$ is discrete DFR and for every realisation $s(t)$ of $\{S(t)\}$ $s(t)$ is decreasing in t.

b) H is DFRA if $(\bar{P}_k)_{k=0}^{\infty}$ is discrete DFRA and for every realisation $s(t)$ of the rate process $\{S(t)\}$ $\left(\int_0^t s(x)dx\right)/t$ is decreasing in t.

c) H is IMRL if $(\bar{P}_k)_{k=0}^{\infty}$ is discrete IMRL and for every realisation $s(t)$ of the rate process $\{S(t)\}$, $s(t)$ is decreasing.

d) H is NWU if $(\bar{P}_k)_{k=0}^{\infty}$ is discrete NWU, and for every realisation $s(t)$ of $\{S(t)\}$, $I(t) = \int_0^t s(x)dx$ is sub-additive and none of the conditional survivor functions
$F_{s(.)}(t) = P(T>t \mid \{S(x)=s(x)\}_0^t)$ cross.

e) H is NWUE if $(\bar{p}_k)_{k=0}$ is discrete NWUE, for every realisation s(t) of {S(t)}, $(\int_0^t s(x)dx)/t$ is decreasing in t and none of the conditional survivor functions $F_{s(.)}(t) = P(T>t \mid \{S(x)=s(x)\}_0^t)$ cross.

f) H is HNWUE if $(\bar{p}_k)_{k=0}^{\infty}$ is discrete HNWUE and for every realisation s(.) of {S(t)}, $(\int_0^t s(x)dx)/t$ is decreasing in t.

g) H belongs to Γ if $(\bar{p}_k)_{k=0}$ belongs to $\bar{G}$ and for every realisation s(t) of {S(t)}, $(\int_0^t s(x)dx)/t$ is decreasing in t.

Proof

The proof follows directly from applying the mixture results of section 1.8 to corollary (2.10.1).

Unfortunately, since none of the classes with "positive ageing" (i.e. IFR - L) are closed under mixture, a similar technique cannot be used toestablish a corresponding result for these classes.

A more explicit result than corollary (2.1E.1) can be obtained if the form of the rate process {N(t)} or, alternatively, the form of the random function $I(t) = \int_0^t S(x)dx$ is specified.

A-Hameed and Proschan (1973) suggest that a plausible model for the (random) mean-value function I(t) is :-

(2.1.31) $$I(t) = \int_0^t S(x)dx = YM(t)$$

where Y is a random variable and M is a continuous deterministic function of the age of the device. In this case the survivor function is given by :-

(2.1.32) $$\bar{H}(t) = E\Sigma_{k=0}^{\infty}(e^{-YM(t)}((YM(t))^k) / k!)P_k$$

and from corollary (2.1D.1) the following result is immediate.

Corollary (2.1E.2)

Let $\{N(t)\}$ be a doubly stochastic Poisson process with a random mean value function given by (2.1.21) then if H is defined by (2.1.32)

(a) H is DFR if $(\bar{p}_k)_{k=0}^{\infty}$ is discrete DFR and $m(t) = M'(t)$ is decreasing in t.

(b) H is DFRA if $(\bar{p}_k)_{k=0}^{\infty}$ is discrete DFRA and $M(t)/t$ is decreasing in t.

(c) H is IMRL if $(\bar{p}_k)_{k=0}^{\infty}$ is discrete IMRL and $m(t) = M'(t)$ is decreasing in t.

(d) H is NWU if $(\bar{p}_k)_{k=0}^{\infty}$ is dicrete NWU, $M(t)$ is subadditive and none of the survivor functions

$$F_Y(t) = \Sigma_{k=0}\left(e^{-yM(t)}\ ((yM(t))^k)\ /\ k!\right) P_k \quad \text{cross.}$$

(e) H is NWUE if $(\bar{p}_k)_{k=0}^{\infty}$ is discrete NWUE $M(t)/t$ is decreasing in t and none of the survivor functions

$$F_Y(t) = \Sigma_{k=0}\left(e^{-yM(t)}\ ((yM(t))^k)\ /\ k!\right) P_k \quad \text{cross.}$$

(f) H is HNWUE if $(\bar{p}_k)_{k=0}^{\infty}$ is discrete HNWUE and $M(t)/t$ is decreasing in t.

(g) H belongs to Γ if $(\bar{p}_k)_{k=0}$ belongs to $\bar{G}$ and $M(t)/t$ decreasing in t.

Another possible model for the rate process {N(t)}, is to assume that {N(t)} is an alternating renewal process of the following kind :-

S(t) alternates randomly between two values s_1 and s_2 so that a sample path of {N(t)} would look like :-

```
          ^
   S(t)   |
          |
   s2     |        ---------o        --------o
          |
          |
          |
   s1     |--------o        --------o
          |
          |
          |
          |
        --|--------------------------------------> t
          |
```

The lengths of the intervals in which S(t) = s_1 or s_2 are random variables and we will suppose that they have density functions a(.) and B(.) respectively. We will call an interval in which S(t) = s_1 a type 1 interval and an interval in which S(t) = s_2 a type 2 interval. When S(t) = s_1 we will say that the system (the device plus its operating environment in state 1 and when S(t) = s_2 will say the system is in state 2. All intervals are assumed to be mutually independent.

Gaver (1963) used such a process, {S(t)} to describe a randomly alternating operating environment in which the failure rate of a device's life distribution is itself a random variable. In particular, if either of s_1 or s_2 is equal to zero the process {S(t)} is a good model of the failure rate for a situation in which a device is alternately in and out of use; when out of use (e.g. while undergoing repair) the failure rate is zero, and when in use the device's failure rate is a constant, $s_1 = s$.

A similar interpretation can be given to {S(t)} in the current context. S(t) is the rate at which shocks arrive at a device and this rate may alternate between two values depending on random fluctuations in the level or intensity of operation of the device, e.g., when not in use the device may be stored safely and not subject to shocks so that s_1, say, is equal to zero, but when in use shocks may occur at a constant rate s_2.

Similarly, the shock arrival rate may alternate between two non-zero values corresponding to two speeds or intensities of the device's operation, e.g., the faster a device works the more prone it may be to shocks.

In order to evaluate :-

$$(2.1.33) \qquad A(t) = E(\Sigma_{k=0}(e^{-\int S(x)dx}((\int S(x)dx)^k)/k!)P_k)$$

where $\{S(t)\}$ is an alternating renewal process as discussed above, an initial condition must be specified. There are two possibilities.

(1) at time t=0 a type 1 interval commences; or

(2) at time t=0 a type 2 interval commences.

Note that the points at which $\{S(t)\}$ has a jump constitute a point process and we are assuming here that this process has a point at the origin, i.e., the point process is synchronous. Looking firstly at case (1) :-

If at time t=0 a type one interval commences then for any interval $(0,t')$ either :-

(i) the same number of type one and two intervals commence in $(0,t')$ and t' is covered by a type 2 interval, or

(ii) the number of type 2 intervals commencing in $(0,t')$ is one less than the number of type one intervals and t' is covered by a type 1 interval.

The probability of the event (i) above is given by :-

$\Sigma_{m=1}^{\infty} \int_0^t$ P(m type 1 intervals commence in (0,t),
m type 2 intervals commence in (0,t),
Total length of type 2 intervals in (0,t) is (t-x),
with t covered by a type 2 interval) dx

= $\Sigma_{m=1} \int_0^t$ P(m type 1 intervals commence in (0,t),
m type 2 intervals commence in (0,t),
Total length of type 2 intervals in (0,t) is (t-x),
with t covered by a type 2 interval) dx

= $\Sigma_{m=1} \int_0^t a^{(m)}(x)$ P(m type 2 intervals commence in (0,t),
Total length of type 2 intervals in (0,t) is (t-x),
with t covered by a type 2 interval) dx

(where a(m)(.) is the m-fold
convolution of a(.) with itself)

$$= \Sigma_{m=1} \int_0^t a^{(m)}(x) \int B^{(m-1)}(v) \int_{t-x-v} B(w)\ dwdvdx$$

$$= \Sigma_{m=1} \int_0^t a^{(m)}(x) \int Q_m(t-x)\ dx$$

where $Q_m(t-x) = \int_0^{t-x} B^{(m-1)}(v) \int_{t-x-v}^{\infty} B(w)\ dwdv$

and $B^{(m)}(.)$ is the m-fold convolution of B(.) with itself.

Similarly, the probability of (ii) is given by :-

$\Sigma_{m=1}^{\infty} \int_0^t$ P(m type 1 intervals commence in (0,t),
m type 2 intervals commence in (0,t),
Total length of type 2 interval in (0,t) is
(t-x), with t covered by a type 1 interval) dx

$$= \Sigma_{m=1} \int_0^t B_{m-1}(t-x) L_m(x) dx$$

where $L_m(x) = \int_0^x \int_{x-v}^{\infty} a^{(m-1)}(v) a(w) dwdv.$

Now for a given x = total length of type 1 intervals in $(0,t)$:-

$$\int_0^t S(T)dT = s_1 x + s_2(t-x)$$

and hence conditional on the first interval being of type one we have :-

$$E\ (e^{-\int_0^t B(x)dx}\ (\int_0^t S(x)dx)^k)\ /\ k!)$$

$$= \Sigma_{m=1}^{\infty} \int_0^t e^{-m_1 x + m_2(t-x)} (s_1 x + s_2(t-x))^k\ /\ k!$$

$$X\quad (L_m(x) B^{(m-1)}(t-x) + Q_m(t-x) a^{(m)}(x))\ dx$$

Consequently, the survivor function is given by :-

(2.1.34)

$$H_1(t) = \Sigma_{k=0}^{\infty} \Sigma_{m=1}^{\infty} \int_0^t e^{-m_1 x + m_2(t-x)} (s_1 x + s_2(t-x))^k\ /\ k!$$

$$X\quad (L_m(x) B^{(m-1)}(t-x) + Q_m(t-x) a^{(m)}(x)) \bar{p}_k\ dx$$

(2.1.35) $$= \Sigma_{k=0}^{\infty} \int \Sigma_{m=1}^{\infty}\ e^{-m_1 x + m_2(t-x)} (s_1 x + s_2(t-x))^k\ /\ k!\ \bar{p}_k$$

$$X\quad (L_m(x) B^{(m-1)}(t-x) + Q_m(t-x) a^{(m)}(x))\ dx$$

The combined effect of integrating over x and summing over m in (2.1.35) is to integrate over all possible sample paths of $\{S(.)\}$ over the interval $(0,t)$.

A similar argument to the one above leads to the following expression for the survivor function given that the first interval is of type 2:-

(2.1.36) $$H_2(t) = \Sigma_{m=1}^{\infty} \int_0^t \Sigma_{k=0}^{\infty}\ e^{-m_1 x + m_2(t-x)} \left((s_1 x + s_2(t-x))^k\ /\ k!\right) \bar{p}_k$$

$$X\quad [B^{(m)}(t-x) L_m(x) + a^{(m-1)}(x) Q_m(t-x)]dx$$

Now, from (2.1.35) and (2.1.36) it is clear that whether the first interval is of type one or two :-

$$\bar{H}_i(t) = E\ [\bar{G}_{B(.)}(t)] \qquad i = 1,2,...$$

where $$\bar{G}_{B(.)}(t) = \Sigma_{k=0}^{\infty}\ e^{-\int_0^t B(x)dx}\ (\int_0^t S(x)dx)^k\ /\ k!\ \bar{p}_k$$

Note that $\overline{G}_{s}(.)$ is just the survivor function for the inhomogeneous Poisson Shock Model with mean value function $I(t) = \int_0^t S(t)dt$ i.e. the survivor function (2.1.35) or (2.1.36) is the mixture of survivor functions of inhomogeneous Poisson Shock Models.

For a particular realistion, s(.), of S(.) :-

$I(t) = s_1 x + s_2(t-x)$ where x is the (given) total length of type one intervals in $(0,t)$ Since $I'(t)$ is constant in t and the DFR and IMRL Classes are closed under mixture we have by corollary (2.1D.1) or corollary (2.1E.1.)

Corollary (2.1E.3).

Let {N(t)} be a doubly stochastic Poisson process with an alternating renewal process {S(.)} as the rate process then if H is defined by (2.1.35) or (2.1.36) we have :-

a) H is DFR if $(\overline{P}_k)_{k=0}^{\infty}$ is discrete DFR

b) H is IMRL if $(\overline{P}_k)_{k=0}^{\infty}$ is discrete IMRL.

A similar result does not in general hold for the classes DFRA, NWU, NWUE, HNWUE and $\underline{\Gamma}$ since $I(t)/t$ is not necessarily decreasing, nor is $I(t)$ necessarily subadditive.

Before studying a special case of the model (2.1.33) we note that the model can be generalised by allowing the rate process {S(.)} to alternate between more than two values. e.g. {S(.)} may alternate between 3 values s_1, s_2 and s_3 in the following manner :-

$$s_1 \rightarrow s_2 \rightarrow s_3 \rightarrow s_1.$$

Such a process is referred to as a cyclic renewal process (c.f. Wang Ann Lee (1975)). The three possible values of S(.) may correspond to three speeds or intensities of the device's operation so that s_i, i = 1,2,3... is the rate at which shocks occur when the device is operating at speed i. In essence, the argument used to investigate :-

$$A(t) = E\left(\Sigma_{k=1}\left(e^{-\int_0^t S(x)dx}\left(\left(\int_0^t S(x)dx\right)^k\right) / k!\right)\bar{P}_k\right)$$

is the same whether S(t) is 2,3,or n valued. First, an initial condition must be specified, e.g., if S(t) is 3 valued the first interval may be of type one, two or three (where an interval is of type i if S(t) = s_i in that interval). Suppose the first interval is of type one then for any interval (0,t) there are three rather than two mutually exclusive possibilities; (i) The same number of type one, type two and type three intervals commence in the interval (0,t) and t is covered by a type 3 interval.

(ii) The same number of type one, and type two intervals commence in (0,t) and one less type three interval commences in (0,t) and t is covered by a type two interval, or

(iii) The same number of type 2 and 3 intervals commence in (0,t) one more type 1 interval commences in (0,t) and type t is covered by a type one interval. Now, by an analysis similar to that used in the two-state case, we have the following expression for the probability of the first of these events :-

$$\Sigma_{m=1}^{\infty}\int_0^t\int_0^{t-x_1} a^{m}(x_1)B^{(m)}(x_2)\ R_m(t-x_1-x_2)\ dx_2dx_1$$

where $$R_m(x) = \int_0^x h^{(m-1)}(v)\int_{x-v}^{\infty} h(w)dwdv$$

and h(.) is the density function of the type three intervals.

All other notation is as in the 2-step case.

Similarly the probability of the second event is given by:-

$$\Sigma_{m=1}^{\infty}\int_0^t\int_0^{t-x_1} a^{(m)}(x_1)h^{(m-1)}(t-x_1-x_2)\ Q_m(x_2)dx_2dx_1$$

and the third probability is given by :-

$$\Sigma_{m=1}^{\infty}\int_0^t\int_0^{t-x_1} B^{(m-1)}(x_2)\ h^{(m-1)}(t-x_1-x_2)\ L_m(x_1)\ dx_2dx_1$$

combining these three probabilities we have the following expression for the for the survivor function conditional on the initial interval being of type 1 :-

$$A_1(t) = E\ (\Sigma_{k=0}^{\infty}\ e^{-\int_0^t s(t)dt}\ ((\int_0^t S(T)dT)^k)\ /\ k!\ \bar{p}_k]$$

(2.1.37) $$= \Sigma_{m=1}\int_0^t\int_0^{t-x_1}\Sigma_{k=0}\ e^{-(s_1x + s_2x + s_3(t-x_1-x_2))}\ X$$

$$(s_1x_1+s_2x_2+s_3(t-x_1-x_2))^k/k!\ \bar{p}_k\ X$$

$$[a^{(m)}(x_1)B^{(m)}(x_2)R_m(t-x_1-x_2)\ +$$

$$a^{(m)}(x_1)\ h^{(m-1)}(t-x_1-x_2)Q_m(x_2)\ +$$

$$B^{(m-1)}(x_2)h^{(m-1)}(t-x_1-x_2)L_m(x_1)]\ dx_2dx_1$$

Of course, similar results can be obtained if the first interval is of type two or three. Moreover, if {S(.)} is an n-step cyclic process so that S(t) alternates between n levels in the order $s_1 \to s_2 \to s_3 \to \ldots \to s_n \to s_1$ and $a_i(.)$ is the density of the i^{th} type of interval and,

$$L_m^i(x) = \int_0^x a_i^{(m-1)}(v)\int_{x-v}^{\infty} a_i(w)\ dwdv$$

we have :-

(2.1.38) $$\overline{H}_1(t) = \Sigma_{m=1}^{\infty} \int_0^t \int_0^{t-x_1} \dots \int_0^{t-\Sigma_{i=1}^{n-1}} \Sigma_{k=0}^{\infty} ($$

$$\exp[-s_1x_1+s_2x_2+\dots+s_{n-1}x_{n-1} +$$

$$+ s_n(t - \Sigma_{i=1}^{n} x_i)] / k!) X$$

$$-[s_1x_1+s_2x_2+\dots s_{n-1}x_{n-1}+s_n(t-\Sigma_{i=1} x_i)]^k/k! \ \bar{p}_k$$

$$[a_1^{(m)}(x_1) \ X \ \dots \ X \ a_{(n-1)}(x_{n-1})L_m^n(t-\Sigma_{i=1}x_i) +$$

$$a_1^{(m)}(x_1) \ X \ \dots \ X \ a_{(n-2)}a_n^{(m-1)}x_nL_m^{n-1}(x_{n-1}) +$$

$$a_2^{(m-1)}(x_2) \ X \ \dots \ X \ a_n^{(m-1)}(t-\Sigma_{i=1}^{n-1}x_i)L_m^1(x_1) \ X$$

$$dx_{n-1}dx_{n-2} \dots dx_1.$$

Despite the complexity of expressions (2.1.37) and (2.1.38) the crucial thing to note is that they are still mixtures of survivor functions of inhomogeneous Poisson Shock Models. Consequently, corollary (2.1D.3) generalises to give :-

Corollary (2.1D.4)

Let {N(t)} be a doubly stochastic process with an n-step cyclic process, {S(.)} governing the rate, then if H is defined by (2.1.38) :-

(a) H is DFR if $(\bar{p}_k)_{k=0}^{\infty}$ is discrete DFR.

(b) H is IMRL if $(\bar{p}_k)_{k=0}^{\infty}$ is discrete IMRL.

We now return to the two-state case, i.e., the case where the shock arrival rate alternates between two values and obtain a more explicit expression for the survivor functions (2.1.35) and (2.1.36) in the special case that the shock survival probabilities are those of a geometric distribution (2.1.39) :-

(2.1.39) $\bar{p}_k = u^k$ $\qquad 0 =< u =< 1, \qquad k = 1,2,\dots.$

We will also assume that the type one and two intervals of the rate process are exponentially distributed with mean $1/\mu_1$ and $1/\mu_2$ respectively.

If the first interval is of type 1 we have from (2.1.34)

$$A_1(t) = \Sigma_{k=0}^{\infty} \Sigma_{m=1}^{\infty} \int_0^t e^{-s_1 x+s_2(t-x)} \frac{(s_1 x+s_2(t-x))^k}{k!} \; X$$

$$L_m(x) B^{m-1}(t-x) + Q_m(t-x) a^{(m)}(x) dx \; \bar{p}_k$$

Now if $\bar{p}_k = u^k$ where $k = 0,1,2,\ldots$ then :-

$$(2.1.40) \qquad \bar{A}_1(t) = \Sigma_{m=1} \int_0^t \exp[-(s_1 x+s_2(t-x))(1-u)] \; X$$

$$L_m(x) B^{m-1}(t-x) + Q_m(t-x) a^{(m)}(x) dx \; \bar{p}_k$$

Now $\quad \Sigma_{k=0} P(N(t)=k) u^k$

$0 =< u =< 1$ is just the probability generating function (p.g.f.) of N(t), hence $\bar{A}_1(t)$ in (2.1.40) is the p.g.f. of N(t) where {N(t)} is a doubly stochastic Poisson process with an alternating renewal process governing the rate. Srinivasan (1979) gives the Laplace transform for the p.g.f. of such a process and hence for $\bar{A}_1(t)$. Thus we have :-

$$A_1^*(s) = \int_0^\infty e^{-wt} \bar{A}_1(t) dt =$$

$$(2.1.41) \qquad 1 \,/\, (1- a^*(w+s_1-s_1u)\, B^*(w+s_2-s_2u)) \quad X$$

$$\frac{1- a^*(w+s_1-s_1u)}{w + s_1 - s_1u} + \frac{a^*(w+s_1-s_1u)}{w + s_2 - s_2u} \quad -$$

$$\frac{a^*(w+s_1-s_1u)\; B^*(w+s_2-s_2u)}{w + s_2 - s_2u}$$

where $a^*(.)$ and $B^*(.)$ are the Laplace transforms of a(.) and B(.), the densities of the type one and two intervals respectively. Now using the assumption of exponentiality for a(.) and B(.) (2.1.41) can be written as :-

(2.1.42)

$$\bar{H}_1(w) = \frac{1}{1 - \dfrac{\mu_1}{\mu_1+w+s_1-s_1u} - \dfrac{\mu_1}{\mu_2+w+s_2-s_2u}} \; X$$

$$X \quad 1 - \frac{\mu_1}{\mu_1+w+s_1-s_1u} \quad +$$

$$+ \quad \frac{\dfrac{\mu_1}{(\mu_1+w+s_1-s_1u)} - \dfrac{\mu_1\mu_2}{(\mu_1+w+s_1-su)(\mu_2+w+s_2-s_2u)}}{w + s_2 - s_2u}$$

(2.1.43)

$$= \frac{w + \mu_1 + \mu_2 + s_2 - s_2u}{w(w+s_2-s_2u+\mu_1+s_1-s_1u+\mu_2+\mu_2(s_1-s_1u) + \mu_1(s_2-s_2u) + (s_1-s_1u)(s_2-s_2u)}$$

(2.1.44)

$$= \frac{w + \mu_1 + \mu_2 + s_2 - s_2u}{(w + k/2 + sqr(k^2/4 - J))(w + k/2 - sqr(k^2/4 - J))}$$

where sqr = square root of (...)

$$k = (\mu_1 + \mu_2 + s_1 - s_1u + s_2 - s_2u)$$

$$J = (\mu_2(s_1-su) + \mu_1(s_2-s_2u) + (s_1-s_1u)(s_2-s_2u)$$

Inverting (2.1.40) gives :-

(2.1.45) $\bar{H}_1(t) =$

$$= \frac{-k/2 + (s_1-s_1u) + sqr(k^2/4 - J))e^{-(k/2 + sqr(k /4 - J)t}}{2\, sqr(k^2/4 - J)} \quad +$$

$$\frac{k/2 - (s_1-s_1u) + sqr(k^2/4 - J))e^{-(k/2 + sqr(k /4 - J)t}}{2\, sqr(k^2/4 - J)}$$

Note that $sqr(k^2/4 - J)$ is real since :-

$$k^2/4 - J = 1/4\,[(\mu_1+\mu_2 - (s_2-s_2u) + (s_1-s_1u) + 4\mu_1\mu_2)] > 0$$

Similarly, if the initial interval is of type two, then we have :-

(2.1.46) $\overline{H_2}(w) =$

$$\frac{w + \mu_1 + \mu_2 + s_1 - s_1u}{(w + k/2 + sqr(k^2/4 - J)(w + k/2 - sqr(k^2/4 - J))}$$

(2.1.47) and

$$H_2(t) = (-k/2 + (s_2 - s_2 u) + \mathrm{sqr}(k^2/4 - J)) \times \frac{\exp[-(k/2 + \mathrm{sqr}(k^2/4 - J)) t]}{2\,\mathrm{sqr}(k^2/4 - J)} + \frac{(k/2 - (s_2 - s_2 u) + \mathrm{sqr}(k^2/4 - J))}{2\,\mathrm{sqr}(k^2/4 - J)} \times \exp[-(k/2 - \mathrm{sqr}(k^2/4 - J)) t]$$

In discussing the homogeneous Poisson Shock Model, it was noted that if the survival probabilities $(\bar{P}_k)_{k=0}^{\infty}$ were of the form $\bar{P}_k = u^k$ for $0 \leq u \leq 1$ then the corresponding life distribution was exponential. Clearly the same is not true in the Doubly Stochastic Poisson Model although (2.1.45) and (2.1.47) can be interpreted as the scaled sums of exponential survivor function. Since the survival probabilities $(\bar{P}_k = u^k)_{k=0}$ are discrete DFR it is immediate from corollary (2.1E.3) that H defined by (2.1.45) or (2.1.47) is DFR.

The mean time to failure of the device can be obtained easily since if T_i is the lifetime of the device conditional on the initial interval being of type i :-

(2.1.48) $\qquad E(T_i) = H_i^*(0)$

so,

$$E(T_1) = \frac{\mu_1 + \mu_2 + s_2 - s_2 u}{\mu_2(s_1 - s_1 u) + \mu_1(s_2 - s_2 u) + (s_1 - s_1 u)(s_2 - s_2 u)}$$

and similarly :

$$E(T_2) = \frac{\mu_1 + \mu_2 + s_1 - s_1 u}{\mu_2(s_1 - s_1 u) + \mu_1(s_2 - s_2 u) + (s_1 - s_1 u)(s_2 - s_2 u)}$$

Note that if $s_1 < (>) s_2$ then $E(T_1) > (<) E(T_2)$

i.e., on the average, a device which begins operation with an interval in which fewer shocks occur will have a longer expected lifetime than a device which begins operation with an interval in which a greater number of shocks occur.

Note that as $u \to 1$ the advantage becomes negligible. This is intuitively appealing since if the probability of surviving each individual shock is close to 1 then it should make little difference to the device's lifelength whether the initial interval is one of low or high shock incidence.

Other moments of the distribution of T_1 and T_2 can be obtained via the formula :-

(2.1.49) $$E(T_i^n) = (-1)^{n-1} n \, d^{(n-1)}/ds^{(n-1)} (\bar{A}_i^*(s))|_{s=0}$$

The special case where one of the rates s_1, s_2 is equal to zero is worthy of special mention. As indicated earlier, such a model corresponds to a situation where a device is alternately in and out of use and while out of use is free from shocks. Thus the rate process is a homogeneous Poisson process interspersed with periods when no shocks occur. Suppose $s_1 = 0$ then :-

$$\bar{A}_1(t) = \frac{(k/2 + \mathrm{sqr}(k^2/4 - J)) \exp[-(k/2 - \mathrm{sqr}(k^2/4 - J))t]}{2\,\mathrm{sqr}(k^2/4 - J)} + \frac{(\mathrm{sqr}(k^2/4 - J) - k/2) \exp[-(k/2 - \mathrm{sqr}(k^2/4 - J))t]}{2\,\mathrm{sqr}(k^2/4 - J)}$$

where the constants k and J are somewhat simplified so that

$$K = \mu_1 + \mu_2 + s_1 - s_2 u$$

and,

$$J = \mu_1(s_1 - s_1 u)$$

and,

$\bar{A}_2(t)$ is as in (2.1.47)

More interestingly,

$$E(T_1) = \frac{\mu_1 + \mu_2 + s_2 - s_2 u}{\mu_1(s_2 - s_2 u)}$$

and,

$$E(T_2) = \frac{\mu_1 + \mu_2}{\mu_1(s_2 - s_2 u)}$$

Clearly, as expected :- $E(T_1) > E(T_2)$
i.e., commencing operation with a period free from shocks increases the lifetime of the device.

Of course, similar results can be obtained if $s_2 = 0$

~2.1F The Poisson Cluster Shock Model

Thall (1981) introduced a model in which shocks arrive at a device in accordance with a homogeneous Poisson cluster process.

As with the Doubly Stochastic Poisson Shock Model, we will see that the results obtained are much less general than for the models considered earlier in this chapter.

As usual, our interest in studying a Poisson Cluster Shock Model is to establish sufficient conditions for the life distribution defined by:-

$$\bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k)\ \bar{P}_k$$

to belong to a particular class of distribution where, in this case $\{N(t)\}$ is a Poisson cluster process.

Firstly, however, we will present some basic properties of the Poisson Cluster process.

The general structure of a homogeneous Poisson cluster process is as follows :-
There is a homogeneous Poisson process $\{N_o(t)\}$ with rate s, say, of cluster centres and associated with each point, t_j, of the process is a subsidiary process or cluster, with occurrences at the points :-

$$\{t_j + x_0, t_j + x_1, t_j + x_2 + \ldots t_j + x_k\}$$

where k is an integer valued random variable with,

$$P(K=k_1) = \pi_{k+1} \quad \text{and} \quad \{X_0, X_1, \ldots\ldots X_k\}$$

is a collection of random variables such that $X_0 = 0$ and $\{X_1, \ldots X_k\}$ have some prescribed distribution W_k on $\mathbb{R}^k$.

We denote the subsidiary process by $\{N^{(f)}(.)\}$ and the Poisson Cluster process $\{N(.)\}$ is the superposition of all the subsidiary processes. We will suppose that $\{N_0(.)\}$, $\{N^{(f)}(.)\}$ and $\{N(.)\}$ have associated probability measures P_0, $P^{(f)}$ and P.

Also of interest is the synchronous version of $\{N(.)\}$ i.e., the process $\{N(.)\}$ conditioned on the occurrence of a shock at the origin. We denote the synchronous process by $\{N\hat{}(.)\}$ and note that the probability generating function of $N(.)$ and $N\hat{}(.)$ are related by the well-known Palm-Khinchin equations as follows :-

Let $\phi(z,t) = E(z^{N(t)})$; $\phi\hat{}(z,t) = e(z^{N(t)})$ $0 =< z < 1$
then,

$$d/dt\ \phi(z,t) = -r_p(1-z)\phi\hat{}(z,t)$$

where r_p = intensity (rate) of $\{N(.)\}$

$$= s \sum_{k=1}^{\infty} \pi_k$$

Now label the occurrence corresponding to $t_j + x_i$ in any cluster as an occurrence of type i and consider a single subsidiary process $\{N^r(.)\}$ with corresponding probability measure P^r having a single point at t = 0. Suppose $\{N^r(.)\}$ is a randomised subsidiary process in the sense that the choice of type of occurrence at the origin is made randomly and the probability that a type i occurrence is chosen is the same as the proportion of type i occurrences in the cluster process $\{N(.)\}$.

The following lemma due to Oakes (1975) will prove useful in establishing the main result of this section.

Lemma (2.1F.1)

If $P^{\wedge}$, P and $P^{(r)}$ are defined as above, then :-

$$P^{\wedge}(.) = P * P^{(r)}(.) ,$$

where * denotes convolution

Thall (1981) gives somewhat complex expressions for the density and failure rate of the Poisson Cluster Shock Model but of more interest to us is the following result :-

Theorem (2.1F.1)

Let $\{N(t)\}$ be a Poisson cluster process and let H be defined by :-

$$\bar{H}(t) = \sum_{k=0}^{\infty} P(N(t)=k)\, \bar{P}_k$$

where $\bar{P}_k = z^k$ and $0 =< z =< 1$

then, H is strictly DFR except when $\{N(t)\}$ exhibits no clustering in which case H is exponential, and the failure rate $r(t)$ is constant.

Proof

It follows from the Lemma (2.1F.1) that the p.g.f's of $N^{\wedge}(.)$, $N(t)$ and $N^{r}(t)$ are related by the equation :-

$$\phi(z,t) = \phi(z,t)\,\phi^{r}(z,t)$$

Now, combining this equation and the Palm-Khinchin equations and noting that :-

$$\bar{P}_k = z^k \quad \text{implies} \quad \bar{H}(t) = \phi(z,t) \quad \text{we have :-}$$

$$r(t) = -\phi'(z,t) \,/\, \phi(z,t)$$

$$= r_p(1-z)\phi^{(r)}(z,t)$$

which is strictly decreasing in t except when there is no clustering in which case $\phi^{r}(z,t) = 1$ and the failure rate $r(t)$ is constant.

Thall (1981) also shows that H is DFR if $(\bar{P}_k)_{k=0}^{\infty}$ is completely monotonic and DFR. A sequence of real constants $(E_k)_{k=0}^{\infty}$ such that $0 =< E_k =< 1$ is said to be completely monotonic if :-

$$D^n(E_k) =^{df} \Sigma_{r=0}^{n} ({}^nC_r)(-1)^r E_{k+r} >= 0$$

for all $n, k = 0,1,2...$ and $D^0(E_k) = E_k$

Thall (1981) also shows, by counter-example, that for the Poisson Cluster Shock Model $(\bar{P}_k)_{k=0}$ discrete IFR does not imply that H is IFR.

~2.2 Generalised Shock Models

In this section, we study the lifetime distribution of a device subject to shocks governed by a generalised renewal process where, here, the term generalised refers to the fact that while the shock interarrival times are independent but are not necessarily identically distributed. We assume, however, that the interarrival times have distributions which belong to the same non-parametric class of distribution.

The aim in studying these models is to establish sufficient conditions on the shock survival probabilities $(\bar{p}_k)_{k=0}$ and on the process governing the arrival of shocks $\{N(t)\}$ for the life distribution, H to belong to one of the claims of life distribution discussed in Chapter one. The main feature of the Poisson and related shock models considered in the earlier part of this Chapter is the relationship between the class to which H belongs and the discrete class to which the survival probabilities $(\bar{p}_k)_{k=0}$ belong. In the case of the generalised shock model we will see that H inherits its class from both the survival probabilities and the shock interarrival time distribution.

Before proceeding, we need to introduce some notation.
Let T_k denote the time of the k^{th} shock, :-

Set $\quad U_k = T_k - T_{k-1}$ where $k = 1,2,3...$, $T_0 = 0$

and let $\quad z_k(t) = P(N(t)=k)$
where $\{N(t)\}$ is a generalised renewal process. Now define A_k by :-

(2.2.1) $\quad A_k =^{df} \int_0^\infty z_k(x)dx = \int_0^\infty P[T_k =< x < T_{k+1}]dx$

$$= \int_0^\infty E[I_{[T_k, T_{k+1})}(x)]dx = E[\int_{T_k}^{T_{k+1}} dx]$$

(2.2.2) $\quad = E(T_{k+1} - T_k) = E(U_{k+1})$

Here I_A is the indicator function of the set A. We will denote by $J_k(.)$ the distribution of U_k.

As usual, we define the life distribution H of a device subject to shocks governed by a generalised renewal process {N(.)} by :-

(2.2.3) $\bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k)\ \bar{p}_k$

$= \Sigma_{k=0}^{\infty} z_k(t)\ \bar{p}_k$

where $\bar{p}_k$ = P(surviving k shocks).

We are now in a position to establish the following theorem :-

Theorem (2.2.1)

Let {N(.)} be a process with independent interarrival times U_k and suppose H is defined by (2.2.3) then :-

(a) H is NBU if the interarrival distribution $J_k(.)$ are increasing in k for each t and NBU, and $(\bar{p}_k)_{k=0}^{\infty}$ is discrete NBU.

(b) H is NBUE if the $(J_k(.))_{k=0}^{\infty}$ are NBUE and :-

(2.2.4) $p_j\ \Sigma_{k=0}^{\infty} \bar{p}_k A_k \ >= \ \Sigma_{k=j}^{\infty} \bar{p}_k\ A_k$,

where $j = 0,1,\ldots$ and $A_k = E(U_{k+1})$

(c) H is HNBUE if the interarrival times distributions $J_k(.)$, $k = 1,2,\ldots$ are HNBUE and :-

(2.2.5) $\Sigma_{j=k}^{\infty} p_j A_j =< a_0 \prod_{j=0}^{k-1} (1-(A_j/a_0)$

where $k = 1,2,\ldots$ and $a_0 = \Sigma_{k=0}^{\infty} \bar{p}_k A_k$

(d) H belongs to L if the $J_k(.)$ $k = 1,2,\ldots$ belong to L and :-

(2.2.6) $\Sigma_{k=0}^{\infty} \bar{p}_k \, \Pi_k(s) \geq a_0 / (1+sa_0)$

where $a_0 = \Sigma_{k=0}^{\infty} \bar{p}_k A_k$ and

$\Pi_0(s) = 1 / (A_0^{-1} + s)$

$\Pi_k(s) = (\Pi_{j=0}^{k-1} \; A_j/A_j+s) \; 1/A_j+s$

where $k = 1,2,3...$

Note : the $(\Pi_k(s))_{k=0}^{\infty}$ are the Laplace transforms of $z_k(t) = P(N^*(t)=k)$ where $\{N(t)\}$ is a stationary pure birth process with birth coefficients $(A_k^{-1})_{k=0}^{\infty}$

<u>Proof</u>

a) The NBU case was established in the course of proving Theorem (2.1A.(d)), the corresponding result for the stationary pure-birth shock model.

b) We have to show that :-

$\int_t^{\infty} \bar{H}(x)dx \leq \mu\bar{H}(t)$ where $= \int \bar{H}(x)dx$

Now, $\bar{H}(t)\int_0^{\infty}\bar{H}(x)dx = \Sigma_{j=0}^{\infty} z_j(t)\bar{p}_j \int_0^{\infty} \Sigma_{k=0}^{\infty} z_k(x)\bar{p}_k dx$

where $z_j(t) = P(N(t)=j)$

$= \Sigma_{j=0}^{\infty} z_j(t) \; \bar{p}_j \; \times$

$\Sigma_{k=0} \bar{p}_k A_k$

$\geq \Sigma_{j=0}^{\infty} z_j(t) \; \Sigma_{k=j}^{\infty} \bar{p}_k A_k$ by (2.7.5)

$= \Sigma_{k=0}^{\infty} \bar{p}_k A_k \; \Sigma_{j=0}^{\infty} z_j(t)$

hence it suffices to show that :-

$$\Sigma_{k=0}^{\infty} \bar{p}_k A_k \quad \Sigma_{j=0}^{\infty} A_j(t) \quad >= \int_t^{\infty} \bar{H}(x)\,dx$$

but $\int_t^{\infty} H(x)\,dx \;=\; \Sigma_{k=0}^{\infty} \bar{p}_k \int_t^{\infty} z_k(x)\,dx$

so we need only show that,

$$A_k \; \Sigma_{j=0}^{k} z_j(t) \quad >= \int_t^{\infty} z_k(x)\,dx$$

i.e.

(2.2.7) $$E(U_{k+1})\; P[N(t)=<k] \quad >= \int_t^{\infty} P(N(x)=k)\,dx$$

Now, recall that the time of occurrence of the k^{th} shock is a random variable, T_k. We will denote by FT_k the distribution function of T_k.

$$P(N(t)=<k) \;=\; P(T_{k+1}>t) \quad \text{and}$$

$$\int_t^{\infty} P(N(x)=k)\,dx \;=\; \int_t^{\infty} P(T_k=<x<T_{k+1})\,dx$$

$$= \int_t^{\infty} E(I_{(T_k,T_{k+1})}(x))\,dx$$

Where $I_A(.)$ is the indicator function of the set A.

$$= \quad E(\int_t^{\infty} I_{(T_k,T_{k+1})}(x)\,dx)$$

$$= \quad E((T_{k+1}-T_k)\; I_{(0,T_k)}(t) \;+\; (T_{k+1}-t)\,I_{(T_k,T_{k+1})}(t))$$

$$= \quad E(U_{k+1})\; P(T_k>t) \;+\; E((T_{k+1}-t)\; I_{(T_k,T_{k+1})}(t))$$

by the assumption of independent inter-arrival times. So (2.2.7) reduces to :-

$$E(U_{k+1}) \; X \; P(T_k=<t<T_{k+1})$$

(2.2.8) $$>= E[(T_{k+1}-t)\; I_{(T_k,T_{k+1})}(t)$$

Now again using the assumption of independent inter-arrival times, we have :-

$$P[T_k =< t < T_{k+1}] \;=\; P[T_{k+1}-T_k > t-T_k,\ T_k =< t]$$

$$= \int P[U_{k+1} > t-T_k,\ T_k =< t \mid T_k =< (s,s+ds) dFT_k(s)$$

(2.2.9) $$= \int P[U_{k+1} > t-s]\ dFT_k(s)$$

Now, for any non-negative random variable X with distribution function F :-

(2.2.10) $$E(XI_{[s,\infty)}(x)) \;=\; s\overline{F}(s) + \int_s^\infty \overline{F}(u)du$$

so $$E[(T_{k+1}-t)\ I_{(T_k,T_{k+1})}(t)] \;=$$

$$= E[(T_{k+1}-t)\ I_{(0,t)}(T_k)\ I_{(t,\infty)}(T_{k+1})]$$

$$= E[(T_{k+1}-T_k-(t-T_k))\ I_{(0,t)}(T_k)\ I_{(t-T_k,\infty)}(T_{k+1}-T_k)$$

$$= \int_0^t E[U_{k+1}-(t-s)\ I_{(0,\infty)}(U_{k+1}-(t-s))dFT_k(s)$$

$$= \int_0^t \int_0^\infty P(U_{k+1}-(t-s)>u)\ dudFT_k(s) \qquad \text{by (2.2.10)}$$

(2.2.11) $$= \int_0^t \int_{t-s}^\infty P(U_{k+1}>u)\ dudFT_k(s)$$

It follows from (2.2.9), (2.2.11) and (2.2.8) that :-

$$E(U_{k+1})\ X\ P(T_k =< t < T_{k+1}) - E[(T_{k+1}-t)\ I_{(T_k,T_{k+1})}(t))$$

$$= E(U_{k+1})P(U_{k+1}>t-s\ dFT_k(s) \;-$$

$$- \int_0^t \int_{t-s}^\infty P(U_{k+1}>u)\ dudFT_k(s)$$

$$= \int_0^t (E(U_{k+1})\ P(U_{k+1}>t-s) \;-$$

$$- \int_{t-s}^\infty P(U_{k+1}>u)\ du)\ dFT_k(s)$$

$>= 0$ since $(U_{k+1})_{k=0}^\infty$ have NBUE distributions.

(c) We first show that if :-

$$K(t) = \Sigma_{k=0}^{\infty} z_k(t)\ \bar{p}_k$$

is the survivor function of a device subject to shocks whose occurrence is governed by a stationary pure birth process $\{N^*(t)\}$ such that :-

$$P(N^*(t)=k) = z_k(t), \qquad \text{then,}$$

(2.2.12) $$\int_t^{\infty} \bar{H}(x)dx =< \int_t^{\infty} \bar{K}(x)dx$$

The result then follows from the fact that K was shown to be HNBUE in Theorem (2.1A.1(f)), under the condition on $\bar{p}_k$ and on A_k of this theorem.

Now, $$\bar{H}(x) = \Sigma_{j=0}^{\infty} z_j(x)\ \bar{p}_j$$

$$= \Sigma_{j=0}^{\infty} (\Sigma_{k=0}^{j} z_k(x))(\bar{p}_j-\bar{p}_{j+1})$$

$$= \Sigma_{j=0}^{\infty} P(\Sigma_{k=1}^{j+1} U_k>x)(\bar{p}_j-\bar{p}_{j+1})$$

and hence,

$$\int_t^{\infty} \bar{H}(x)dx = \Sigma_{j=0}^{\infty} (\int_t^{\infty} P(\Sigma_{k=1} U_k>x)\ dx(\bar{p}_j-\bar{p}_{j+1})$$

Denote by V_k where k = 1,2,... the inter-arrival times of a stationary pure birth process with birth coefficients $(A_k)_{k=0}^{\infty}$ so that:-

$$P(V_k>x) = \exp(-x/A_{k-1}), \qquad k = 1,2,...$$

Since the U_k are HNBUE it follows that :-

$$\int_t^{\infty} P(U_k>x)dx =< \int_t^{\infty} P(V_k>x)dx$$

Now, Theorem 4.2 of Marshall and Proschan (1970) states that if :-

$$\int_t^\infty F_i(x)dx \quad =< \quad \int_t^\infty G_i(x)dx \qquad\qquad i = 1,2,\ldots n$$

where F_i, G_i are distributive functions and,

$$F(t) = F_1 * F_2 * \ldots * F_n(t) \quad , \quad G(t) = G_1 *\ldots* G_n(t)$$

$$\text{then} \qquad \int_t^\infty F(x)dx \quad =< \quad \int_t^\infty G(x)dx$$

Using this result and the independence of inter-arrival times we have from :-

(2.2.13) $\quad P(\Sigma_{k=1}^{\infty} U_k>x)dx \quad =< \int P(\Sigma_{k=1}^{\infty} V_k>x)dx$

i.e. $\quad \Sigma_{j=0}^{\infty} (\int P(\Sigma_{k=1}^{\infty} U_k>x)dx \ (\bar{p}_j-\bar{p}_{j+1})$

$$=< \quad \Sigma_{j=0}^{\infty} (\int P(\Sigma_{k=1} V_k>x)dx \ (\bar{p}_j-\bar{p}_{j+1}))$$

i.e. $\quad \int_t^\infty \bar{H}(x)dx \quad =< \int_t^\infty \bar{K}(x)dx$

and since by Theorem (2.1A.1(f)), K is HNBUE under the condition of this theorem it follows that H is HNBUE.

d) In a similar fashion to proof of part (c) of this Theorem we first show that :-

(2.2.14) $\quad \int_0^\infty \bar{H}(t)\ e^{-st}\ dt \quad >= \quad \int_0^\infty \bar{K}(t)\ e^{-st}\ dt$

where K(.) is the survivor function of the stationary pure birth shock model. We then show that under the condition on $(p_k)_{k=0}^{\infty}$ and $(A_k)_{k=0}^{\infty}$ of this theorem K belongs to L and consequently the desired result follows. (2.2.14) follows from an argument very similar to that used in the HNBUE case to show that :-

$$\int_t^\infty \bar{H}(x)dx \quad =< \quad \int_t^\infty \bar{K}(x)dx$$

The one additional step required is that :-

$$\int_0^{\infty} e^{-st} P(\Sigma_{k=1}^{j} U_k > t)dt \;>=\; \int_0^{\infty} e^{-st} P(\Sigma_{k=1}^{j} V_k > t)dt$$

if $\int_0^{\infty} e^{-st} P(U_k > t)dt \;>=\; \int P(V_k > t)\, e^{-st}\, dt$

for $k = 1,2,3,...$

but this is a well known result of Laplace transforms.

Now, under the conditions of the theorem :-

$$\Sigma_{k=0}^{\infty} \bar{p}_k \, \pi_k(s) \;>=\; a_0/(1+sa_0)$$

where $a_0 = \Sigma_{k=0}^{\infty} \bar{p}_k A_k$ and the π_k $k = 0,1,2,...$ are the Laplace transforms of $z_k(t) = P(N^*(t)=k)$ $k = 1,2,...$ where $\{N^*(.)\}$ is a stationary pure birth process.

i.e. $$\Sigma_{k=0}^{\infty} p_k \pi_k(s) \;=\; \int_0^{\infty} e^{-st} z_k(t)dt \;>=\; a_0/(1+sa_0)$$

So K, and by (2.2.13) H, both belong to L.

Part (a) of Theorem (2.2.1) above is due to A-Hameed and Proschan (1975) part (b) is due to Block and Savitts (1978) and parts (c) and (d) to Klefsjo (1981, 1983), although in the L case the condition on $(p_k)_k$ and $(A_k)_k$ is weaker than that used by Klefsjo.

In parts (b), (c) and (d) of Theorem (2.2.1) quite general conditions have been imposed on $(p_k)_k$ and $(A_k)_k$ in order to emphasise the relationship between the class to which the inter-arrival distribution $F_k(.)$ $k = 1,2,...$ belongs and the class to which H(.) belongs.

In the theorems to follow, stronger conditions on $(p_k)_k$ and on $(A_k)_k$ will be imposed and these will better illustrate the relationship between the classes of H(.) and the discrete survival probabilities $(\bar{p}_k)_{k=0}$.

The obvious analog to Theorem (2.2.1) holds for the dual classes NWU, NWUE, HNWUE and $\bar{L}$.

By imposing stronger conditions on $(p_k)_{k=0}^{\infty}$ and on $(A_k)_{k=0}^{\infty}$ than in Theorem (2.2.1) the following result can be obtained.

Theorem (2.2.2)

Suppose the shock inter-arrival times U_k are independent and H is defined by (2.2.3) then :-

(a) H is NBUE (NWUE) if the interarrival time distribution J(.) k = 1,2,... are NBUE (NWUE), ($A_k = E(U_{k+1})$), k = 0,1,2,... is decreasing (increasing) in k and $(\bar{P}_k)_{k=0}$ discrete NBUE (NWUE)

(b)(i) H is HNBUE if the $J_k(.)$ are HNBUE k = 1,2,... $(A_k)_{k=0}^{\infty}$ is decreasing and :-

$$\Sigma_{j=k}^{\infty} \bar{P}_j =< \Sigma_{j=0}^{\infty} p_j \prod_{j=0}^{k-1} (1-(A_j/a_0))$$

where k = 1,2,3... and

$$a_0 = \Sigma_{j=0} \bar{P}_j A_j$$

(b)(ii) H is HNWUE if the $J_k(.)$ are HNWUE, $(A_k)_{k=0}^{\infty}$ is increasing to a finite limit,

$$\Sigma_{j=k}^{\infty} p_j >= \Sigma_{j=0}^{\infty} p_j \prod_{j=0}^{k-1}(1-(A_0/a_0))$$

and there exists a k_0 such that $a_0 >= A_k$ for every $k >= k_0$ for which $\bar{P}_k > 0$

(c) H belongs to L ($\bar{L}$) if $J_k(.)$ belongs to L ($\bar{L}$) n = 1,2,..., $A_k =< (>=) A_0$ k = 1,2,... and $(\bar{P}_k A_k)_{k=0}^{\infty}$ is decreasing and belongs to G ($\bar{G}$).

Proof

We will consider only the NBUE, HNBUE and L classes here. The proof ofthe corresponding results for the NWUE, HNWUE and L classes are quite similar.

(a) By Theorem (2.2.1) we need only show that $(A_k)_{k=0}^{\infty}$ decreasing and $(\bar{p}_k)_{k=0}^{\infty}$ discrete NBUE implies :-

$$\bar{P}_j \ \Sigma_{k=0}^{\infty} \bar{p}_k A_k \ \ >= \ \Sigma_{k=j}^{\infty} \bar{p}_k A_k \quad j = 0,1,2,...$$

but this was shown in the course of proving Theorem (2.1A.1(e)).

(b) By Theorem (2.2.1(c)) it suffices to show that $(A_k)_{k=0}$ decreasing and :-

$$\Sigma_{j=k}^{\infty} \bar{P}_j =< \Sigma_{j=0}^{\infty} \bar{P}_j \ \Pi_{j=0}^{k-1} (1-A_j/a_0), \quad k=1,2,3,...$$

implies $\quad \Sigma_{j=k} \bar{P}_j A_j =< \Sigma_{j=0} p_j A_j \ \Pi_{j=0}^{K-1} (1-A_j/a_0)$

We will find it convenient to define B_k by :-

$$B_k =^{df} \Pi_{j=0}^{k-1}(1-A_j/a_0) \ , \quad k=1,2,... \quad \text{(Note that } B_k =< 1)$$

Now , $(A_k)_{k=0}^{\infty}$ decreasing implies that $\lim_{k->\infty} A_k = A$ exists, consequently,

$$(2.2.15) \qquad B_k \ \Sigma_{j=0}^{\infty} \bar{P}_j A_j - \Sigma_{j=k} \bar{p}_j A_j =$$

$$= [B_k \ \Sigma_{j=0} \bar{P}_j A - \Sigma_{j=k} \bar{p}_j A] +$$

$$B_k \ \Sigma_{j=0} \bar{P}_j (A_j - A) - \Sigma_{j=k} \bar{p}_j (A_j - A)$$

By assumption, the expression in brackets in (2.2.15) is non-negative, since :-

$$\Sigma_{j=k}^{\infty} \bar{P}_j (A_j - A) = \Sigma_{v=k}^{\infty} A_v - A_{v+1} \ \Sigma_{j=k}^{v} \bar{p}_k$$

we have :-

$$B_k \ \Sigma_{j=0}^{\infty} \bar{P}_j (A_j - A) = \Sigma_{j=k} \bar{p}_j (A_j - A)$$

$$= B_k \ \Sigma_{v=0}^{k-1} (A_v - A_{v+1}) \ \Sigma_{j=0}^{v} \bar{P}_j - \Sigma_{v=k}^{\infty} (A_v - A_{v+1}) \ \Sigma_{j=k}^{v} \bar{P}_j$$

$$(2.2.16) \quad = B_k \Sigma_{v=0}^{k-1} (A_v - A_{v+1}) \Sigma_{j=0}^{v} \bar{p}_j + \Sigma_{v=k}^{\infty} (A_v - A_{v+1}) \; X$$

$$[B_k \Sigma_{j=0}^{v} \bar{p}_j - \Sigma_{j=k}^{v} \bar{p}_j]$$

The first expression on the right hand side of (2.2.16) is non-negative since $(A_k)_{k=0}^{\infty}$ is decreasing. The second term is also non-negative by the following argument.

Let $\quad \phi_k(v) = B_k \Sigma_{j=0}^{v} \bar{p}_j - \Sigma_{j=k} \bar{p}_j$

then $\quad \phi_k(v) - \phi_k(v+1) = \bar{P}_{v+1} (1-B_k) >= 0$

since $B_k =< 1$, consequently $\phi_k(v) >= \lim_{v->\infty} \phi_k(v)$

but by the assumption of this theorem :-

$$\lim_{v->\infty} \phi_k(v) = B_k \Sigma_{j=0}^{\infty} \bar{p}_j - \Sigma_{j=k}^{\infty} \bar{p}_j >= 0$$

So $\phi_k(v) >= 0$ and consequently,

$$B_k \Sigma_{j=0}^{\infty} \bar{p}_j A_j >= \Sigma_{j=k}^{\infty} \bar{p}_j A_j \qquad k = 0,1,2,\ldots$$

as required.

(c) By Theorem (2.2.1(d)) it suffices to show that $A_k =< A_0$ and $(\bar{p}_k A_k)_{k=0}^{\infty}$ is decreasing and belongs to G implies :-

$$\Sigma_{k=0} \bar{p}_k \pi_k(s) >= a_0/(1+sa_0)$$

where $a_0 = \Sigma_{k=0}^{\infty} \bar{p}_k A_k$ and $\pi_k(s)$ is the Laplace transform of $P(N^*(t)=k)$ where $\{N^*(.)\}$ is a stationary pure birth process.

This was shown in the course of proving Theorem (2.1A.1(g))

Now, if the shock inter-arrival times all have the same mean, i.e. $A_k = A$ where $k = 0,1,2,\ldots$ the following corollary to Theorem (2.2.2) is immediate.

Corollary (2.2.1)

Let {N(t)} be a generalised renewal process with independent interarrival times U_k such that $E(U_k) = A$ where $k = 1,2,...$ then if H is defined by (2.2.3)

a) H is HNBUE (NWUE) if the shock inter-arrival time distributions $J_k(.)$ $k = 0,1,2,...$ are NBUE (NWUE) and $(\bar{P}_k)_{k=0}^{\infty}$ is discrete NBUE ($\overline{\text{NWUE}}$).

b) H is HNBUE (HNWUE) if the $J_k(.)$ are HNBUE (HNWUE) and $(\bar{P}_k)_{k=0}^{\infty}$ discrete HNBUE (HNWUE).

c) H belongs to L (Γ) if the $J_k(.)$ belongs to L ($\bar{\Gamma}$) $k = 1,2,...$ and $(\bar{P}_k)_{k=0}^{\infty}$ belongs to G ($\bar{G}$).

In the NBU (NWU) case, it is possible to relax the assumption of independent interarrival times whilst still obtaining a useful result. This result is contained in the following Theorem due to Block and Savitts (1978).

Theorem (2.2.3)

Let H be defined by (2.2.3) then H is NBU (NWU) if $(\bar{P}_k)_{k=0}^{\infty}$ is discrete NBU (NWU) and the stochastic process {N(t)} satisfies :-

(2.2.17) $P[N(x+y =< j+k, N(x)=k] =<$

$(>=)$ $P[N(y) =< j] P(N(x)=k]$

i.e.

(2.2.18) $P[N(x+y) > j+k \mid N(x)=k] >= (=<) P(N(y)>j)$

Proof

For all x, y > 0 :-

$$\bar{H}(x+y) = \Sigma_{i=0}^{\infty} \bar{P}_i \, P[N(x+y)-i]$$

$$= \Sigma_{i=0}^{\infty} \bar{P}_i \, \Sigma_{k=0}^{i} P(N(x+y)=i,\ N(x)=k)$$

$$= \Sigma_{k=0}^{\infty} \Sigma_{i=k}^{\infty} P[N(x+y)=i,\ N(x=k] \, \bar{P}_i$$

and putting j = i-k gives :-

$$\bar{H}(x+y) = \Sigma_{k=0}^{\infty} \Sigma_{j=0}^{\infty} P[N(x+y)=j+k,\ N(x)=k] \, \bar{p}_{j+k}$$

$$=< \Sigma_{k=0}^{\infty} \Sigma_{j=0}^{\infty} P[N(x+y)=j+k,\ N(x)=k] \, \bar{p}_j \bar{p}_k$$

$$=< \Sigma_{k=0}^{\infty} \Sigma_{j=0}^{\infty} P[N(x+y)=j+k,\ N(x)=k] \, \bar{p}_j \bar{p}_k$$

since $(\bar{p}_k)_{k=0}$ is discrete NBU,

$$= \Sigma_{k=0}^{\infty} p_k \, \Sigma_{j=0}^{\infty} P[N(x+y)=j+k,\ N(x)=k] \, \Sigma_{i=j}(\bar{P}_i-\bar{P}_{i+1})$$

$$= \Sigma_{k=0}^{\infty} \bar{p}_k \, \Sigma_{i=0}^{\infty} (\bar{P}_i-\bar{P}_{i+1}) \, \Sigma_{j=0}^{i} P[N(x+y)=j+k,\ N(x)=k]$$

$$=< \Sigma_{k=0}^{\infty} p_k \, \Sigma_{i=0}^{\infty} (\bar{P}_i-\bar{P}_{i+1}) \, P[N(x+y)=<i+k,\ N(x)=k]$$

$$=< \Sigma_{k=0}^{\infty} p_k \, \Sigma_{i=0}^{\infty} (\bar{P}_i-\bar{P}_{i+1}) \, P[N(y)=<i] \, P[N(x)=k]$$

(by assumption (2.2.17),

$$= \Sigma_{k=0}^{\infty} P(N(x)=k)\bar{p}_k \, \Sigma_{i=0}^{\infty} P[N(y)=<i] \, (P_i-P_{i+1})$$

$$= \bar{H}(x) \, \Sigma_{i=0}^{\infty} P[N(y)=i) \, P_i$$

$$= \bar{H}(x)\bar{H}(y)$$

i.e. $\bar{H}(x+y) =< \bar{H}(x)\bar{H}(y)$

The proof in the NWU case follows similarly.

Note that the single condition (2.2.18) in the above Theorem replaces the conditions on $\{N(t)\}$ in Theorem (2.2.1(a)) of mutually independent interarrival times with distributions $J_k(.)$ $k = 1,2,...$ which are increasing and NBU. In fact it has been shown by Block and Savits (1978) that the condition (2.2.18) is weaker than the conditions used in Theorem (2.2.1(a)).

We have seen that in the case of the NBU, NBUE, HNBUE and L Classes of lifetime distribution and their duals, the NWU,NWUE, HNWUE and $\bar{L}$ Classes, the lifetime distribution of a device subject to shocks governed by a generalised renewal process inherits its Class from the shock inter-arrival time distributions and the shock survival probabilities $(\bar{p}_k)_{k=0}^{\infty}$.

Unfortunately, the same is not true in the IFR, IFRA, DFR or DFRA Classes. A-Hameed and Proschan (1975) provide counter examples in the IFR and IFRA Classes and an obvious modification of these examples yields counter examples for the DFR and DFRA Classes.

To date, the preservation of the DMRL and IMRL Classes in the context of generalised shock models appears not to have been considered in the literature.

CHAPTER 3 : PHYSICALLY MOTIVATED MODELS FOR FAILURE

~3.0 Introduction

In Chapter two we investigated the model :-

$$\bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k)\bar{p}_k$$

under a variety of prescribed structures on the process $\{N(t)\}$. The nature of the survival probabilities was largely ignored; we assumed only that $(\bar{p}_k)_{k=0}^{\infty}$ formed a decreasing sequence and that $\bar{p}_0 = 1$.

In this Chapter we take some account of the method by which a device may fail, e.g., failure may occur on the first occasion a shock exceeds some critical magnitude or failure may occur when the total accumulated damage due to shocks exceeds some critical threshold.

In the context of the model $\bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k)\bar{p}_k$ accounting for the method of failure is equivalent to imposing some structure on the survival probabilities $(\bar{p}_k)_{k=0}^{\infty}$. Moreover, classifying shock models according to the means by which failure occurs allows us to consider more general models in which there is some dependency between the probability of surviving k shocks and the shock interarrival times or in which failure may occur at times other than on the occurrence of a shock.

Considerations such as these render the model :-

$$\bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k)\bar{p}_k$$

no longer applicable and we will see that an alternative approach is required.

In this Chapter we associate with each shock the physical quantity :- magnitude of shock, or amount of damage inflicted. This by no means compromises the generality of our models since, for example, the magnitude of a shock may be negative, corresponding to a partial repair of the device.

The structure of this Chapter is as follows :- We begin by considering Maximum Shock Threshold Models in which failure occurs on the first occasion a shock exceeds some critical magnitude. We return briefly to the model :-

$$\bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k)\bar{P}_k$$

and structure the $\bar{P}_k$ so as to accommodate a maximum shock threshold model. Models where the shock magnitudes are correlated to the shock interarrival times are then considered.

Next, we consider Cumulative Damage Models in which damage caused by shocks accumulates and failure occurs when the total accumulated damage exceeds some critical threshold. Again, we begin the study of such models by returning to the model :-

$$\bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k)\bar{P}_k$$

and structuring the $(\bar{P}_k)_k$ appropriately. We then consider models in which damage still accumulates but between shocks, either wear continues or the device is partially repaired.

Finally, we consider accumulative damage models in which the critical threshold is random.

Throughout, our main aim is to obtain sufficient conditions for the life distribution associated with each of these models to belong to a particular class of life distribution.

~3.1 Maximum Shock Threshold Models

In this section, we suppose that shocks arrive at a device but cause no damage unless the magnitude of the shock is greater than some threshold value, in which case failure of the device is instantaneous. Esary, Marshall and Proschan (1973) suggest that such a model provides a plausible description of the fracture of brittle glass.

~3.1A The Standard Maximum Shock Threshold Model

We will assume that the Shock Magnitudes X_i $i = 1,2,...$ are random with distribution $F_i(.)$ $i = 1,2,.....$ If the magnitudes are mutually independent and independent of the shock interarrival times, then for a critical threshold z, the probability of the device surviving k shocks is given by :- $\bar{p}_k = \prod_{i=1}^{k} F_i(z)$ $k = 1,2,....$ will assume $\bar{p}_0 = 1$.

Consequently, the corresponding survivor function is :-

$$\bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k)\bar{p}_k$$
$$= \Sigma_{k=1}^{\infty} P(N(t)=k) \prod_{i=1}^{k} F_i(z) + P(N(t)=0)$$

Clearly, if {N(t)} is a homogeneous Poisson process, a stationary or non-stationary pure birth process or a generalised renewal process, the results of Chapter two apply with the condition on $(\bar{p}_k)_{k=0}$ replaced by the identical condition on the sequence :- $(1, \prod_{i=1}^{k} F_i(z)$ $k = 1,2,...)$

In some cases, it is possible to establish sufficient conditions on the $F_i(z)$ for the sequence :- $(1, \prod_{i=1}^{k} F_i(z)$ $k = 1,2,...)$ to belong to a discrete Class of survival probabilities. Thus we have the following Lemma :-

Lemma (3.1A.1)

The sequence $(1, \prod_{i=1}^{k} F_i(z), \quad k=1,2,\ldots)$ is :-

(a) discrete IFR (DFR) if $(F_i(z))_{i=1}^{\infty}$ is decreasing (increasing) in i

(b) discrete IFRA (DFRA) if :-

$$(\prod_{i=1}^{K} F_i(z))^{1/k} \;>=\; (=<)\; F_{k+1} \qquad k = 1,2,..$$

(c) discrete NBU (NWU) if :-

$$\prod_{i=k+L}^{\infty} F_i(z) \;=<\; (>=)\; \prod_{i=1}^{\infty} F_i(z)$$
$$k = 1,2,\ldots \quad L = 1,2,\ldots$$

Proof

In each case the proofs follow directly from the definition of the Class in question.

As a consequence of the above lemma, if $\{N(t)\}$ is a non-stationary pure birth process with :-

$$(3.1.1) \qquad P(N(t+d)-N(t)) = 1 \mid N(t)=k) = s_k s(t)d+o(d)$$
$$P(N(t+d)-N(t)) > 1 \mid N(t)=k) = o(d),$$
where d is small.

the following theorem follows directly from Theorem (2.1C.1) :-

Theorem (3.1A.1)

Define H by :-

$$H(t) = \Sigma_{k=1}^{\infty} P(N(t)=k) \prod_{i=1}^{K} F_i(z) + P(N(t)=k)$$

where $\{N(t)\}$ is a non-stationary pure birth process with probabilities given by (3.1.1)

a) H is IFR (DFR) if $(s_k)_{k=0}^{\infty}$ is increasing (decreasing), s(t) is decreasing (increasing) and $(F_i(z))_{i=1}^{\infty}$ is decreasing (increasing)in i.

b) H is IFRA (DFRA) if $(s_k)_k$ is increasing (decreasing) in k, $\int_0^t (s(x)dx)/t$ is increasing (decreasing) in t and :-

$$(\prod_{i=1}^{k} F_i(z))^{1/k} >= (=<)\ F_{k+1}(z)\ ,\ k = 1,2,...$$

c) H is NBU (NWU) if (s_k) is increasing (decreasing) in k s(x)dx is superadditive (sub-additive) in t and :-

$$\prod_{i=k+L}^{\infty} F_i(z) =<\ (>=)\ \prod_{i=1}^{L} F_i(z)$$

$$k = 1,2,...;\ L = 1,2,...$$

By setting $s_k = s$ where k = 1,2,... a corresponding result can be obtained for the case where shocks arrive according to a non-stationary Poisson process and by setting s(t) = 1 the corresponding result for the stationary pure birth process can be obtained.

If {N(t)} is a generalised renewal process, Theorem (2.2.1(a)) applies with the condition :-

$$\prod_{i=k+1}^{\infty} F_i(x) =< \prod_{i=1} F_i(x)$$

in place of the condition that $(\bar{P}_k)_{k=0}^{\infty}$ is discrete NBU.

The special case where the shock magnitudes X_i are i.i.d is of some independent interest.

In this case, if the magnitudes have common distribution F and the critical threshold below which shocks cause no damage and above which failure occurs is denoted by z , the probability of surviving k shocks is given by :-

(3.1.2) $$\bar{P}_k = (F(z))^k.$$

Consequently, if shocks arrive according to a homogeneous Poisson process of rate s the survivor function of the device in question is given by :-

$$\begin{aligned}\bar{H}(t) &= \Sigma_{k=0}^{\infty} P(N(t)=k)\ \bar{P}_k \\ &= \Sigma_{k=0}^{\infty} e^{-st} \frac{(st)^k}{k!} (F(x))^k \\ &= e^{-st}(1-F(x))\end{aligned}$$

That is, the lifetime distribution for the maximum shock threshold model with i.i.d shock magnitude is exponential whenever the process governing the arrival of the shocks is a homogeneous Poisson process. More formally, we have :-

Theorem (3.1A.2)

Let $1-H(t) = \bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k)\bar{P}_k$ where $\bar{P}_k = [F(x)]^k$ then H is exponential if $\{N(t)\}$ is a homogeneous Poisson process.

Since the survival probabilities $\bar{P}_k = (F(x))^k$ are both discrete IFR and discrete DFR, we have in a similar vein to Theorem (3.1.2) above, the following results which follow directly from Theorem (2.1A.1) corollary (2.1D.1) and Theorem (2.1C.1) respectively.

Theorem (3.1A.3)

Let $1-H(t) = \overline{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k)(F(z)^k$.
If $\{N(t)\}$ is a stationary pure birth process with birth coefficients $(s_k)_{k=0}^{\infty}$ then H is IFR (DFR) if $(s_k)_{k=0}^{\infty}$ is increasing (decreasing).

Theorem (3.1A.4)

Define H(t) by :-

$$\bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k)(F(z))^k$$

Let $\{N(t)\}$ be a non-stationary Poisson process with (mean-value function $I(t) = \int_0^t s(x)dx$ then :-

(a) H is IFR (DFR) if $s(t)$ is increasing (decreasing)

(b) H is IFRA (DFRA) if $I(t)/t$ is increasing (decreasing)

(c) H is NBU (NWU) if $I(t)$ is superadditive (subaddditive).

<u>Theorem (3.1A.5)</u>

Define H by :-

$$\bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k)\ (F(z)^{k}$$

and let {N(t)} be a non-stationary pure birth process with :-

$$P(N(t+d)-N(t) = 1 \mid N(t)=k) = s_k s(t)d + o(d)$$

and $$P(N(t+d)-N(t) > 1 \mid N(t)=k) = o(d),$$

where d is small.

then :-

(a) H is IFR (DFR) if $(s_k)_{k=0}^{\infty}$ is increasing (decreasing) and s(t) is increasing (decreasing) in t.

(b) H is IFRA (DFRA) if s_k is increasing (decreasing) in k and $\int_0^t s(x)dx/t$ is increasing (decreasing) in t.

(c) H is IFRA (DFRA) if s_k is increasing (decreasing) and s(x)dx is superadditive (subadditive).

In the case that {N(t)} is a generalised renewal process we have the following result directly from Theorems (2.2.1) and (2.2.2) :-

<u>Theorem (3.1A.6)</u>

Define H(t) by :-

$$\bar{H}(t) = 1-H(t) = 1-\Sigma_{k=0}^{\infty} P(N(t)=k)\ (F(z))^{k}\ ,\ z > 0$$

where {N(t)} is a generalised renewal process; then :-

(a) H is NBU if the shock inter-arrival time distributions $(J_i(t))_{i=1}^{\infty}$ are increasing (decreasing) in i for each t and NBU (NWU)

(b) H is NBUE (NWUE) if the inter-arrival time distribution $(J_i(t))_{i=1}^{\infty}$ are NBUE (NWUE) and their first moments A_i, i = 1,2,... form a decreasing (increasing) sequence.

If we impose the additional condition that the interarrival times all have the same first moment, then a similar result to the one above holds for the HNBUE and L Classes (c.f. Corollary (2.2.1))

Esary, Marshall and Proschan (1973) identified the following practical application of Theorem (3.1A.1(a)) in the case that the interarrival times are i.i.d. Suppose that each shock causes a change in the threshold which does not depend on the shock magnitudes. If the successive thresholds are denoted by z_1, $z_2,\ldots$ and F is the common magnitude distribution then $\bar{P}_k = \prod_{i=1}^{k} F(z_i)$ which is very similar to the form $\bar{P}_k = \prod_{i=1}^{k} F_i(z)$.

In fact, Theorem (3.1A.1) applies with $F_i(z)$ replaced by $F(_z)$ e.g., in the special case that $s_k = s$, $k = 0,1,2,\ldots$ and $s(t) = 1$ so that shocks arrive according to a homogeneous Poisson process, it follows from Theorem (3.1A.1(a)) that if successive shocks cause a lowering (raising) of the threshold i.e. $(z_i)_{i=1}^{\infty}$ decreasing (increasing) then the life distribution H is IFR (DFR).

One possible generalisation of the Maximum Shock Threshold Model is to the case where the threshold level is itself a random variable. Suppose the threshold level has a distribution G and retaining the assumption of mutually independent shock magnitudes, which are independent of the shock interarrival times, it is clear that the probability of surviving k shocks is given by :-

(3.1.3) $$\bar{P}_k = \int_0^{\infty} \prod_{i=1}^{k} F_i(z)\, dG(z)$$

and the survivor function for this random threshold model is given by :-

(3.1.4) $$\bar{H}(t) = \int_0^{\infty} \sum_{k=1}^{\infty} P(N(t)=k) \prod_{i=1}^{k} F_i(x)\, dG(z) + P(N(t)=0).$$

Now by the closure properties of the DFR, DFRA and NWU Classes under the mixture operation, we have the following ccrollary to Theorem (3.1.1).

Corollary (3.1A.1)

Let {N(t)} be a non-stationary pure birth process with transition probabilities as in Theorem (3.1.1), then if H is defined by (3.1.4) :-

(a) H is DFR if s(t) decreasing $(s_k)_{k=0}^{\infty}$ is decreasing and $F_i(z)$ is increasing in i for every z > 0.

(b) H is DFRA if $\int_0^t s(x)dx/t$ is decreasing in t, $(s_k)_{k=0}^{\infty}$ is decreasing in k and :-

$$(\prod_{i=1}^{k} F_i(x))^{1/k} \leq F_{k+1}(z) \qquad k = 1,2,\ldots, \; z > 0.$$

(c) H is NWU if $\int_0^t s(x)dx$ is subadditive $(s_k)_{k=0}^{\infty}$ is decreasing in k and:-

$$\prod_{i=k+L}^{\infty} F_i(z) \geq \prod_{i=1}^{L} F_i(z)$$
$$k = 1,2,\ldots, \; L = 1,2,\ldots, \; z > 0$$

and none of the functions $\sum_{k=1}^{\infty} P(N(t)=k)$, $\prod_{i=1}^{k} F_i(z)$, cross where z > 0

In the special case that the shock magnitudes are i.i.d. similar corollaries to Theorem (3.1A.2) - (3.1A.6) can be obtained. In particular, we note that if {N(t)} is a homogeneous Poisson process :-

$$\bar{H}(t) = \int_0^{\infty} \sum_{k=1}^{\infty} e^{-st} (st)_k/k! \prod_{i=1}^{k} F_i(s) \, dG(x) + P(N(t)=0)$$
$$= \int_0^{\infty} e^{-st(1-F(x))} \, dG(x) + P(N(t)=0)$$

and consequently, since a mixture of exponential distribution is logarithmically convex, then H is logarithmically convex, i.e., log H is a convex function i.e., H is DFR.

~3.1B <u>A Maximum Shock Threshold Model with Shock Magnitudes Correlated with the Preceding Interarrival Time</u>

A more interesting type of maximum shock threshold model arises if the shock magnitudes X_i i = 1,2,... are allowed to depend on the process {N(t)} which governs the arrival of shocks, or, more particularly, on the interarrival times of the process. In this case, the model :-

$$\bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k)\bar{p}_k$$

studied in Chapter two and section 3.1A may not be applicable. Consequently, we shall study the lifetime distribution of this more general maximum shock threshold model by using a direct analysis of the system-failure time, T_z, although it will be shown that $\bar{H}_z(t) = P(T_z > t)$ can be written in the form :-

$$\bar{H}_z(t) = \Sigma_{k=0}^{\infty} P(N_z(t)=k)\bar{p}_k$$

where $\{N_z(.)\}$ is a stochastic process related to the process governing the arrival of shocks.

The models discussed in this and the following section have been studied extensively by Shanthikumar and Sunita (1983, 1984) and the following discussion is based largely on their work.

We will first consider the case where the magnitude of the n^{th} shock, X_n, is correlated with the time between the $(n-1)^{th}$ and n^{th} shocks, Y_n.

We will suppose that the random vectors (X_n, Y_n) are independent pairwise and have a common joint distribution function given by :- $F_{X,Y}(x,y) = P[X_n =< x, Y_n =< y]$ where n = 0,1,2...

For each n, X_n and Y_n may be correlated Shanthikumar and Sumita (1983, 1984) term such a sequence of random vectors a correlated pair of renewal sequences. If z is the threshold below which shocks cause no damage and above which shocks cause immediate failure, then the lifetime distribution of the device in question is given by:-

(3.1.5) $H_z(t) = P(T_z =< t) = P(M(t) > z)$
where $M(t) = \max_{(0 =< j =< N(t))} (X_j)$
and the system failure time T_z is given by :-

(3.1.6) $T_z = \inf\{t : M(t) >= z\}$.

Although our main aim is to establish sufficient conditions for the distribution of T_z to belong to a particular class of life distribution, we will also present some results concerning the moments of T_z and the limiting properties of its distribution. To this end, the following notaion will prove useful. Let :-

(3.1.7) $V(z,t) = P(M(t) =< z)$
thus :-
(3.1.8) $H_z(t) = 1 - V(z,t)$

We will denote the Laplace transforms of $V(z,t)$ and $H_z(t)$ by $V^*(z,s)$ and $H_z^*(s)$ respectively. Throughout the following discussion, we will assume that for each n the random variables X_n and Y_n have finite first and second moments and are absolutely continuous with joint probability density function given by :-

$$f_{X,Y}(x,y) = \frac{d^2}{(dxdy)} F_{X,Y}(x,y)$$

The corresponding marginal distribution survivor and density functions are given by :-

$$F_X(x) = F_{X,Y}(x,\infty) \; ; \; F_Y(y) = F(\infty,y)$$
$$\overline{F}_X(x) = 1-F_X(x) \; ; \; F_X(y) = 1-F_Y(y)$$
$$f_X(x) = \int_0^\infty f_{X,Y}(x,y)dy \; ; \; f_Y(y) = \int_0^\infty f_{X,Y}(x,y)dx$$

For this model it will also be assumed that $X_0 = Y_0 = 0$ w.p.1.

It will prove convenient to define the following functions:

(3.1.1) $G_X(x,y) =^{df} \int_0^x f_{X,Y}(t,y)dt,$
$G_Y(x,y) =^{df} \int_0^y f_{X,Y}(x,t)dt,$
$\overline{G}_X(x,y) =^{df} \int_x^\infty f_{X,Y}(t,y)dt,$
and $\overline{G}_Y(x,y) =^{df} \int_y^\infty f_{X,Y}(x,t)dt$

Loosely speaking $G_X(x,y)$ is the probability that a shock occurring a time y since the last shock is of magnitude less than or equal to x, and $G_X(x,y)$ is the probability that a shock of magnitude x occurs following an interval of length smaller than or equal to y since the previous shock.

Note that :-

(3.1.10) $\bar{G}_X(x,y) = f_Y(y) - G_X(x,y)$

and

(3.1.11) $\bar{G}_Y(x,y) = f_X(x) - G_Y(x,y)$

The results concerning the moments of T_z will be established using Laplace transform techniques and, consequently, the following Lemma will prove useful.

<u>Lemma (3.1B.1)</u>

Let $V(z,t)$ and $H_z(t)$ be defined as in (3.1.7) and (3.1.5) and let $H_z(t) = d/dt\, H_z(t)$ then :-

a) $V^*(z,s) = \dfrac{1 - f_Y^*(s)}{s(1-G_X^*(z,s)}$ Re(s) >= 0

b) $h^*(s) = \dfrac{\bar{G}_X^*(z,s)}{1-G_X^*(z,s)}$ Re(s) >= 0

<u>Proof</u>

a) $V(z,t) = \mathcal{P}(M(t) =< z)$. Conditioning on the first renewal time Y_1 gives :-

(3.1.12) $V(z,t) = \int_0^\infty P(M(t) =< z \mid Y_1 \in (y,y+d)\, f_Y(y)dy$

and using the regenerative property of the paired process $\{(X_n,Y_n)\}$ at Y_1 we have :-

(3.1.13) $V(z,t) = F_Y(t) + \int_0^t V(z,t-y)\, G_X(z,y)dy.$

Now, taking Laplace transforms on both sides of (3.2.13) yields :-

$$V^*(z,s) = 1/s - (f^*_Y(s))/s + \int_0^\infty \int_0^t e^{-st} V(z,t-y)\, G_X(z-y)\, dydt$$

$$= 1/s\ (1-f^*_Y(s)) + \int_0^\infty \int_y^\infty e^{-st} V(z,t-y)\, G_X(z,y)\, dtdy$$

$$= 1/s\ (1-f^*_Y(s)) + \int_0^\infty \int_0^\infty e^{-s(t-y)} V(z,t)\, G_X(z,y)\, dtdy$$

$$= 1/s\ (1-f^*_Y(s)) + G^*_X(z,s)\, V^*(z,s)$$

hence
$$V^*(z,s) = \frac{1-f^*_Y(s)}{s(1-G^*_X(z,s))}$$
as required.

b) Now, $H_z(t) = 1-V(z,t)$

hence $H_z^*(s) = 1/s - V^*(z,s)$

and
$$h_z^*(s) = sH_z^*(s)$$

$$= 1 - \frac{1-f^*_Y(s)}{1 - G^*_X(z,s)}$$

$$= \frac{f_Y^*(s) - G_X^*(z,s)}{(1-G_X^*(z,s))}$$

$$= \frac{\bar{G}_X^*(z,s)}{1-G_X^*(z,s)}$$
by (3.1.10).

$h_z^*(s)$ is of course the Laplace Transform of the density of T_z and consequently by differentiating $h_z^*(s)$ and equating s to zero, the moments of T_z can be obtained. Thus we have the following result :-

Theorem (3.1B.1)

a) $E(T_z) = E(y) / \bar{F}_X(z)$

b) $$E(T_z^2) = \frac{E(y^2)}{\bar{F}_X(z)} + \frac{2F_X(z)E(y)}{(\bar{F}_X(z))^2} E(Y_1 X < Z)$$

c) $Var(T_z) =$

$$\frac{E(y^2)}{\bar{F}_X(z)} + \frac{E(y)}{(\bar{F}_X(z))^2} (2F_X(z)\ E(Y \mid X < z) - E(Y))$$

The above theorem and proceeding lemma are both due to Shanthikumar and Sumita (1983). In the same paper, it is shown that as $z \to \infty$ (i.e. as the threshold becomes large) :-

$$P\,(T_z/E(T_z)\ \geq\ t)\ \to\ e^{-t}$$

provided $0 < F_{X,Y}(x,y) \leq 1$ for $0 < x < \infty$, $0 < y < \infty$.
i.e. the distribution of $T_z/E(T_z)$ approaches that of an exponential variate of mean one as $z \to \infty$.

One further aspect of the distribution of T_z worth noting is that as the threshold z becomes large, the probability of a system failure tends to zero since the probability of any shock magnitude exceeding z decreases as z increases. So for some fixed t, $A_z(t) = P(T_z < t)$ is increasing in z.

This is formalised in the following theorem, but first a definition is required.

<u>Definition (3.1B.1)</u>

A random variable X is stochastically larger than a random variable Y

$$(X \geq Y) \text{ if } P(X>t) \geq P(Y>t)\quad -\infty < t < \infty.$$

<u>Theorem (3.1B.2)</u> (Shanthikumar and Sumita (1983))

T_z as defined by (3.1.6) is stochastically increasing in z,
i.e. $0 \leq z_1 \leq z_2 \implies T_{z1} \leq_{st} T_{z2}$

<u>Proof</u>

Inverting the Laplace transform :-

$$h_z^*(s) = \frac{\bar{G}_X^*(z,s)}{1-G_X^*(z,s)}$$

gives the following real-domain form for $h_z(t)$:-

(3.1.14) $$h_z(t) = \bar{G}_X(z,t) + \bar{G}_X(z,t) * \Sigma\, G_X^{(k)}(z,t)$$

where $$G_X^{(k+1)}(z,u) = \int_0^t G_X(z,t-u)\, G_X^{(k)}(z,u)\,du$$

and the asterisk denotes similar convolution in t.

Now

$$P(T_z >= t) = \bar{H}_z(t) = \int_t^\infty h_z(u)du$$

and it follows that :-

$$\bar{H}_z(t) = \bar{F}_x(t) + \bar{F}_x(t) * \Sigma_{k=1} G_x^{(k)}(z,t).$$

Clearly, $\bar{H}_z(t)$ is non decreasing in z i.e. $T_{z1} =< T_{z2}$ whenever $z1 =< z2$.

We now turn our attention to the question of determining the class of distributions to which $H_z(t)$ belongs. The general aim is to establish conditions on the distributions of X_n and Y_n and/or on the bi-variate distribution of (X_n, Y_n) which guarantee that $H_z(t) = P(T_z =< t)$ belongs to a particular class of distributions.

Definition (3.1B.2) **(Barlow and Proschan (1975) p. 142)**

For given random variables X and Y, X is said to be stochastically increasing in Y (SI(X | Y)) if P (X>x | Y=y) is increasing in y for all x.

Similarly, if P(X>x | Y=y) is decreasing in y for all x we say X is stochastically decreasing in y (SD(X | Y))

Definition (3.1B.3) (Barlow and Proschan (1975) p. 142)

For given random variables X and Y, X is said to be Right-tail increasing in y (RTI(X | y)) if P(X>x | Y>y) is increasing in y for all x.

X is right tail increasing in Y (RTD (X | y)) if P(X>x | Y>y) is decreasing in y for all x.

In establishing conditions under which $H_z(t)$ belongs to the NBUE, NWUE, HNBUE and HNWUE Classes, the function

(3.1.15) $E_x(y,t) =^{df} P(X>x \mid Y>y+t) - P(X>x \mid Y>y)$

$x,y,t >= 0$

(3.1.16) $$= \frac{F_x(x)-F_{x,Y}(x,y)}{\bar{F}_Y(y)} - \frac{F_x(x)-F_{x,Y}(X,y+t)}{\bar{F}_Y(y+t)}$$

is of some importance. It is clear that :-

(3.1.17) $RTI\ (X \mid Y) \iff E_x(y,t) >= 0$; $x,y,t >= 0$

(3.1.18) $RTD\ (X \mid Y) \iff E_x(y,t) >= 0$ $x,y,t >= 0$

We are now in a position to present the main result for the general maximum shock threshold model. The result is due to Shanthikumar and Sumita (1984).

<u>Theorem (3.1B.3)</u>

Let $(Y_n)_{n=0}^{\infty}$ be the sequence of shock interarrival times, $(X_n)_{n=0}^{\infty}$ the sequence of shock magnitudes and T_x the failure time of a device subject to $(X_n,Y_n)_{n=0}^{\infty}$:-

(a) If the Y_n all have distributions which are NBU and $SI\ (X_n \mid Y_n)$ $n = 0,1,2,\ldots$ then $H_x(t) = P(T_x<t)$ is NBU.

(b) If the Y_n all have distributions which are NBUE and for X_n,Y_n $E_x(0,t) >= 0$ for all $x,t >=0$; $n = 0,1,2,\ldots$ then $H_x(t)$ is NBUE

(c) If the Y_n all have distributions which are HNBUE and for X_n,Y_n $E_x(0,t) >= 0$ for all $x,t,n >= 0$ then $H_x(t)$ is HNBUE

<u>Proof</u>

(Note since this proof is quite involved many of the details have been ommitted here but can be found in Shanthikumar and Sumita (1984)

a) We have to show that :-

$$A_x(t+s) =< A_x(t)\ \bar{A}_x(s) \quad ; \quad s,t >= 0$$

i.e. $$P(T_x > t+s \mid T_x > t) =< P(T_x>s); \quad s,t >= 0$$

The following notation will prove useful :-

Let :- (3.1.19) $R_L = 0$; $L = 0$

$R_L = \Sigma_{j=1} Y_j$; $L >= 1$

and let $\mu_L = t - r_L$

Further, let :-

(3.1.20) $P(s,u) = P(T_z > s+u \mid u<Y_1)$

Now by using the regenerative property of the renewal sequence $(Y_n)_{n=0}^{\infty}$ together with some straight-forward computation (see Shanthikumar and Sumita (1984)) it proves possible to write :-

$$\frac{\bar{H}_z(s+t)}{\bar{H}_z(t)} = P(T_z > s+t \mid T_z > t)$$

as follows :-

(3.1.27)
$$\frac{\bar{H}_z(s+t)}{\bar{H}_z(t)} = P(s,t)\, F_Y(t) + \Sigma_{i=1} P(N(t)=L \mid T_z > t)$$
$$X \int_0^t P(s,\mu_L)\, dP(\Sigma_{j=1} Y_j =< r_L \mid T_z > t,\ N(t)=L]$$

By (3.1.27) it is clear that the desired result holds provided it can be shown that $P(s,t) =< \bar{H}_z(s)$ for any $t>0$ This last step is established via a rather detailed argument which is omitted. We note however that the argument relies on the regenerative and NBU properties of the renewal sequence $(Y_n)_{n=0}^{\infty}$ and the fact that X_n is stochastically increasing in Y_n, $n = 0,1,2,...$ i.e. $SI(X_n \mid Y_n)$

(b) To show that $H_z(t)= P(T_z < t)$ is N.B.U.E. we must establish that :-

$$\frac{\int_0^{\infty} \bar{H}_z(s+t)ds}{\bar{H}_z(t)} =< E(T_z)$$

By (3.1.27) it suffices to show that :-

$$\int_0^{\infty} P(s,t)ds =< E(T_z) \quad \text{for any } t > 0$$

where $P(s,t)$ is as defined by (3.1.20) and this follows by using the conditions of the theorem :-

$$E_X(0,t) = \frac{F_X(z) - F_X(z)\, F_{X,Y}(z,t)}{\bar{F}_Y(t)} >= 0$$

and $F_Y(.)$ is NBUE : and by noting that :-

$$E(T_z) = E(Y)/\bar{F}_X(z) \qquad \text{by Theorem 3.1B.1(a)}$$

(c) To show that $H_z(t) = P(T_z =< t)$ is HNBUE we must show that $H_z(t)$ satisfies the inequality :-

$$\int_t^{\infty} H_z(s)ds =< E(T_z) \exp[-t \mid E(T_z)]$$

now conditioning on X in (3.1.13) gives :-

(3.1.33) $\bar{H}_z(t) = \bar{F}_Y(t) + F_X(z) \int_0^t H_z(t-y)\, dF_z(y)$

where $F_z(y) = P(Y =< y \mid X =< z)$

Repeated substitution of (3.1.33) into the H_z of the right hand side of (3.1.33) yields :-

(3.1.34) $\bar{H}_z(t) = \Sigma_{n=\infty} (\bar{F}_z^{(n)}(t)\ \bar{F}_Y(t) -$
$\bar{F}_z^{(n)}(t))(F_X(z))^n$

where * denotes convolution and $F_z^{(n)}(t)$ the nth fold convolution of $F_z(t)$ with itself, the result follows by integrating both sides of (3.1.34) with resect to t over the range (t,∞), and by using the conditions of the Theorem via two technical lemmas due to Shanthikumar and Sumita (1984) (Lemmas 1.0.1 and 1.0.2)

It is interesting to observe that the expression obtained for the survivor function in (3.1.34) is of the form :-

$\Sigma_{n=\infty}^{\infty} P(N^*(t)=n)\ \bar{p}_n$ with the expression in brackets on the right hand side of (3.1.34) being the : P(exactly n shocks of magnitude less than or equal to z occur in time t) $= P(N^*(t)=n)$

The following Corollary to Theorem (3.1B.9) follows directly from (3.1.16)

Corollary (3.1B.1)

a) If for n = 1,2,... Y_n has an NBUE distribution and
$RTI(X_n \mid Y_n)$ $H_z(t) = P(T_z < t)$ is NBUE.

b) If for n = 1,2,..., Y_n has a HNBUE distribution and
$RTI(X_n \mid Y_n)$ item $H_z(t)$ is HNBUE.

We have seen that for the NBU, NBUE and HNBUE Classes of life distributions, the distribution H_z inherits its class from the distribution of the shock interarrival times provided certain conditions on the paired sequence (X_n, Y_n) are satisfied. In an exactly analagous fashion, corresponding results can be established in the NWU, NWUE and HNWUE cases. For completeness, these results are presented below.

Theorem (3.1B.4)

Let $(Y_n)_{n=0}^{\infty}$ be a sequence of shock interarrival times (with $(Y_0 = 0)$), $(X_n)_{n=0}^{\infty}$ be the shock magnitude and let T_z be the failure time of a device subject to $(X_n, Y_n)_{n=0}^{\infty}$ then we have :-

(a) If the $(Y_n)_{n=0}^{\infty}$ all have distributions which are NWU
and $SD(X_n \mid Y_n)$ $n = 0,1,2,\ldots$ then
$H_z = P(T_z =< t)$ is NWU.

(b) If the $(Y_n)_{n=0}^{\infty}$ all have distributions which are NWUE
and $E_x(0,t) =< t$ for every x and $t >= 0$
then $H_z(t) = P(T_z =< t)$ is NWUE.

(c) If the $(Y_n)_{n=0}^{\infty}$ all have HNWUE distribution and
$E_x(0,t) =< 0$ for all x and $t >= 0$ then
H_z is HNWUE for every $z > 0$

Corollary (3.1B.2)

(a) If the Y_n all have NWUE distributions and
$RTD(X_n \mid Y_n)$ $n = 0,1,2,\ldots$
then H_z is NWUE for every $z > 0$

(b) If the Y_n all have HNWUE distributions and
$RTD(X_n \mid Y_n)$ $n = 0,1,2,\ldots$ then H_z is HNWUE
for every $z > 0$

*3.1C A Maximum Shock Threshold Model with Shock Magnitudes Correlated with the succeeding Inter-arrival Times

The model just considered has dealt with the case where the magnitude of the nth shock is correlated with the length of the interval between the $(n-1)^{th}$ and n^{th} shock. In this section we consider the situation where the nth shock affects the length of the following interval. This model is closely related to the one just studied but some important differences do arise. For example, the lifetime distribution H_z has a mass $F_x(z) = P(X_0 > z)$ at the origin because of the zero'th shock, whereas in the previous model we were able to assume that $X_0 = Y_0 = 0$ with probability 1. Another difference is that the successive failure times T_z form a renewal sequence for the model discussed in section 3.1B but that is not the case for the current model.

Despite these differences, results very similar to those obtainedin section 3.1B can be obtained for the moments of the distribution of T_z and for the Class to which it belongs.

We will see that as in the proof of Theorem (3.1B.3(c)) it proves possible to write the survivor function $H_z(t)$ in the form :-

$$\bar{H}_z(t) = \Sigma_{k=0}^{\infty} P(N_z(t)=k)\ \bar{P}_k$$

for some process $\{N_z(.)\}$, and consequently a result from Chapter two can be applied to determine the Class of distributions to which $H_z(.)$ belongs.

Firstly, however, some results concerning the moments of the distribution of T_z are presented. Laplace transform techniques are again used and consequently the following Lemma is of some importance. (All notation is the same as in section 3.1B).

Lemma (3.1C.1)

Let $V(z,t) = (P(M(t) < z)$

$H_z(t) = P(T_z =< t) = P(M(t) >= z)$

and $h_z(t) = d/dt\ H_z(t)$

then

(a) $$V^*(z,s) = \frac{F_X(z) - G_X^*(z,s)}{s(1-G_X^*(z,s)}$$

and

(b) $$h_z^*(s) = \frac{\bar{F}_X(z)}{1-G_X^*(z,s)}$$

Proof

As in the proof of Lemma (3.1B.1) we condition on the first renewal time, Y_0 in this case, and use the regenerative property of the paired process (X_n, Y_n) to obtain :-

(3.1.35) $$V(z,t) = \int_t^\infty G_X(z,y)dy + \int_0^\infty G_X(z,y)V(z,t-y)dy$$

(3.1.36) $$= F_X(z) - F_{X,Y}(z,t) + \int_0^t G_X(z,y)V(z,t-y)dy$$

and taking Laplace transforms of both sides of (3.1.35) yields the desired result. Part (b) of the lemma follows from the fact that : $h_z^*(s) = 1 - s\ V_z^*(s)$

Now as for Theorem (3.1B.1), the corresponding result in section 3.1B, differentiating $h_z^*(s)$ at $s = 0$ yields the following result :-

Theorem (3.1C.1)

a) $$E(T_z) = E(Y \mid X < z)\ \frac{F_X(z)}{\bar{F}_X(z)}$$

b) $$E(T_z^2) = E(Y^2 \mid X < z)\ \frac{F_X(z)}{\bar{F}_X(z)} + 2(E(T_z))^2$$

c) $$\text{Var}\ T_z = E(Y^2 \mid x < z)\ \frac{F_X(z)}{\bar{F}_X(z)} + (E(T_z))^2$$

Note that the condition that $X < z$ in the above theorem is really a condition on the zeroth shock which ensures that the lifetime, T_z, is strictly positive. Hence, in studying this model, the conditional lifetime $T_z \mid X_0 < z$ is of some importance. We will denote the conditional lifetime by T_z^+.

We note also that inverting $h_z^*(s)$ yields the following real-domain form for $h_z(t)$

(3.1.38) $$h_z(t) = \overline{F}_X(z) \, (\delta(t) + \Sigma_{k=1} G_X^{(k)}(z,t))$$

(where δ is the Diriac delta function) and analagously to Theorem (3.1.8) we have :-

<u>Theorem (3.1C.2)</u>

T_z is stochastically increasing in z :-

i.e. $0 =< z1 =< z2 ==> T_{z1} =<_{st} T_{z2}$

<u>Proof</u> as for Theorem (3.1B.2)

The above results are due to Shanthikumar and Sumita (1983) who also established that under the conditions :-

$$0 < F_{X,Y}(x,y) < 1 \quad \text{for } 0 < x < \infty, \; 0 < y < \infty$$
$$\text{and } E(Y) < \infty$$
$$P(T_z \,/\, E(T_z) > t) \;->\; e^{-t} \quad \text{as } z -> \infty$$

Thus it is clear that the moment and limiting properties of the lifetime distribution of the current model are analogous to those of the model in which shock magnitudes are correlated with the length of the interval preceding the shocks, studied in section 3.1B.

We turn now to the question of establishing sufficient conditions for $H_z(t)$ to belong to particular classes of lifetime distribution. We first note that :-

(3.1.39) $H_z(t) = P(T_z =< t)$
$= P(T_z =< t \mid T_z >= 0)\ F_x(z) + \bar{F}_x(z)$

and hence H_z belongs to the same class as H_z^+ , the distribution of the conditional lifetime T_z^+ Therefore, it suffices to study the conditional lifetime H_z^+.

The following lemma shows that it is possible to write $H_z^+(t)$ in the form : $\bar{H}_z^+(t) = \Sigma_{k=0}^{\infty} P(N_z(t)=k)\ \bar{p}_k$ for some process $N_z(.)$ and sequence of survival probabilities $(\bar{p}_k)_{k=0}$.

<u>Lemma (3.1C.2)</u>

Let Y_z have a distribution $F_z(y) = P(Y < y \mid X < z)$ with corresponding density function :-

$$f_z(y) = \frac{G_x(z,y)}{F_x(z)}$$

then

$$\bar{H}_z^+(t) = P(T_z^+ > t)$$
$$= \bar{F}_x(z) \quad \Sigma_{k=0}\ F_z^{(k+1)}(t)\ (F_x(z))^k$$

<u>Proof</u>

$$h_z^{+*}(s) = (h_z^*(s) - \bar{F}_x(z)) \ / \ F_x(z)$$
$$= \frac{\bar{F}_x(z)}{F_x(z)} \times \frac{G_x^*(z,s)}{1-G_x^*(z,s)} \qquad Re(s) >= 0$$

by Lemma (3.1C.1(b))

$$= \frac{\bar{F}_x(z)\ f_z^*(s)}{1-F_x(z) f_z^*(s)} \qquad Re(s) >= 0$$

inverting gives

$$h_z^+(t) = \bar{F}_x(z) \quad \Sigma_{k=0}\ f_z^{(k+1)}(t)(F_x(z))^k$$

and interpreting both sides of the above yields the desired result.

Now, if $\{N_x(t)\}$ is the counting process associated with the sequence of intervals $(Y_{x,n})_{n=0}^{\infty}$ so that :-

$$P(N_x(t)=k) = (\bar{F}_x(t) \qquad k = 0$$
$$= (F_x^{(k+1)}(t) - F_x^{(k)}(t) \ ; \ k >= 1$$

then it is clear from (3.1.40) that :-

(3.1.41) $\quad H_x^{+}(t) = \Sigma_{k=0} P(N_x(t)=k) \ (\bar{F}_x(z))^k$

and since $\bar{P}_k = (F_x(z))^k$ is discrete IFR and consequently discrete NBU, discrete NBUE, discrete HNBUE and also a member of the Class G the following result follows from Corollary (2.21) and equations (3.1.39) and (3.1.41) since the intervals between the events of $\{N(t)\}$ are independent.

<u>Theorem (3.1C.3)</u>

(a) If $F_x(t)$ as defined in Lemma (3.1C.2) is NBU (NWU) then H_x^{+} is NBU (NWU) and consequently so is H_x.

(b) If $F_x(t)$ is NBUE (NWUE) then H_x^{+} is NBUE (NWUE) and consequently so is H_x.

(c) If $F_x(t)$ is HNBUE (HNWUE) then H_x^{+} is HNBUE (HNWUE) and consequently so is H_x.

(d) If $F_x(t)$ belongs to L (Γ) then so does H_x^{+}.

Proof

Directly from corollary (2.2.1) and (3.1.39) and (3.1.41) since the intervals $\{Y_{x,n}\}_{n=0}^{\infty}$ are i.i.d. and hence have constant mean.

Note that for this model the lifetime distribution inherits its class not from the class of the interarrival times of the process governing the arrival of the shocks but from the interarrival times of a related process. We should also remark that writing H_x^* in the form :-

$$\Sigma_{k=0}^{\infty} P(N_x(t)=k)\bar{p}_k$$

has allowed us to extend the result concerning H_x's class to the L and $\bar{L}$ Classes. This extension is not so easily made for the model of section 3.1B since although we saw in the proof of Theorem (3.1B.3(c)) that the survivor function of that model could be written as :-

$$\bar{H}_x(t) = \Sigma_{k=0}^{\infty} P(N^*(t)=k)\bar{p}_k$$

for some counting process $N^*(t)$ the expression $P(N^*(t)=k)$ is quite complicated. Further, it is not clear that an independent sequence of intervals corresponding to $\{N^*(t)\}$ exists. If however, a sequence of i.i.d. interarrival times corresponding to the counting process N^* can be found then Corollary (2.2.1) could again be applied and not only could Theorem (3.1C.3) be extended to include the L Class but the proofs of parts (a), (b) and (c) could be greatly simplified. In fact, the sequence of interarrival times need not be i.i.d , all that is required is that their first moments be identical.

To conclude this section, it should be noted that the first three parts of the previous theorem are presented in Shanthikumar and Sumita (1984) but the extension to the L and $\bar{L}$ Classes is new.

~3.1D Minimum Shock Threshold Models

The previous three sections, 3.1A, 3.1B and 3.1C, have dealt with models where system failure occurred on the first occurrence of a shock of magnitude greater than some critical threshold value. In an analogous fashion, some systems or devices may fail the first time the magnitude of a shock falls below some critical level, e.g., consider a stochastic clearing system in which constant demand occurs after random time intervals.— If the accumulated quantity reaches the demand level before the occurrence of such an epoch, production is terminated and the demand fulfilled. If, on the other hand, the accumulated quantity is insufficient to fill the demand, a severe penalty may be imposed. In this case, the shock magnitudes $(X_n)_{n=0}^{\infty}$ are the accumulated quantities at the random times of demand $(Y_n)_{n=0}^{\infty}$ Rather than failure time, we are interested in the time until a penalty is imposed. If $\{N(t)\}$ is the counting process associated with the renewal sequence $(Y_n)_{n=0}^{\infty}$ and

$$M(t) = \text{Min}_{0<j<N(t)} \{Xj\}$$

then the time to imposition of a penalty, T_i , say, is given by :-

$$T_i = \inf \{t:m(t) =< i\}$$

The results of sections 3.1A, 3.1B and 3.1C concerning the distribution of the failure time T_x of the maximum shock threshold model carry over in an obvious way to the distribution of T_i.

~3.2 Cumulative Damage Shock Models

In this section, the lifetime distribution of a device subject to shocks which each cause a random amount of damage is considered. The damages accumulate and the device fails when the accumulated damage exceeds some critical threshold. The threshold may be fixed or it may be regarded as varying randomly.

Note that in the previous section it was the shock magnitudes which were considered important and shocks inflicted no damage unless their magnitude exceeded some critical level. By contrast, it is assumed in this section that each shock causes damage and it is the amount of damage caused that is crucial in analysing the life distribution of the device subject to shock.

~3.2A The Standard Cumulative Damage Model

Initially, we return to the model :-

(3.2.1) $\bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k)\bar{p}_k$

where $\{N(t)\}$ is the stochastic counting process governing the arrival of shocks and $\bar{p}_k$ is the probability of surviving k shocks. In view of the results of Chapter two, if $\{N(t)\}$ is a homogeneous or non-stationary Poisson process, a stationary or non-stationary pure birth process, or a generalised renewal process, we need only establish sufficient conditions for the $(\bar{p}_k)_{k=0}^{\infty}$ to belong to a discrete class of distributions for the lifetime distribution H(.) to belong to the corresponding continuous class (provided, for the more general processes, that certain conditions on the process are met).

Firstly, however, we must structure the $(\bar{p}_k)_{k=0}^{\infty}$ so as to accommodate a cumulative damage threshold model.

In this section we will let X_i denote the amount of damage inflicted by the i^{th} shock and we will suppose that $F_{X_i}(x) = P(X_i =< x)$ is its distribution function. We will further suppose that damage accumulates additively and that the damages $(X_i)_{i=1}$ are independent of the process $\{N(t)\}$. For a fixed threshold z:-

(3.2.2) P (surviving k shocks) $= P(\Sigma_{i=1} X_i =< z)$ $k >= 1$

$= 1$ $k = 0$

If the X_i are i.i.d. with common distribution F then :-

(3.2.3) $\bar{p}_k = F^{(k)}(z)$ where $F^{(k)}$ is the k-fold convolution of F with itself.

It is possible that successive shocks may become increasingly effective at causing damage even though the damages themselves remain independent. In this case, the X_i while remaining independent can not be assumed to be identically distributed. Thus we have :-

(3.2.4) $p_k = F_1 * F_2 * \dots * F_k(z)$ $k \geq 1$ and $\bar{p}_0 = 1$ where $(F_i(z))_{i=1}^{\infty}$ is decreasing in i for each z.

More generally, the X_i may be neither independent nor identically distributed as would be the case if an accumulation of damage resulted in a loss of resistance to further damage. Thus, in this case, the amount of damage inflicted by any one shock depends to some extent on the damage inflicted by previous shocks. In order to make any progress with this more general cumulative damage model, it is necessary to make the following reasonable assumptions :-

(3.2.5) $P(X_k \leq x \mid X_1, \dots X_{k-1})$ depends on $X_1 \dots X_{k-1}$ only via $Z_{k-1} = X_1 + \dots + X_{k-1}$

(3.2.6) $P(X_k \leq x \mid Z_{k-1} = z)$ is decreasing in $z \geq 0$.

(3.2.7) $P(X_k \leq x \mid Z_{k-1} = z) \geq P(X_{k+1} \leq x \mid Z_{k-1} = z)$, $z \geq 0$, $k = 1,2,\dots$, $Z_0 = 0$ w.p. 1

Condition (3.2.6) says that an accumulation of damage lowers resistance to further damage while (3.2.7) says that for any given accumulation of damage, later shocks are likely to have a more severe effect.

For some fixed threshold, z, conditions (3.2.5.), (3.2.6) and (3.2.7) give :- $\bar{p}_0 = 1$

(3.2.8) $\bar{p}_k = F^{[k]}(z)$

where $F^{[k]}(z) = P(X_1 + X_2 + \dots X_k \leq z)$

$= P(X_k \leq z - x \mid Z_{k-1} = x) \; dF^{[k-1]}(x)$

For each of the three models for the survival probabilities $(\bar{p}_k)_{k=0}$ outlined above Esary, Marshall and Proschan (1973) and Barlow and Proschan (1975, pp. 94-96) have shown that the sequence $(\bar{p}_k)_{k=0}$ is discrete IFRA, i.e. $(\bar{p}_k^{1/k})_k$ is decreasing in k.

Now, it was shown in Chapter two that, in general, the lifetime distribution, H(.) , of a device inherits its class from the survival probabilities ($\bar{P}_k$). Consequently, the result below follows directly from the IFRA property of the $(\bar{P}_k)_{k=0}$, Theorems (2.1A.1), (2.1C.1) and Corollaries (2.1B.1) and (2.1D.1).

Theorem (3.2A.1)

Let H(.) be a life distribution defined by :-

$$\bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k)\bar{P}_k$$

where $(\bar{P}_k)_{k=0}^{\infty}$ satisfies one of (3.2.3), (3.2.4) or (3.2.8) under the assumptions (3.2.5)-(3.2.7) then H is IFRA wherever :-

(a) {N(t)} is a homogeneous Poisson process of rate s,

(b) {N(t)} is a non-stationary Poisson process with mean-value function $I(t) = \int_0^t s(x)dx$ satisfying I(t)/t is increasing in t,

(c) {N(t)} is a stationary pure birth process with birth coefficients $(s_k)_{k=1}^{\infty}$ which are increasing in k,

or

(d) {N(t)} is a non-stationary pure birth process with

$$P\ (\text{shock in } (t,\ t+d)\ |\ k \text{ shocks in } (0,t))$$
$$= s_k s(t)d + o(d)\ ; \quad \text{where } d \text{ is small.}$$

and $(s_k)_{k=1}^{\infty}$ is increasing and $(\int_0^t s(x)dx)/t$ is increasing in t.

If {N(.)} is a generalised renewal process it is still possible to obtain a useful result by recalling that the discrete IFRA class is contained by the discrete NBU, discrete NBUE, discrete HNBUE Classes and the Class G. Hence the next result follows directly from Corollary (2.2.1).

<u>Theorem (3.2A.2)</u>

Let $H(t)$ be defined by :-

$$\overline{H}(t) = \Sigma_k P(N(t)=k)\overline{P}_k \quad \text{where} \quad (\overline{P}_k)_{k=0}^{\infty}$$

satisfies one of (3.2.3), (3.2.4) or (3.2.8) under the assumption (3.2.5)-(3.2.7) and $\{N(t)\}$ is a generalised renewal process with independent inter-arrival times whose distributions have a common first moment, then :-

(a) H is NBUE if the inter-arrival time distributions are NBUE

(b) H is HNBUE if the inter-arrival time distributions are HNBUE

(c) H belongs to L if the inter-arrival time distributions belong to L.

In the NBU case, we have the following result directly from the IFRA property of the $(\overline{P}_k)_{k=0}^{\infty}$ and Theorem 2.2.1(a).

<u>Theorem (3.2A.3)</u>

Let $H(t)$ be defined by $H(t) = \Sigma\, P(N)(t)=k)\overline{P}_k$ where $(\overline{P}_k)_{k=0}^{\infty}$ satisfies one of (3.2.3), (3.2.4) or (3.2.8) under the assumptions (3.2.5)-(3.2.7) and $\{N(t)\}$ is a generalised renewal process with independent inter-arrival times whose distributions $G_k(t)$ are increasing in k for each t then H is NBU if $G_k(t)$ is NBU, $k = 1,2,3,...$

In the above discussion, we have assumed that damage due to shocks accumulates additively. Ross (1981) has considered the more general case where the accumulation of damage is governed by a function which is symmetric and increasing. More formally, if shocks cause a random amount of damage X_i and exactly n shocks occur up to time t the amount of damage sustained by the device up to time t is given by the function :-

(3.2.9) $D_t(x_1,...x_n,0)$ which satisfies the condition :-

$$D_t(x_1,...x_n,0) = D_t(i_1,i_2,...i_n,0)$$

where $i_1,i_2,...i_n$ is a permutation of 1,2,...n and is increasing in each of its arguments.

One example of such a function is :-

$$D_t(x) = \begin{cases} \Sigma x_i & \max =< A \\ \max (101, E & \max > A \end{cases}$$

If the critical threshold above which failure occurs is 100 this suggests that the device fails when any single shock causes damage greater than an amount A or if the sum of all damages exceeds 100.

The survivor function of a device subject to shocks causing damage which accumulates according to an increasing symmetric function $D_t(x)$ is given by :-

(3.2.10) $H(t) = P(D_t(x_1,...X_n;0) < z)$

where z is the (fixed) threshold.

Ross (1981) studies the distribution defined by (3.2.10) in the case that the shocks arrived according to a non-stationary Poisson Process and the relevant result is contained in the following Theorem :-

<u>Theorem (3.2A.4)</u>.

Let H(t) be defined by (3.2.10) where $D_t(x)$ is increasing and symmetric in each of its arguments. Suppose shocks arrive according to a non-stationary Poisson Process {N(t)} and that the damages are independent of the process.

If the mean-value function of {N(t)} , $\int_0^t s(x)dx$, is starshaped i.e., $(\int_0^t s(x)dx)/t$ is increasing in t then H is IFRA.

Proof

The proof requires the notion of an IFRA stochastic process which can be simply defined as a stochastic process $\{X(t)\}$ which has first passage times $T_a = \inf\{t: X(t) > a\}$ which have IFRA distributions.

For our purposes, the following example of such a process will prove useful :-

$$X(t) = \begin{cases} \max(X_1, \ldots X_{N(t)}) & , N(t) >= 1 \\ 0 & N(t) = 0 \end{cases}$$

where X_i is a value associated with the i^{th} event of the process $\{N(t)\}$. Since it is assumed that $\{N(t)\}$ has a mean-value function :-

$$\int_0^t s(x)\,dx$$

which is starshaped in t and the failure rate of :-

$H_a(t) = P(T_a =< t)$ is given by :-

$$r(t) = s(t)\ (1 - G(a))$$

where $G(.)$ is the common distribution of the $(X_i)_{i=1}^{\infty}$, it follows that H_a is IFRA, and consequently $\{X(t)\}$ is an IFRA process.

We will refer to $\{X(t)\}$ as a record process with value distribution G and intensity function s(t). An event of the record process, $\{X(t)\}$ will be said to occur whenever an event of the associated non-stationary Poisson process $\{N(t)\}$ occurs even though the occurrence of an event may not change the value of $\{X(t)\}$. Now, let m be large and fixed and consider m independent record processes $\{X_i(t)\}$ $i = 1, \ldots m$ each having value distribution G and intensity function $s(t)/m$. A Cumulative Damage Shock Model can be generated from these record processes by defining a shock to occur whenever an event from any of the m record processes occurs and by letting the damage inflicted be the value associated with the corresponding Poisson event.

Ross (1979) has shown that if $(X_i(t))_{i=1}^{\infty}$ are m increasing IFRA functions and $\phi(x)$ is an increasing function then $\phi(X_1(t), X_2(t), \ldots X_m(t))$ is also an increasing IFRA function hence $D_{(t)}(X_1,(t), \ldots X_m(t), 0)$ is an IFRA process.

Now, let N be the number of shocks that occur up until the device fails. If the first N shocks all come from different record processes, the time at which the device fails is just the first passage time of the process :-

$\{D_t(X_1(t), X_2(t), \ldots X_n(t), 0)\}$ to the threshold, z.

Since we have shown that $\{D_t(X_1(t), -- X_n, (t), 0)\}$ is an IFRA process, this first passage time must have an IFRA distribution. Now, the probability that all shocks up until failure come from different record processes can be made arbitrarily close to one by taking m large enough. Hence, the result follows by letting m go to infinity and noting that the limit of IFRA distributions is also IFRA.

Although the condition of symmetry on the function $D_t(x)$ in the above proof is not used explicity, it is required to ensure that the accumulated damage at time t is indeed given by $D_t(X_1(t), \ldots X_m(t), 0)$ since even if the first m shocks to occur all come from different record processes, there is no guarantee that the i^{th} shock to occur comes from the i^{th} process.

~3.2B Cumulative Damage Models with Wear or Recovery

The models considered so far have assumed that wear to a device is caused only by shocks and, further, that once inflicted any damage caused by the shocks continues to impair the performance of the device for the duration of the device's lifetime. In this section, we consider a model in which the wear on a device is allowed to increase or decrease between shocks in some deterministic fashion, i.e., either damage continues to accumulate between shocks or recovery takes place. Failure occurs on the total accumulated wear exceeding some critical fixed threshold. Since wear (or recovery) is allowed between shocks a device may fail at times other than immediately following a shock and thus the model $\bar{H}(t) = \Sigma_{k=0}^{\infty} (P(N(t) = k)\bar{P}_k$ is not appropriate in the current context. To investigate the lifetime distribution, $H_z(t)$, of a device subject to shocks with wear or recovery between shocks, it is helpful to introduce the notion of a damage process.

Let $\{Z(t)\}$ be a stochastic process such that for every $t > 0$, $Z(t)$ is the amount of wear on the device at time t. It is clear that for a fixed threshold z :-

(3.2.11) $H_z(t) = P(T_z(Z)<t)$

where $T_z(Z)$ is the first passage time of the process $\{Z(t)\}$ to the level z . Thus it is the first passage times of the process $\{Z(t)\}$ which are of interest in determining the clas of distributions to which $H_z(t)$ belongs. The analogy with the maximum shock threshold models of section 3.1B and 3.1C is quite strong. For those models, the lifetime distribution was determined by the first passage time of the maximum process :-

$$\{M(t) = \max_{0=<j=<N(t)} X_j\}$$

where X_j is the magnitude of the j^{th} shock.

In studying the cumulative damage model of this section we will, as in sections 3.1B and 3.1C, make use of the correlated pair of renewal sequences $(X_n,Y_n)_{n=0}^{\infty}$ where, in this case, X_n denotes not the magnitude of the n^{th} shock but the amount of damage inflicted by the n^{th} shock. Y_n is the length of the interval between the $(n-1)^{th}$ and n^{th} shocks and we will assume that $X_0 = Y_0 = 0$. For any n X_n and Y_n may be correlated.

We will denote the epoch at which the n^{th} shock occurs by :-

$$\begin{cases} R_n = \Sigma_{i=1} Y_i & n >= 1 \\ R_0 = 0 & \end{cases}$$

and our interest will focus on the pair $\{Z(t),(R_n)_{n=0}^{\infty}\}$ which will be abbreviated to $\{Z,R\}$.

The first passage time (i.e. the failure time) $T_z(z)$ will sometimes be abbreviated to T_z. We will show that under certain conditions on the correlated pair of renewal sequences $(X_n,Y_n)_{n=0}^{\infty}$ $H_z = P(T(z)=<t)$ is NBU. In particular, it will be shown that H_z inherits the NBU property from the inter-arrival times $(Y_n)_{n=0}^{\infty}$.

The NBU property of $H_z(t)$ will be established under quite general conditions but for more practical purposes, it will be seen that it suffices to study a special case. We will largely follow the approach of Shanthikumar (1984) who generalised the earlier work of Marshall and Shaked (1983). Marshall and Shaked (1983) assumed that for each n, the damage, X_n and the inter-arrival time Y_n were independent but Shanthikumar (1984) relaxed this assumption.

Initially we will assume that the sequence of shock inter-arrival times $(Y_n)_{n=0}^{\infty}$ forms a renewal sequence but later, more general sequences will be considered.

Now, in order to characterise mathematically a cumulative damage shock model with wear or recovery between shocks, it is necessary to define the damage process $\{Z(t)\}$ so that at each epoch R_n it jumps an amount X_n and in between jumps moves deterministically in a manner dependent on the time and magnitude of the previous jumps. This must all be subject to the condition that (for all $t >= 0$) $Z(t) >= 0$

The most convenient way to achieve this is to consider the sequence of real-valued monotone functions :-

$$\begin{cases} h_0(t) \\ (h_j(X_1,\ldots X_j \;;\; Y_1,\ldots Y_j \;;\; t-R_j).\,)_{j=1}^{\infty} \end{cases}$$

defined on $\mathbb{R}^+$ and to define :-

(3.2.13) $$Z(t) = [h_j\,(X_1,\ldots X_j;\; Y_1,\ldots Y_j;\; t-R_j),]^+ \qquad R_j =< t < R_{j+1} \qquad j = 0,1,2,\ldots$$

Clearly, $h_j(.)$ decreasing on the internal (R_j,R_{j+1}) corresponds to a device exhibiting recovery between shocks while if $h_j(.)$ is increasing on (R_j,R_{j+1}) $j = 0,1,2,\ldots,$ the model (3.2.13) corresponds to a situation where wear continues between shocks. Of course, it is possible that in some intervals recovery takes place while in others wear continues to accumulate. However, we consider here only the case where the $h_j(.)$ are either all increasing or all decreasing.

In most applications, it can be assumed that the damage process $\{Z(t)\}$ satisfies the condition :-

(3.2.14) $$[Z(R_n^-) + X_n]^+ = Z(R_n)$$

where $$Z(R_n^-) = \lim_{t\to R_n} (Z(t)),$$

and for any function g(t) :-

$$[g(t)]^+ = \begin{cases} g(t) & , \; g(t) > 0 \\ 0 & \;\; g(t) =< 0 \end{cases}$$

From (3.2.14) it follows that we are interested in sequences of functions $(h_j(.))_{j=0}$ which satisfy :-

(3.2.15) $$[h_{j-1}\,(X_1,\ldots X_{j-1},\;\; Y_1,\ldots Y_{j-1},\; R_j-R_{j-1}) + X_j]^+ = h_j(X_1,\ldots X_j;\;\; Y_1,\ldots Y_j;\;\; R_j-R_j)$$

so that the process $\{Z(t)\}$ does indeed jump an amount X_j at time R_j, subject always to the requirement of non-negativity.

We will see, however, that neither of the conditions (3.2.14) or (3.2.15) are required to establish the NBU property of the lifetime distribution of a device subject to a damage process $\{Z(t)\}$. On the other hand, the requirement that $\{Z(t)\}$ remains non-negative is required to ensure that if, for example, a situation involving recovery between shocks is being modelled, recovery stops once the wear on a device returns to its original level, i.e. once $Z(t) = Z(0)$. Thus recovery cannot improve a device's performance beyond the original level. The shock magnitudes can, of course, be negative, corresponding to repairs of the device but the non-negativety requirement ensures that no amount of repair can improve the device to a state better than its original one.

In order to establish that the life distribution $H_z(t)$ defined by (3.2.12) is NBU, it is necessary to introduce some notation and to establish some preliminary results. Firstly, some notions of conditional first passage time will prove useful :-

$$T_z(Z) = \inf\{t : Z(t) > z\}$$

is the unconditional first passage time. Conditional first passage times for the process $\{Z(t)\}$ may be defined as follows :-

(3.2.16) $$T_z(Z) =^{df} [T_z(Z)-s \mid T_z(Z)>s] , \quad s > 0$$

(3.2.17) $$T_z(Z,R) =^{df} [T_z(Z)-s \mid T_z(Z)>s, R_1>s] , \quad s > 0.$$

We now restate the definition of the NBU Class of life distributions in terms of the stochastic comparison relation, $>=_{st}$, which is defined as follows :-

<u>Definition (3.2B)</u>

Let X and Y be two non-negative random variables :-

$$X >=_{st} Y \iff P(X>=x) >= P(Y>x) \text{ for every } X > 0$$

A random variable X has a NBU distribution if :-

(3.2.18) $X \geq_{st} [X-s \mid X>s]$ for every $s > 0$.

For convenience, a stochastic process with NBU first passage times will be referred to as a NBU process. By (3.2.18) above, it follows that $\{Z(t)\}$ is a NBU process iff :-

(3.2.19) $T_z(Z) \geq_{st} T_z^s(Z)$ $\quad s > 0, \; Z > 0$

Hence, in order to establish that the life distribution

$$H_z(t) = P(T_z(Z) < t)$$

of a device with damage threshold z subject to the damage process $\{Z(t)\}$ is NBU, it suffices to show that (3.2.19) holds.

If :-

(3.2.20) $T_z(Z) \geq_{st} T_z^s(Z,R)$ $\quad s > 0, \; z > 0$

we will say that (Z,R) is a single step N.B.U. process (s.s. NBU). This definition is somewhat artificial but is useful for establishing that a stochastic process is NBU since , as is shown later, a process $\{Z(t)\}$ is NBU (under appropriate conditions) if $\{Z,R\}$ is s.s. N.B.U.

The notion of stochastic dominance of one process over another will also be of some use and is defined below.

A stochastic process $Z(t)$ stochastically dominates a process $W(t)$, written $Z(t) \geq_{(st)} W(t)$ if :-

(3.2.21) $E(f(Z) \geq E(f(W)$ for every non-decreasing functional, f, for which the expectations exists.

A conditional type of stochastic dominance can also be defined as follows :-

A process $\{Z,R\}$ stochastically dominates a process $\{W,S\}$ over a single step written $\{Z,R\} \geq_{(s)} \{W,S\}$ if :-

(3.2.22) $\{Z(t+s) \mid Z(u) = Z(u),\ R_1 > s\}$

$>=_{(st)} \{W(t+s) \mid W(u) = w(u),\ S_1 > s\}$,

for all t and s > 0 whenever $Z(u) >= W(u)$

$0 <= u <= S$

and

(3.2.23) $P(Z(u) >= W(u) \mid R_1 > s,\ S_1 > s) = 1;\quad 0 =< u =< s$

It follows from (3.2.21) and (3.2.22) that :-

(3.2.24) $\{Z,R\} >=_{(*)} (W,S) => T_z{}^u(Z,R)$

$=<_{st} T_z{}^u(W,S)$

for every u, z > 0

The following transformations of the process will also prove useful :-

Define $\{Z_s(.)\}$ by :- $Z_s(t) =_{st} Z(t+s)$, t > 0, s > 0

This is equivalent to shifting the origin to s while the next transformation shifts the origin to R_n ;

Define $\{Z^n,R^n\}$ by :-

(3.2.25) $(Z_n,R_n) =_{(st)} (Z(t+R_n),\ (R_k{}^n = R_{n+k}-R_n)_{k=0}$

for every t > 0 and n >= 1

The final transformation of interest traces the historical maximum of the process $\{Z(t)\}$

Define $\{Z\hat{}(t)\}$ by :-

(3.2.26) $Z\hat{}(t) = Max_{0<u<t}\ \{(Zu)\}$ for every t > 0

We will also find it useful to be able to refer to the "history" of the process $\{Z,(t)\}$ and its transformations. To this end, define $Hist(Z(.),t)$ to be the history of the process $\{Z(.)\}$ up to time t i.e., the realisation of $Z(w,u)$ over the interval $(0,t)$ where w belongs to Ω, the sample space on which $\{Z(.)\}$ is defined.

We will find $Hist(Z,R_n)$ useful in discussing the process $Z^n,R^n\}$ and $Hist(Z,s)$ useful in discussing $\{Z_s(.)\}$

Shanthikumar (1984) states that stochastic dominance over a single step is a stronger property than ordinary stochastic dominance and is generally difficult to establish directly. The following Theorem, due to Shanthikumar, provides a means of establishing the property of stochastic dominance over a single step.

Theorem (3.2B.1)

Suppose there exist two stochastic processes :-

$$\{Z^*(.), R^*\} \quad \text{and} \quad \{W^*(.), S^*\}$$

defined on the same probability space $(\Omega,), P)$ such that $R_1^* = S_1^*$.

If $\Omega_s =^{df} \{w : R_1^*(w) > s \ ; \ w \in \Omega\}$ $\quad s > 0$

and for every $s > 0$

then

$$Z^*(t,w) >= W^*(t,w), \quad t > s, \quad \text{for every } w \in \Omega_s$$

and

$$\min_{w \in \Omega_s} \{Z^*(u,w)\} >= \max_{w \in \Omega_s} \{W^*(u,w)\} \ ; \qquad 0 =< u < s$$

and further if :-

$$\{Z,R\} =_{st} \{Z^*,R^*\} \quad \text{and}$$

$$\{W,S\} =_{st} \{W^*,R^*\}$$

then $\{Z,R\} >=_{(st)} \{W,S\}$

We are now in a position to establish the main result of this section : that the life distribution of a device subject to shocks with wear or recovery between shocks and a fixed damage threshold level is NBU.

Theorem (3.2B.2)

Let $\{Z(t)\}$ be a damage process as defined in (3.2.13) and let $H_z(t) = P(T_z(Z)<t)$ be the life distribution of a device subject to $\{Z(t)\}$ and with a fixed damage threshold z. Suppose :-

i) $(X_n, Y_n)_{n=0}^{\infty}$ is a correlated pair of renewal sequences.

ii) Y_n has a NBU distribution $n = 1,2,...$

iii) $h_0(t) >= 0$; $t >= r_0$

iv) $h_j\ (X_i, Y_i, i=1,2,...j;t) >= h_{j-1}\ (X_i, Y_i, i=2,3,...j;t)$
where $t > 0$, $j = 2,3,...$

v) $h_j\ (X_1, Y_1, (x_i, y_i,)\ i=2,...j;t)$ is Stochastically Increasing (SI) on R_1 i.e. on Y_1

Then $H_z(t)$ is NBU for every $Z > 0$

Proof

The proof of this Theorem is quite involved so we will begin with an outline of the proof :-

Outline of Proof of Theorem (3.2B.2)

To show that $H_z(t)$ is NBU we must show that the process $\{Z(t)\}$ has first passage times which have NBU distributions, i.e., we must show that $\{Z(t)\}$ is an NBU process. Now, conditions (i) and (iv) allow us to construct a process $\{W,S\}$ such that :-

(3.2.27) $\{W(.),S\} =_{(st)} \{Z(.),R\}$

and

(3.2.28) $\{W(.),S\} =<_{(*)} Z^n(.),R^n\}$

Theorem (3.2B.1) is required to show the stochastic dominance over a single step of $\{Z^n(.)R^n\}$ over $\{W(.),S\}$. The following Theorem, (3.2.27) and (3.2.28) imply that $\{Z(t)\}$ is NBU whenever $\{Z(.),R\}$ is s.s. N.B.U.

Theorem (3.2B.3)

Suppose that for each $n \geq 1$ and every history $Hist(Z,R_n)$ there exists a stochastic process $\{W,(.),S\}$ such that :-

(i) $(W,(.),S\} \leq_{(*)} Z^n(.),R^n\}$

and

(ii) $\{Z(.),R\} =_{(st)} \{W(.),S\}$

(or $\{Z(.),R\} \leq_{(*)} \{W(.),S\})$

then $\{Z(.)\}$ is an NBU process wherever :-

$\{Z(.),R\}$ is s.s N.B.U.

The proof of this result is omitted here but is given in full in Shanthikumar (1984).

The s.s. N.B.U. property of $\{Z(.),R\}$ is established using conditions (ii) and (v) of the Theorem as well as an additional Lemma.

We can now proceed with the proof of Theorem (3.2B.2) :-

Define $S_k = R_{n+k} - R_n \qquad n = 0,1,2,\ldots$

$W(t) = h(X_{n+1},\ldots X_{n+j};Y_{n+1},\ldots Y_{n+j} ; t-S_j)$

where $S_j \leq t < S_{j+1}$

Now $\{W,S\} =_{(st)} \{Z,R\}$ since $(Y_n)_{n=0}^{\infty}$ is a renewal sequence. Thus condition (ii) of Theorem (3.2B.3) is satisfied. To show that condition (i) of Theorem (3.2B.3) is satisfied,

i.e., that $\{Z^n,R^n\} \geq_{(*)} \{W,S\}$ for every history $Hist(Z_n,R_n)$.

We note that $\{Z(t+R_n),(R_j^n)_{j=1}^{\infty}\} =_{(st)} Z^n,R\}$ by definition

and $R_1^n = R_{n+1} - R_n = S_1$

Further $Z(t+R_n) = h_{j+n}(X_i,Y_i;i=1,\ldots j+n; t-(R_{j+n}-R_n))$

and

condition (iv) of Theorem (3.2B.21) implies (via a simple induction on j) that for any $0 < L \leq j$ and $t \geq 0$

(iv') $h_j(X_i, Y_i \; ; \; i=1,2,\ldots j \; ; \; t) \; >=$
$h_{j-i} \; (X_i, Y_i, \; i= L+1, L+2, \ldots j \; ; \; t)$

In the case $L = j$ this reduces to :-

(iv") $h_\emptyset(t) =< h_j \; (X_i, Y_i \; i=1,2,\ldots j \; ; \; t)$

Now applying condition (iv) in the above expression for $Z(t+R_n)$ yields:-

$$\begin{aligned} Z(t+R_n) &>= h_{(j+n)-n} \; (X_i, Y_i; \; i=n+1, \ldots n+j; \; (t-R_{n+j} - R_n) \\ & \qquad\qquad \text{where } R_{j+n}-R_n =< t =< R_{n+j+1}-R_n \\ &= h_j \; (X_i, Y_i; \; i=n+1, \ldots n+j; \; t-S_j) \\ & \qquad\qquad\qquad \text{where } S_j =< t < S_{j+1} \\ &= W(t) \qquad\qquad\qquad\qquad S_j =< t < S_j \end{aligned}$$

For $t < S < R_1{}^n = S_1$

$$\begin{aligned} W(t) = h_\emptyset(t) &=< h_n(X_i, Y_i \; i=1, \ldots n \; ; \; t) \\ & \qquad\qquad \text{by condition (iv")} \\ &= Z(t+R_n) \; ; \; t < s < R_1{}^n = s_1, \end{aligned}$$

Now by taking :-

$$\{W^*, S^*) = (W, S\} \quad \text{and} \quad \{Z^*, R^*\} = \{Z(t+R_n), R^n{}_k\}$$

in Theorem (3.2B.1) it follows that :-

(3.2.29) $\{Z^n, R^n\} >=_{(*)} \{W, S\}$

and consequently, by Theorem (3.2B.2) $\{Z(.)\}$ is NBU whenever $\{Z,R\}$ is s.s. NBU. Thus to establish the NBU property of $\{Z(.)\}$ and hence of $H_z(.)$ we now need only to establish that $\{Z,R\}$ is s.s. NBU. To this end, define τ_z by :-

$$\tau_z = \inf \{t : h_\emptyset(t) > z, \quad t \in \mathbb{R}^+\}$$

so that τ_z is the first time $h_\emptyset(t)$ exceeds z. If no such τ_z exists then set $\tau_z = \infty$

Define $\quad Y_1{}^x =^{df} Y_1$ if $Y_1 =< \tau_x$
$\qquad\qquad =^{df} T_x$ if $Y_1 > \tau_x$

and

$$T_z{}^* =^{df} T_z(Z) - Y_1{}^x$$

(so $\quad T_z(z) = T_z{}^* + Y_1{}^x)$

Condition (v) of Theorem (3.2B.2) implies that T_x^* is stochastically decreasing on Y_1 and hence on Y_1^x i.e.:-

(3.2.30) $T_x^* \geq_{st} (T_x^* \mid Y_1^x > s)$ for every $s > 0$

From condition (ii) of the Theorem we know that Y_1 has a NBU distribution :-

i.e. $Y_1 \geq_{st} (Y_1 \mid Y_1 > s - s)$ for all $s > 0$

and it follows that :-

(3.2.31) $Y_1^x \geq_{st} (Y_1^x \mid Y_1 > s - s)$ $0 < s < \tau_x$

hence since if $Y_1 > s > T_x$ then $Y_1^x = Y_1$ then we conclude that Y_1^x also has an NBU distribution.

Now by convolving (3.2.30) and (3.2.31) it can be shown that :-

(3.2.32) SD $(T_x^* \mid Y_1^x)$ (c.f Definition 3.1B.2)

by the following Lemma it follows that :-

$T_x^* + Y_1^x \geq_{st} [(T_x^* \mid Y_1 > s + Y_1^x \mid Y_1^x > s) - s]$

i.e. $T_x(Z) \geq_{st} T_x^* (Z,R)$

Thus by (3.2.20) {Z,R} is s.s. NBU as required. Hence by (3.2.29) and Theorem (3.2B.3) {Z(t)} is NBU and so is $H_x(t) = P(T_x(Z) < t)$.

<u>Lemma (3.2B.1)</u> (Shanthikumar (1984))

Let X and Y be two non-negative random variables.

If :-

(i) X is stochastically decreasing in Y (c.f. Definition (3.1B.2)

(ii) Y has a NBU distribution.

Then $X + Y \geq_{st} (X+Y \mid Y > s) - s$, $s > 0$

Some remarks on the statement of Theorem (3.2B.2) are in order. The non- negativity of $\{Z(t)\}$ follows from conditions (iii) and (iv) of the Theorem. Shanthikumar does not explicitly include condition (iii) but it has been included here since in our context any damage process $\{Z(t)\}$ must satisfy this condition.

Shanthikumar (1984) notes that Theorem (3.2B.3) still holds if conditions :- (i) and (ii) are replaced by :-

(i)' $\{W,S\} =<_{(*)} \{Z^{\wedge n},R_n\}$

(ii)' $\{Z,R\} =<_{(*)} \{W,S\}$

We have seen that under some quite general conditions on the correlated pair of renewal sequences $(X_n,Y_n)_{n=1}^{\infty}$ the lifetime distribution of a device subject to a damage process $\{Z(t)\}$ generated by a sequence of shocks with magnitudes $(X_n)_{n=1}^{\infty}$ and interarrival times $(Y_n)_{n=1}^{\infty}$, and in which deterministic wear or recovery takes place between shocks is NBU.

We now present some examples of such processes by specifying the sequence of functions : $h_j(X_1,-X_j;Y_1,-Y_j;t-R_j)$ $j=0,1,2,...$ of the definition (3.2.13). :-

Example 1

If $h_j(X_1,...X_j;Y_1,...Y_j;t...R_j) = \Sigma_{i=1} X_i$

$j = 1,2,3...$

and $Z(t) = h_j(X_1,...X_j;Y_1,...Y_j;\ t-R_j)$

$R_j =< t < R_{j+1}$

as in (3.2.13) then the model reduces to the cumulative damage model of section 3.2A in which no wear or recovery is allowed between shocks. Note, however, that condition (iv) of Theorem (3.2B.2) holds only if the damages X_i are all non-negative. Provided this condition and the other conditions of Theorem (3.2B.2.) are met, the life distribution of a device subject to $\{Z(t)\}$ as defined above is NBU.

If the shock interarrival times are i.i.d. exponential random variables, i.e., shocks arrive according to a Poisson process and for each n X_n and Y_n are independent, then by Theorem (3.2A.1) the lifetime distribution of the device in question is IFRA.

Many damage processes can be modelled as the random interruption of a continuous monotonic function so that the deterministic movement of the process between shocks is governed by the same function in each interval. This is the special case of interest in most practical applications. The following examples are all generated by the random interuption of a continuous function and we will see that the fact that the deterministic movement of the process between shocks is govened by the same functions in each interval ensures that condition (iv) of Thereom 3.2B.2 is met.

We will begin by presenting a set of examples which have in common the fact that the rate of wear or recovery between shocks depends only on the age of the devices, not on the amount of wear or on the sequence of shocks and intervals.

Example 2

Consider a damage process $\{Z(t)\}$ with deterministic wear between shocks which is governed by the continuous strictly monotone increasing function, g, defined on $\mathbb{R}^+$. Now, if no shocks occur then at time t Z(t) must equal g(t) and if shocks do occur then Z(t) is equal to the accumulated damage due to shocks plus the amount of damage that would have accumulated had no shocks occurred, i.e., plus g(t). Since g is monotone increasing the $(h_j(.))_{j=1}^{\infty}$ of (3.2.13) must also be all increasing. (If we are considering a model with recovery between shocks, then g would be monotone decreasing and so would the $(h_j(.))_{j=1}^{\infty}$)

The $(h_j(.))_{j=1}^{\infty}$ are of course related to the function g and it is the nature of this relationship that ensures that condition (iv) of Theorem (3.2B.2) is met. A simple example may make this clear.

<u>Example 2a</u> (Linear Wear)

Suppose $g(t) = t$ so that between shocks wear continues as a linear function of time. A typical sample path of the process would look like fig. (3.2B.1).

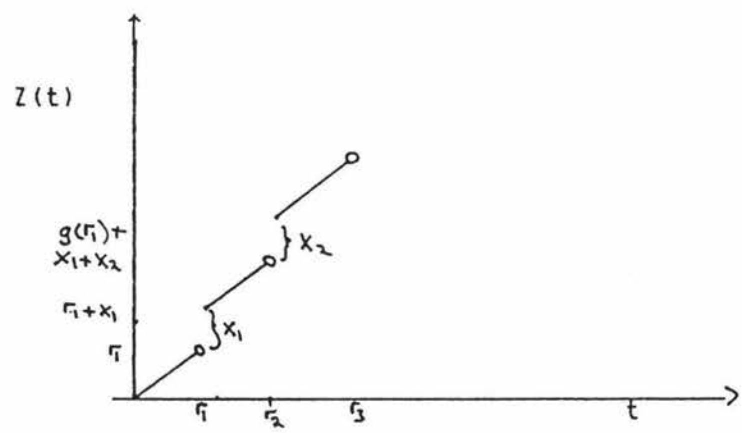

Fig. (3.2B.1) Sample Path of a Damage Process with Linear Wear between Shocks

For the moment let us suppose that all damages are non-negative. Under this assumption it is clear that :-

$$h_0(t) = t = g(t) \; ; \qquad t >= 0$$

$$h_1(x_1, y_1, \; ; \; t-ri) = t + x_1 = g(t)+x_1; \qquad t >= r_1$$

(3.2.35)
$$h_2(x_1, x_2; y_1, y_2; \; t-r_2) = t + x_1 + x_2$$
$$= g(t)+x_1+x_2 \qquad t >= r_2$$

$$(h_n(x_1, \ldots x_n; y_1, \ldots y_n; \; t-r_n) = t + \Sigma_{i=1} x_i$$
$$= g(t) + \Sigma_{i=1} X_i$$
where $t >= r_n$

It is clear that the $(h_n(.))_{n=1}^{\infty}$ thus defined satisfy :-

$$h_n(x_1, \ldots x_n; y_1, y2, \ldots y_n; \; t-r_n)$$
$$>= h_{n-1} \; (x_2, \ldots x_n; y_2, \ldots y_n; \; t-r_n)$$

and hence condition (iv) of Theorem (3.2B.2) is met.

Note that equation (3.2.35) can be written recursively as follows:-

$$h_0(t) = g(t) \qquad t >= 0$$

$$h_1(x_1,y_1;\ t-r_1) = g(t)+(h_0(r_1) + x_1 - g(r_1)) \qquad t >= r_1$$

(3.2.36) $$h_2(x_1,x_2;y_1,y_2;\ t-r_2) = $$
$$= g(t) + (h_1(x_1,y_1;r_2-r_1) + x_2-g(r_2)) \qquad t >= r_2$$

—

$$h_n(x_1,\ldots x_n;\ y_1,\ldots y_n;\ t-r_n) =$$
$$= g(t) + (h_{n-1}(x_1,\ldots x_{n-1};y_1,\ldots y_{n-1};r_n-r_{n-1}) + x_n-g(r_n)) \qquad t >= r_n$$

Equations (3.2.36) hold for any continuous monotone increasing function g such that g(0) =0 provided that the assumption of non-negative damages is met. The terms in parenthesis on the RHS of (3.2.31) represent the difference between the height of the process, {Z(t)} at the beginning of the n^{th} interval and the height {Z(t)} would have attained (i.e. the amount of wear on the device at time t) had no shocks occurred.

To allow for the possibility that some or all of the damages may be negative (3.2.36) can be generalised as follows :-

$$h_0(t) = g(t)$$

$$h_1(x_1,y_1,;t-r_1) = g(t) + ([h_0(r_1)+x_1]^+ - g(r_1)) \qquad t >= r_1$$

(3.2.37)

$$h_n(x_1,\ldots x_n;y_1,\ldots y_n;\ t-r_n) =$$
$$= g(t) + [h_{n-1}(x_1,\ldots x_n;y_1,\ldots y_n;\ r_n-r_{n-1}) + x_n]^+ - g(r_n) \qquad t >= r_n$$

These equations hold for any continuous monotone increasing g and, in the special case that g(t) = t they reduce to :-

$$h_0(t) = t$$

$$h_1(x_1,y_1;\ t-r_1) = t + [r_1+x_1]^+ - r_1\ ; \qquad t >= r_1$$

(3.2.38) $h_n(x_1,\ldots x_n;y_1,\ldots y_n;\ t-r_1) =$

$$= t + [h_{n-1}(x_1,\ldots x_{n-1};y_1,\ldots y_{n-1};r_n-r_{n-1}) + xn]^+ - r_n \qquad t >= r_n$$

From (3.2.37) and the definition (3.2.13) it follows that:-

$$d/dt\ Z(t) = g'(t) \qquad R_n =< t < R_n \quad n = 0,1,2,\ldots$$

i.e., the wear rate of the device in between shocks is indeed a function only of the age of the device.

Now in this more general case where damages are allowed to be negative, it can be shown by the following straight- forward induction that condition (iv) of Theorem (3.2B.2) is met, provided we impose on the process {Z(t)} the additional condition :-

(3.2.39) $P(X_1>=0) = 1$

To show that the sequence of functions $(h_n(.))_{n=1}$ satisfy condition (iv) of Theorem (3.2B.2) we proceed by induction on n.

Condition (iv) of Theorem (3.2B.2) holds for n = 1 since g is increasing and X_1 >= 0 with probability 1 (w.p.1.). Now suppose condition (iv) holds for n = j-1 then :-

$$h_j(x_1,\ldots x_j;y_1,\ldots y_j;\ t-r_j) =$$
$$= g(t) -g(r_j) +[h_{j-1}(x_1,\ldots x_{j-1};y_1,\ldots y_{j-1};r_j-r_{j-1}) +x_j]^+$$
$$>= g(t) -g(r_j) +[h_{j-2}(x_2,\ldots x_{j-1};y_2,\ldots y_{j-1};r_j-r_{j-1}) +x_j]^+$$

(by the inductive hypothesis) :-

$$= h_{j-1}(x_2,\ldots x_j;y_2,\ldots y_j;\ t-r_j) \qquad \text{as required.}$$

We now illustrate equations (3.2.37) by a damage process with exponential wear between shocks :-

<u>Example 2b</u> (exponential wear)

Let $g(t) = (e^{@t} - 1)$ $\qquad @ > 0,\ t >= 0$

then by (3.2.37)

$$h_0(t) = e^{@t} - 1$$

$$h_1(x_1, y_1; t - r_1) = e^{@t} + [\exp(@r_1) - 1 + x_1]^+ - \exp(@r_1) \qquad t >= r_1$$

$$h_n(x_1, \ldots x_n; y_1, \ldots y_n;\ t - r_n) =$$

$$= e^{@t} + [h_{n-1}(x_1, \ldots x_{n-1}; y_1, \ldots y_{n-1}; r_n - r_{n-1}) + x_n]^+ - \exp(@r_n) \qquad t >= r_n$$

If it can be assumed that the shock magnitudes are all non-negative, this reduces to :-

$$h_0(t) = e^{@t}\ ; \qquad t >= r0 = 0$$

$$h_1(x_1, y_1;\ t - r_1) = e^{@t} - 1 + x_1\ ;\ t >= r_1$$

$$h_n(x_1, \ldots x_n; y_1, \ldots y_n; t - r_n) = (e^{@t} - 1) + \Sigma_{i=1}\ x_i \qquad t >= r_n$$

Note that in this case the rate at which wear accumulates between shocks is given by :-

$$d/dt\ Z(t) = d/dt\ (h_n(x_1, \ldots x_n; y_1, \ldots y_n;\ t - r_n)$$

$$r_n < t < r_{n+1}\ ;\ n = 0, 1, 2, \ldots$$

$= @e^{@t}$, a function of the age of the device.

Before turning our attention to examples of shock models with recovery between shocks, it is pertinent to remark that for the damage process $\{Z(t)\}$ defined by (3.2.13) with the $(h_j(.))_{j=1}^{\infty}$ as in (3.2.37) it follows that if the device subject to $\{Z(t)\}$ is ever fully repaired and the level of damage reset to zero, the damage process does not begin anew but, rather, damage continues to accumulate at the same rate as it would had no repair taken place. This reflects the feeling that a device which has been damaged and subsequently repaired is likely, on account of its age, to wear faster than a new device.

If, on the other hand, the wear rate was a function only of the amount of damage sustained by a device, it would be reasonable to assume that following a complete repair damage would begin to accumulate as if the device were new. To model such a situation adequately, involves shifting the monotonic function g by appropriate amounts. For an example of such a model, see example 3.

Illustrating a model with a recovery between shocks but with a recovery rate dependent only on the age of the device, also invovles some shifting of the continuous monotonic function g as is shown below.

This case differs from the previous one in that since recovery begins anew immediately after each shock, the monotone decreasing functions g(.) must be shifted by varying amounts when the $(h_j(.))_{j=1}$ are written in terms of g(.)

A typical sample path of a damage process with recovery between shocks is shown in Figure (3.2B.2) :-

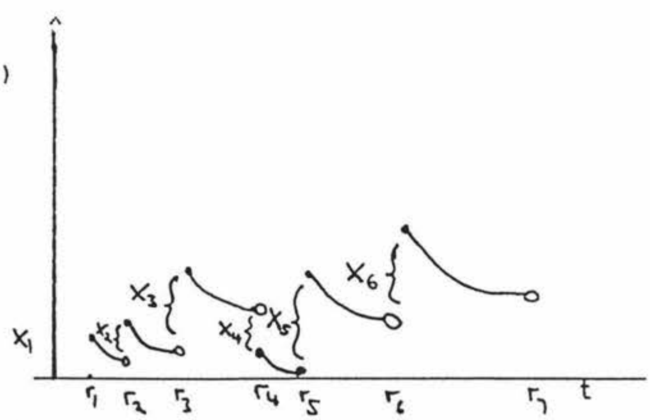

Figure (3.2B.2). Sample path of a damage process with recovery between shocks.

The fact that g is monotonic decreasing implies that all the functions $(h_j(.))^{\infty}_{j=0}$ are decreasing. That is, damage to the device is caused only by shocks and it follows that:

$$h_0(t) = 0, \qquad t >= 0.$$

We will only consider continuous monotone decreasing functions g such that $g(0)=0$.

Under these assumptions, the $(h_j(.))_{j=0}$ of (3.2.13) can be related to g(.) as follows :-

$$h_0(t) = 0 \; ; \qquad t >= 0$$

$$h_1(x_1,y_1;t-r_1) = g(t-r_1) + Z(r_1); \qquad t >= r_1$$

(3.2.40) $$h_2(x_1,x_2;y_1,y_2;t-r_2) = g(t-r_2) + Z(r_2^-) \qquad t >= r_2$$

$$h_n(x_1,\ldots x_n;y_1,\ldots y_n;t-r_n) = g(t-r_n) + Z(r_n^-) \qquad t >= r_n$$

Equation (3.2.35) can be written recursively as follows :-

$$h_0(t) = 0$$

$$h_1(x_1,y_1;t-r_1) = [g(t-r_1) + x_1]^+ \; ; \qquad t >= r_1$$

(3.2.41) $$h_2(x_1,x_2;y_1,y_2;\, t-r_2) = [g(t-r_2) + h_1(x_1,y_1;r_2-r_1) + x_2]^+ \qquad t >= r_2$$

$$h_n(x_1,x_2,\ldots x_n;y_1,\ldots y_n;\, t-r_n) = [g(t-r_n)+(h_{n-1}(x_1,\ldots x_n;y_1,\ldots y_n;r_n-r_{n-1}) +x_n]^+ \qquad t >= r_n$$

By an inductive argument very similar to that used for the case of g increasing but without the additional condition $X_1 >= 0$ w.p.1. it is easily shown that the sequence of functions $(h_j(.))_{j=0}$ defined by (3.2.41) satisfies condition (iv) of Theorem (3.2B.2).

Consequently, if g(.) is either :-

(a) decreasing

or

(b) increasing and $X_1 >= 0$ w.p.1

then the following corollary follows from Theorem (3.2B.2).

Corollary (3.2B.1)

Let $Z(t)$ be defined by (3.2.13) with the $h_j(.)$ as in (3.2.37) or (3.2.41). If

(i) $(X_n,Y_n)_{n=0}^{\infty}$ are a correlated pair of renewal sequences.

(ii) Y_n has a NBU distribution

(iii) X_n is stochastically increasing on Y_n, R_n and $Z(R_n-)$
$n = 1,2,...$

then the failure time of a device subject to $\{Z(t)\}$ without a fixed damage threshold has a NBU distribution.

Proof

The corollary follows directly from Theorem (3.2B.2) Conditions (i) and (ii) of the corollary are identical to conditions (i) and (ii) of Theorem (3.2B.2). As has already been shown, conditions (iii) and (iv) of Theorem (3.2B.2) follow from the definition of $(h_j(.))_{j=0}$ and their relationship to g. Condition (v) of the Theorem follows from condition (iii) of the corollary. Thus all the conditions of Theorem (3.2B.2) are satisfied and the corollary follows.

Note that if g is decreasing and the $H_j(.)$ are defined by (3.2.41) then :-

$$d/dt\ Z(t) = g'(t-r_n)\ ; \quad r_n =< t < r_{n+1}\ ;\ n = 0,1,2,...$$

so that the rate of recovery between shocks is a function only of the age of the device.

Recall that in the case of a damage process with wear between shocks (e.g. examples 2a and 2b) it was shown that:

$$d/dt\ Z(t) = g'(t)\ ; \quad r_n =< t < r_{n+1}\ ;\ n = 0,1,2,...$$

In fact, the wear or recovery rate together with the condition $Z(R_n) = [Z(R_n-) + X_n]^+$ where $n = 0,1,2,...$ can be used instead of (3.2.13) to define the damage process $\{Z(t)\}$

By specifying $g(t)$ we can study a shock model with recovery between shocks more closely, for example :-

<u>Example 2c</u> (Linear recovery)

Let $g(t) = -t$

then by (3.2.36) we have :-

$$h_0(t) = 0 \; ; \qquad t >= 0$$

$$h_1(x_1,y_1;t-r_1) = [g(t-r_1) + x_1]^+ \; ; \qquad t >= r_1$$

$$= [r_1-t+x_1]^+ \qquad t >= r_1$$

$$h_n(x_1,x_2,\ldots x_n;y_1,\ldots y_n;t-r_n) =$$

$$[r_{n-t} +(h_{n-1}(x_1,\ldots x_n;y_1,\ldots y_n;r_n-r_{n-1}) +x_n)]^+$$

$$t >= r_n$$

Note that recovery is complete, i.e.,

$$h_n(x_1,x_2,\ldots x_n;y_1,\ldots y_n;r_n-r_{n-1}) = 0 \quad \text{when}$$

$$t >= r_n + h_{n-1}(x_1,\ldots x_n;y_1,\ldots y_n;r_n-r_{n-1}) +x_n$$

$$= r_n +(Z(r_n^-) +x_n) = r_n +Z(r_n)$$

and further, when recovery is not complete, i.e., when

$$t < r_n + Z(r_n^-)$$

the recovery rate is $d/dt\ Z(t) = -1$

<u>Example 2d</u> (Exponential recovery)

Let $g(t) = e^{-@t} -1$ then $g(0) = 0$

and by (3.2.41)

$$h_0(t) = 0 \; ; \qquad t >= 0$$

$$h_1(t) = [\exp(-@(t-r_1)) -1 + x_1]^+ \; ; \qquad t >= r_1$$

$$h_n(x_1,\ldots x_n;y_1,\ldots y_n;\ t-r_n) =$$

$$= [\exp(-@(t-r_n) -1 +h_{n-1}(x_1,\ldots x_n;y_1,\ldots y_n;r_n-r_{n-1})]^+ +x_n$$

$$t >= r_n$$

$$= [\exp(-@(t-r_n) -1 + Z(r_n^-) +x_n]^+ \; ; \qquad t >= r_n$$

$$= [\exp(-@(t-r_n) -1 + Z(r_n)]^+$$

In this example, recovery is complete only if :-

$$Z(r_n) =< 1 \quad \text{and} \quad exp(-@(t-r_n) + Z(r_n^-) =< 1$$

i.e.

$$Z(r_n) =< 1$$

$$t >= -1/@ \ ln(1-Z(r_n^-) + r_n))$$

If $Z(r_n) > 1$ then recovery cannot be completed before the next shock. In between shocks, recovery takes place at a rate of :-

$$d/dt \ Z(t) = -@exp(-@(t-r_n) \ ; \ r_n<t<r_{n+1}; \ n=0,1,2,...$$

Example 3

The next series of examples are all of the following form : The rate of change of the damage process {Z(t)} depends only on the height of the process. This is a common situation in practice since, often, subjecting a device to wear increases is susceptibility to further damage.

More generally, if we suppose that r(x) is the wear or recovery rate, then whether :-

$r(x) =< 0$ (recovery takes place between shocks)

or

$r(x) >=0$ (wear continues between shocks)

{Z(t)} can be defined by :-

(3.2.42)

$$d/dt \ (Z(t) = r(Z(t)) \ ; \quad r_n<t<r_{n+1}; \ n=0,1,2,...$$

$$Z(R_n) = Z(R_n^- + X_n)$$

As with example 2, such a process can be thought of as the random interruption of a monotonic function but in this case, the relationship between the sequence of functions $(h_j(.))_{j=0}^{\infty}$ of (3.2.13) and the monotonic function, g, say, is different.

From (3.2.42) it follows that the $(h_j(.))$ are either all increasing or all decreasing and if we suppose that the monotonicity is strict where the function is positive and that $h_j(.)$ is continuous, $j = 1,2,...$ then because the rate of wear or recovery is an implicit function of the device's age, it follows (from the Implicit Function Theorem) that there exists a continuous non-negative monotone function g such that:-

$$(3.2.43) \quad h_j(x_1,...x_j;y_1,...y_j;t-r_j) = $$
$$= g(t-r_j + g^{-1}[h_{j-1}(x_1,...x_{j-1};y_1,...y_{j-1};r_j-r_{j-1}) + x_j]^+) \qquad t >= r_j$$

(c.f. Marshall and Shaked (1983))

As in example two it seems reasonable to suppose :-

$$(3.2.44) \quad h_0(t) = 0 \quad \text{if g decreasing}$$
$$h_0(t) = g(t) \quad \text{if g increasing}$$

It should be noted that the monotone, continuous function g of (3.2.43) and (3.2.44) is also required to be non- negative. This extra condition was not required in Example 2.

A straight-forward inductive argument very similar to that used in example 2, shows that $(h_j(.))_{j=0}^{\infty}$ defined by (3.2.43) and (3.2.44) above satisfy condition (iv) of Theorem (3.2B.2). Consequently, we have the following Corollary to Theorem (3.2B.2).

<u>Corollary (3.2B.2)</u>

If Z is defined as in (3.2.40) (or by (3.2.13) with the $h_j(.)$ as in (3.2.43) and (3.2.44) and

(i) $(X_n,Y_n)_{n=1}^{\infty}$ is a correlated pair of renewal sequences

(ii) Y_n is NBU

(iii) X_n is stochastically increasing on Y_n and on $Z(R_n^-)$ n = 1,2,... then the lifetime distribution of a device, with a fixed damage threshold, and subject to $\{Z(t)\}$ is NBU.

Proof

Conditions (i) and (ii) of the Corollary are the same as conditions (i) and (ii) of Theorem (3.2B.2). Conditions (iii) and (iv) follow from the definition of $\{Z(t)\}$ (i.e. equation (3.2.43) and (3.2.44). Condition (v) of Theorem (3.2B.2) follows from Condition (iii) of the corollary. Hence, the result follows from Theorem (3.2B.2).

The above Corollary is very similar to Corollary 31 of Shanthikumar (1984).

By specifying g(.) we can illustrate the above with the following examples :-

Example 3a (Linear Wear)

Let $g(t) = t ; \qquad t >= 0$

then by (3.2.41) and (3.2.42) we have :-

$$h_0(t) = g(t) \qquad t >= 0$$

$$h_1(x_1,y_1;t-r_1) = g(t-r_1 + g^{-1}([h_0(r_1) + x_1]^+))$$
$$= (t-r_1) + [h_0(r_1) + x_1]^+ ; t >= r_1$$

$$h_n(x_1,\ldots x_n;y_1,\ldots y_n;\ t-r_n)$$
$$= [h_{n-1}(x_1,\ldots x_{n-1};y_1,\ldots y_{n-1};\ r_n-r_{n-1})+x_n]^+ + (t-r_n) \qquad t >= r_n$$
$$= [Z(r_n)]^+ + (t-r_n)$$

Note that this is exactly the same as linear wear in the case of example (2a) c.f. equation (3.2.38). By (3.2.13) we have :-

$$Z(t) = Z(r_n) + (t-r_n) \; ; \qquad r_n < t < r_{n+1}$$

and as in example 2a :-

$$d/dt \; Z(t) = 1 \; ; \qquad r_n < t < r_{n+1}$$

Example 3b (Exponential Wear)

Let $g(t) = e^{@t-1}$ $\qquad t >= 0$

then by (3.2.43) and (3.2.44) we have :-

$$h_0(t) = e^{@t} - 1 \qquad t >= 0 \; ; \; @ > 0$$

$$h_1(x_1,y_1;t-r_1) = g(t-r_1+g^{-1}([\exp(@r_1) - 1+x_1]^+))$$

$$= g(t-r_1 + 1/@ \ln([\exp(@r_1) - 1+x_1]^+ + 1))$$

$$X \quad \exp[@(t-r1) + \ln([\exp(@r_1) - 1 + x_1]^+ + 1)] - 1$$

$$= \exp(@(t-r_1))([\exp(@r_1) - 1 + x_1]^+ + 1) - 1$$

$$= \exp(@(t-r_1)) - 1$$

$$x_1 > 1-\exp(@r_1) \; ; \; t >= r_1$$

$$= e^{@t} + x_1 e^{@(t-r_1)} - 1$$

$$x_1 > 1-\exp(@r_1) \; ; \; t >= r_1$$

$$h_n(x_1,\ldots x_n;y_1,\ldots y_n;\; t-r_n) =$$

$$= g(t-r_n + g^{-1}[h_{n-1}(x_1,\ldots x_{n-1};y_1,\ldots y_{n-1};\; X \; r_n-r_{n-1}) + x_n]^+)$$

$$= \exp[@(t-r_n) + \ln([h_{n-1}(x_1,\ldots x_{n-1};y_1,\ldots y_{n-1};\; X \; r_n-r_{n-1}) + x_n]^+ + 1)] - 1$$

$$= ([h_{n-1}(x_1,\ldots x_{n-1},y_1,\ldots y_{n-1};r_n-r_{n-1}) + x_n]^+ + 1) \exp(@(t-r_n)) - 1$$

$$= (Z(r_n) + 1) \exp(@(t-r_n)) - 1$$

so by (3.2.13)

$$Z(t) = (Z(r_n) + 1) \exp(@(t-r_n)) - 1 \; ; \quad r_n < t < r_{n+1}$$

and

$$d/dt \; Z(t) = @(Z(r_n) + 1) \exp(@(t-r_n)); \; r_n < t < r_{n-1}$$

$$= @(Z(t) + 1)$$

a function of the height of the process, Z(t).

Note that exponential wear in this case is quite different from exponential wear in example (2(b)).

Example 3c (Linear Recovery)

Let $g(t) = [-t]^+$

then by (3.2.41) and (3.2.42) we have :-

$$h_0(t) = 0 \; ; \qquad t >= 0$$

$$h_1(x_1,y_1,t-r_1) = g(t-r_1 + g^{-1}[0+x_1]^+)$$
$$= g(t-r_1 - [x_1]^+)$$
$$= [[x_1]^+ - (t-r_1)]^+$$

$$h_n(x_1,...x_n;y_1,...y_n;\ t-r_n) =$$
$$= g(t-r_n + g^{-1}[h_{n-1}(x_1,...x_{n-1};y_1,...y_{n-1};\ X$$
$$r_n-r_{n-1}) + x_n]^+)$$
$$= [[h_{n-1}(x_1,...x_n;y_1,...y_{n-1};\ \ r_n-r_{n-1}) +x_n]^+ -(t-r_n)]^+$$
$$= [Z(r_n) - (t-r_n)]^+ \quad r_n<t<r_{n+1}$$

If all damages are positive then this example is identical to Example 2c. Of course if in Example 2c we had used : $g(t) = [-t]^+$ instead of $g(t) = -t$ then because of the linearity of g the two examples would be identical regardless of the sign or the damages.

We have used $g(t) = [-t]^+$ here in order to satisfy the requirement that g be non-negative.

In this example, as in Example 2c, the rate of recovery between shocks is :-

$$d/dt\ Z(t) = -1 \qquad r_n < t < r_{n+1}$$

and recovery is complete when $t >= r_n + Z(r_n)$

Example 3d (Exponential recovery)

Let $g(t) = [e^{-@t} -1]^+$

then for $x >= 0$; $g^{-1}(x) = -1/@ \ln(x+1)$

and by (3.2.43) and (3.2.44) we have :-

$$h_0(t) = 0 \qquad t >= 0$$

$$h_1(x_1,y_1;\ t-r_1) = g(t-r_1 + g^{-1}\ [h_0(r_1)+x_1]^+)$$
$$= g(t-r_1 + g^{-1}\ [x_1]^+)$$
$$= g(t-r_{n1} - 1/@\ \ln([x_1]^+ + 1))$$
$$= [\exp(-@(t-r_1))\ ([x_1]^+ + 1) - 1]^+$$

$$h_n(x_1,\ldots x_n;y_1,\ldots y_n;\ t-r_n) =$$
$$= g(t-r_n - 1/@\ \ln([h_{n-1}\ (x_1,\ldots x_{n-1};y_1,\ldots y_{n-1};\quad X$$
$$r_n-r_{n-1}) + x_n]^+ + 1))$$
$$= [\exp(-@(t-r_n))\ ([h_{n-1}(x_1,\ldots x_{n-1};y_1,\ldots y_{n-1};\quad X$$
$$r_n-r_{n-1}) + x_n]^+ + 1) - 1]^+$$
$$= [\exp(-@(t-r_n))\ (Z(r_n) + 1) - 1]^+\ ;\quad t >= r_n$$

so $Z(t) = [\exp(-@(t-r_n))\ (Z(r_n)+1)-1]^+;\quad r_n < t < r_{n+1}$
and recovery between shocks occurs at a rate of :-

$$d/dt\ (Z(t) = -@\exp(-@(t-r_n))\ (Z(-r_n) + 1) - 1$$
$$= -@(Z(t) + 1)$$

Note that recovery is complete whenever :-

$$t >= r_n + 1/@\ \ln(Z(r_n) + 1)$$

Again, we note that this example is quite different from the corresponding example in the case of a damage process where recovery rate depends only on the age of the device.

Example 4

It may be the case that the rate at which wear or recovery occurs between shocks is a function both of the age of the device and the amount of damage sustained (i.e. the height of the process $\{Z(t)\}$.) In this case, the damage process $\{Z(t)\}$ can be defined as follows :-

(3.2.44) $d/dt\ Z(t) = r(Z(t),t)$; $r_n < t < r_{n+1}$, $n = 0,1,2,...$

$Z(r_n) = [Z(r_n^-) + X_n]^+$ $n = 0,1,2,...$

where $r(x,t)$ is a real-valued function defined on $\mathbb{R}^+ \times \mathbb{R}^+$ such that either :-

(3.2.45) (i) $r(x,t) >= 0$ and increasing in t

(wear continuous between shocks)

or

(ii) $r(x,t) =< 0$ and is increasing in t

(recovery takes place between shocks)

It is, in principle, possible to define $\{Z(t)\}$ in terms of a sequence of functions :-

$(h_j(X_1,...x_j;Y_1,...y_j;\ t-R_j))_{j=1}^{\infty}$

as in (3.2.13), but (3.2.43) and (3.2.44) provide a more convenient definition in this case. That a device with a fixed damage threshold and subject to $\{Z(t)\}$ defined by (3.2.44) and (3.2.45) has a NBU life distribution can be seen by comparing Corollaries (3.2B.1) and (3.2B.2) and again applying Theorem (3.2B.2). Thus we have the following result which is actually due to Shanthikumar (1984).

Corollary (3.2B.3)

Let $\{Z(t)\}$ be defined by (3.2.44) and (3.2.45) and let H_z be the lifetime distribution of a device which is subject to the damage process $\{Z(t)\}$ and has a fixed damage threshold Z. If :-

(i) $(X_n,Y_n)_{n=1}^{\infty}$ is a correlated pair of renewal sequences

(ii) Y_n is NBU

(iii) X_n stochastically increasing on Y_n then H_z is NBU.

Proof.

This corollary is easily verified from corollaries (3.2B.1) (3.2B.2) and Theorem (3.2B.2).

Example 5 (A Random-Repair-time process)

One application of the cumulative damage shock model with wear between shocks which has been studied in some detail is the random-repair time process. Moshe and Shaked (1983) and Shanthikumar (1984) both discuss this model.

Shocks, in this model, are all assumed to cause negative damage and during the random intervals between shocks it is assumed that wear continues to accumulate. The shocks can be thought of as repairs and the (negative) damages as the amount of damage repaired. We note here that the title Random-Repair-time process is somewhat misleading since it is not the repair-times which are random but rather the times at which repair takes place. The rate at which wear occurs between shocks is assumed to depend on the height of the (damage) process and possibly on the age of the device. We will consider first the case where rate of wear is independent of time. Let g be a monotonically increasing function and as usual let X_n denote the "damage" caused by the n^{th} shock and let $(Y_n)_{n=0}$ be the sequence of shock interarrival times.

If (X_n,Y_n) is a correlated pair of renewal sequences and the damage process {Z,R} is generated by the random interruption of g by (X_n,Y_n) as in example 3, then it follows from Corollary (3.20.2) that the process {Z(t)} is an NBU process whenever :-

(i) Y_n is NBU

and (ii) $SI(X_n \mid Y_n)$, $SI(X_n \mid Z(R_n)$ $n = 1,2,...$
(cf. Defn. 3.1B.2)

Recall that $\{Z(t)\}$ is NBU is equivalent to the lifetime distribution of a device with a fixed damage threshold and subject to $\{Z(t)\}$ belonging to the NBU Class of distribution.

Mosne and Shaked (1983) established the NBU property of $\{Z(t)\}$ under the alternative condition :-

(i)′ Y_n i.i.d. with a common exponential distribution
(ii)′ for each n X_n and Y_n are independent

Shanthikumar (1984) relaxed the condition that Y_n be exponential to (i)" Y_n has a DFR distribution.

In the case where the wear rate depends on both the height of the damage process and the age of the device we have as in Example 4 :-

$$d/dt\ (Z(t) = r(z(t),t) \quad \text{where } R_n < t < R_{n+1},\ n = 1,2,\ldots$$

$$Z(R_n) = [Z(R_n^-) + X_n]$$

where $r(x,t) >= 0$ and increasing in t.

Thus from corollary (3.2B.3) it follows that $\{Z(t)\}$ is NBU (i.e. the life distribution of a device subject to $\{Z(t)\}$ with a fixed damage threshold is NBU) wherever :-

(i) Y_n has a NBU distribution
(ii) X_n is stochastically increasing on Y_n, $Z(R_n^-)$ and R_n , $n = 1,2,\ldots$

Up to now, in this section (3.2B.2) it has been assumed that the shock interarrival times $(Y_n)_{n=0}^{\infty}$ form a renewal sequence. We now consider the more general case in which the Y_n $n = 0\ldots\infty$ while remaining independent are allowed to have different distributions. That is $(Y_n)_{n=0}^{\infty}$ forms a generalised renewal sequence in the sense of the generalised renewal processes of section (2.2). For convenience, we will term the associated damage process a generalised damage process. We will restrict ourselves to the case where the damage process $\{Z(t)\}$ is generated by the random interruption of a monotonic function g by the paired sequence of random variables $(X_n,Y_n)_{n=0}^{\infty}$.

Thus, as in examples 2 - 4, the deterministic movement of the damage process between shocks is governed by the same function in each interval.

It will be shown that under appropriate condition on the damages $(X_n)_{n=0}^{\infty}$ and interarrival times $(Y_n)_{n=0}^{\infty}$ the lifetime distribution of a device with a fixed threshold and subject to the (generalised) damage process $\{Z(t)\}$ is NBU. Since the $(Y_n)_{n=0}^{\infty}$ no longer form a renewal sequence, the regenerative property of renewal sequences cannot be used to establish this result. Instead, a condition on the maximum process ,Z, is used.

The damage process $\{Z(t)\}$ that we are interested in here could be defined as in (3.2.13) but since we are dealing only with the case where $\{Z(t)\}$ is generated by the random interruption of a monotonic function, it is more convenient to use the following definition :-

(3.2.47) Define $\{Z(t)\}$ by :-

(i) $Z(0) = 0$

(ii)a) $d/dt\ Z(t) = d/dt\ g(t)$ $\qquad r_n < t < r_n+1;$

$n = 0,1,2,...$

where $g(t)$ is a continuous, monotone function defined on $\mathbb{R}^+$ or :-

(ii)b) $d/dt\ Z(t) = r(x)\ |x=z(t);$ $\qquad r_n < t < r_n+1$

where $r(x) =< 0$, for all $x > 0$

or $r(x) > 0$, $x > 0$

and

iii) $Z(R_n) = [Z(R_n^-) + X_n]^+$ $\qquad n = 0,1,2,...$

We know from examples 2 and 3 that if either of conditions ii a) or ii b) in the above are satisfied, then the sequence of functions $(h_j(.)j = 1,2,...$ of the definition (3.2.13) can be related to the monotonic function g.

From the point of view of establishing the NBU property of the lifetime distribution of a device subject to $\{Z(t)\}$ it does not matter which of ii a) or ii b) is satisfied. The crucial point is that the behaviour of the process in between shocks is governed by the same function (subject possibly to shifting) in each interval.

Before establishing the NBU propery of the process {Z(t)} defined by (3.2.47) it should be noted that Shanthikumar (1984) discusses this generalised model only in the case where recovery takes place between shocks; we will also consider the case where damage continues to accumulate but only under the condition that all the damages are positive with probability 1.

In order to establish the NBU property of the generalised damage process {Z(t)} and hence of the lifetime distribution of a device with a fixed damage threshold, subject to {Z(t)}, some preliminary results are required. These results are analogous to those established for the case where the shock interarrival times form a renewal sequence.

Firstly, we have a result which is quite similar to Theorem (3.2B.3). The proof is omitted here but can be found in Shanthikumar (1984). All notation is the same as used previously.

Theorem (3.2B.4)

Suppose $(Y_n)_{n=0}^{\infty}$ forms a geeralised renewal sequence and that the stochastic process {Z(t)} is generated by the random interruption of a continuous monotonic function g by the paired sequence of random variables $(X_n, Y_n)_{n=0}^{\infty}$. If for each n and every History Hist(Z, R_n) there exists a stochastic process (W,S) satisfying :-

(i) $\quad Z^{\wedge} \;=<_{*t}\; W^{\wedge}$

(ii) $\quad \{W,S\} \;=<_{(*)}\; \{Z_n, R_n\}$

and

(iii) $\quad$ {W,S} is s.s. NBU

then $\quad$ {Z} is NBU.

The next Theorem provides a means of verifying condition (i) of the previous Theorem.

Theorem (3.2B.5)

Let $\{Z(t)\}$ and $\{W(t)\}$ be two stochastic processes generated by the random interruption of a continuous monotonic function g, by the paired sequences of random variables $(X_n, Y_n)_{n=1}^{\infty}$ $(X_n', Y_n')_{n=1}^{\infty}$ respectively. Suppose further that either g is decreasing ; or g is increasing and $P(X_n >= 0) = P(X_n' >= 0) = 1$ $\quad n = 0,1,2,\ldots$:-

If (i) $X_n' >=_{st} X_n$ $\quad n = 1,2,\ldots$

and (ii) $Y_n' =<_{st} Y_n$ $\quad n = 1,2,\ldots$

Then $\{Z^\wedge(t)\} =<_{st} \{W^\wedge(t)\}$

Proof

It is obvious from the condition of the Theorem and the condition on g that for all $t > 0$:-

$$P(W^\wedge(t) >= Z^\wedge(t)) = 1 \qquad \text{(see Figure 3.2B.3)}$$

it follows that $W^\wedge(t) >=_{st} Z^\wedge(t)$

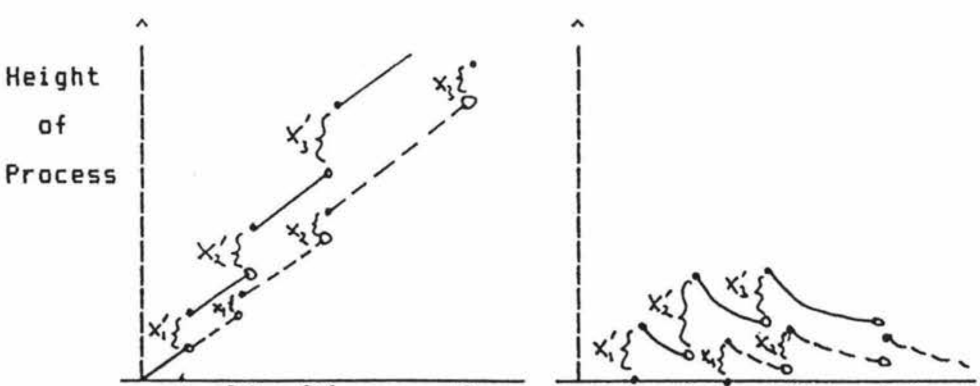

Figure (3.2B.2) : Sample paths of the damage process, W (unbroken line) and Z (broken line), under the conditions of Theorem (3.2B.5)

Theorems (3.2B.4) and (3.2B.5) can now be used to establish the NBU property of generalised damage processes.

Theorem (3.2B.6)

Let $(Y_n)_{n=0}^{\infty}$ be a generalised renewal sequence and let $\{Z(t)\}$ be a damage process generated by the random interuption of the continuous monotonic function g by the paired sequence of random variables $(X_n,Y_n)_{n=0}^{\infty}$ so that $\{Z(t)\}$ satisfies (3.2.47). Further, suppose that either g is decreasing; or g is increasing and $P(X_n>=0) = 1$; $n = 1,2,...$

If :-

(i) X_n is stochastically increasing in n

i.e. $X_{n+1} >=_{st} X_n$ $n = 1,2,..$

(i) Y_n is stochastically decreasing in n

and

(iii) Y_n has a NBU distribution $n = 0,1,2,...$

then $\{Z(t)\}$ is a NBU process and hence the lifetime distribution of a device with a fixed threshold and subject to $\{Z(t)\}$ is also NBU.

Proof

The proof of this Theorem is quite similar to the proof of Theorem (3.2B.2) and consequently some of the details are omitted. As with Theorem (3.2B.2) we begin the proof with an outline :-

Outline of proof of Theorem (3.2B.6)

Conditions (i) and (ii) of the Theorem allow the construction of a process {W,S} which can be shown to satisfy conditions (i) and (ii) of Theorem (3.2B.4). Theorem (3.2B.5) is used to show that {W,S} satisfies condition (i) of Theorem (3.2B.4).

The process {W,S} can be shown to be s.s. NBU by using condition (iii) of this Theorem and hence the desired result follows from Theorem (3.2B.4).

To construct the process $\{W,S\}$, we proceed as follows :-

Let $S_k = R_{n+k} - R_n \qquad k = 0,1,2,...$

$X'_k = X_{n+k} \qquad k = 0,1,2,...$

$X'_0 = 0$

and let $Y'_k = S_k - S_{k-1} \qquad k = 1,2,...$

$Y'_0 = 0$

Suppose that $\{W,S\}$ is generated by the random interruption of g, by $(X'_n, Y'_n)_{n=0}^{\infty}$ and that $\{W,S\}$ satisfies (3.2.47).

By condition (i) of the Theorem it follows that :-

$(X'_n) >=_{st} X_n) \qquad n = 1,2,...$

and by condition (ii)

$Y'_n =<_{st} Y_n \qquad n = 1,2,...$

so by Theorem 3.2B.5 it follows that :-

$\{W^{\wedge}(t)\} >=_{(st)} \{Z^{\wedge}(t)\}$

We now note that the processes $\{Z^n,R^n\}$ and $\{W,S\}$, where $\{Z^n,R^n\}$ is as defined in the earlier part of this section (see (3.2.25)) are similar in that they both commence at time R_n. They differ, however, in that the height of $\{W,S\}$ at time R_n is given by $Z(0) = 0$ whereas $\{Z^n,R^n\}$ is of height $Z(R_n)$ at time R_n .

Since $0 = Z(0) =< Z(R_n)$ and by our conditions on the monotonic function g and/or on, the sequence of damages $(X_n)_{n=1}$ and the fact that from time R_n on, the sequences of shocks and intervals of the two processes $\{Z_n,R_n\}$ and $\{W,S\}$ are identical, it follows from Theorem (3.2B.1) that $\{Z^n,R^n\} >=_{(*)} \{W,S\}$, where the relation $(>=_{(*)})$ is as defined earlier.

Before appealing to Theorem (3.2B.4) we must show that $\{W,S\}$ is a s.s. NBU process. This follows by first noting that if :-

$T_x^* = T_x(w) - Y'_1$

then $T_x^* >= T_x^* \mid Y_1' > S \qquad$ for every $S > 0$

and that Y_1' has a NBU distribution (by condition (iii) of this Theorem). That (W,S) is s.s. NBU then follows by an exactly analogous argument to that used in the proof of Theorem (3.2B.2).

Thus all the conditions of Theorem (3.2B.4) are satisfied and consequently {Z(t)} is a NBU process as required.

Theorems (3.2B.5) and (3.2B.6) are essentially due to Shanthikumar (1984) although, as mentioned previously, he considered only the case where g(t) was decreasing. In the case that g is increasing, the condition $P(X_n >= 0) = 1$ is required on the damages to ensure that Theorem 3.2B.5 holds.

Shanthikumar has shown that condition (iii) of Theorem (3.2B.6) can be weakened to :-

(iii)' $\quad Y_1 >=_{st} (Y_n \mid Y_n > u) - u \; ; \quad u > 0, \; n = 1,2,...$

As in the case where the shock interarrival times form a renewal sequence, it may be that the rate of wear or recovery is dependent on both the height of the damage process and on the age of the device. In this case we can define {Z(t)} as follows :-

(3.2.50) $\quad Z(0) = 0$

$$d/dt\,(Z(t)) = r(Z(t),t) \qquad R_n < t < R_n+1 \qquad n = 0,1,2,...$$

$$Z(R_n) = [Z(R_n^-) + X_n]^+, \qquad n = 0,1,2,...$$

where here $r(x,t)$ satisfies one of :-

(3.2.51) $\quad r(x,t), =< 0, \quad x >= 0, \quad t >= 0$ and increasing in t

or

(3.2.52) $\quad r(x,t) >= 0, \quad x >= 0, \quad t >= 0,$
$r(x,t)$ is increasing in t and
$P(X_n >= 0) = 1 \qquad n = 0,1,2,...$

Note that if (3.2.51) holds, then recovery takes place between shocks and if (3.2.52) holds, damage continues to accumulate between shocks.

For a damage process defined by (3.2.50) nothing in the proof of Theorem (3.2B.6) is changed. Consequently, the following Corollary holds :-

Corollary (3.2B.4)

Let $\{Z(t)\}$ be generated by the random interruption of the continuous monotonic function g by the paired sequence of random variables $(X_n, Y_n)_{n=0}^{\infty}$ where $(Y_n)_{n=0}^{\infty}$ is a generalised renewal sequence. Further, suppose that $\{Z(t)\}$ satisfies (3.2.50) then if :-

(i) (X_{n-1}) stochastically increasing in n

(ii) (Y_n) stochastically decreasing in n

(iii) Y_n has a NBU distribution, $n = 0,1,2,\ldots$

then $\{Z(t)\}$ is a NBU process i.e., the lifetime distribution of a device with a fixed damage threshold and subject to the damage process $\{Z(t)\}$ is NBU.

~3.2C Cumulative Damage Shock Models with Random Threshold

It is often the case that a population of otherwise identical devices exhibits a significant amount of variation in the ability of individual devices to withstand damage. In addition, there may be no practical way to inspect an individual item to determine its damage threshold. In these circumstances, it is often appropriate to regard the threshold level as a random variable. Such a model is the topic of this section.

We will return to the short model proposed by Esary Marshall and Proschan (1973) i.e. if H(t) is the lifetime distribution of a device subject to shocks which arrive according to the stochastic counting process {N(t)}, then:-

$$1 - H(t) = \bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k)\ \bar{p}_k$$

where $\bar{p}_k$ = P(surviving k shocks).

Now, if failure occurs on the first occasion that the total ccumulated damage due to shocks exceeds some random threshold with distribution G(.) such that G(0) = 0 and the damages X_1, X_2,... have distributions F_1, F_2,... then:-

(3.2.53) $$\bar{p}_k = \int_0^{\infty} F_1 * F_2 * \ldots F_k(x)dG(x)$$

where * denotes convolution.

Thus in this section we are primarily interested in the survivor function :-

(3.2.54) $\bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k) \int_0^{\infty} F_1 * F_2 * \ldots F_k(x)dG(x)$ It will be assumed that damage accumulates linearly. As usual, the principal aim in studying this model is to establish conditions on the $(\bar{p}_k)_{k=0}^{\infty}$ defined by (3.2.53) which are sufficient for H(t) as defined by (3.2.54) to belong to one of the classes of lifetime distribution discussed in Chapter one. In view of the results of Chapter two, however, we know that it is often the case that H(t) inherits its class from the survival probabilities $(\bar{p}_k)_{k=0}^{\infty}$.

Consequently, if it is possible to establish conditions on the threshold distribution G and on the damage distribution F_i $i = 1,2,...$ which are sufficient for the $(\bar{P}_k)_{k=0}^{\infty}$ to belong to a discrete class of distributions, then we will be able to conclude that H(.) belongs to the analogous continuous class. In addition we will, of course, have to impose appropriate conditions on the process governing the arrival of shocks.

We will begin by studying the simplest case where damages $(X_i)_{i=1}$ are i.i.d. with common distribution F. In this case :-

$$(3.2.55) \qquad \bar{P}_k = \int_0^\infty F^{(k)}(x)\, dG(x)$$
$$= \int_0^\infty \bar{G}(x)\, dF^{(k)}(x)$$

where $F^{(k)}(x)$ is the k-fold convolution of F with itself.

It is convenient to assume that $F^{(k)}$ and G have no common discontinuities in which case there is no problem in writing

$$(3.2.56) \qquad P_k = E(G(X_1 + X_2 + \dots X_k)$$

In this case :-

$$(3.2.57) \qquad H(t) = \Sigma_{k=0}^{\infty} P(N(t)=k) \int_0^\infty F^k(x) dG(x)$$

and the following result is almost immediate.

<u>Theorem (3.2C.1)</u> (Esary, Marshall and Proschan (1975))

Let {N(t)} be a homogeneous Poisson process and let H(t) be defined by (3.2.57), then H is exponential for any F iff G is exponential.

Proof

If $\{N(t)\}$ is a homogeneous Poisson Process it is clear that $H(t)$ is exponential iff

$$\int_0^\infty F^k(x)\,dG(x) = [\int_0^\infty F(x)\,dG(x)]^k$$

for any F.

Now, suppose that G is exponential then :-

$$\int_0^\infty F(x)\,dG(x) = \int \bar{G}(x)\,dF(x)$$

which is the Laplace transform of F. Hence, by a basic property of Laplace transforms :-

$$\int_0^\infty \bar{G}(x)\,dF^k(x) = [\int_0^\infty \bar{G}(x)\,dF(x)]^k$$
$$= [\int_0^\infty F(x)\,dG(x)]^k.$$

Conversely if $\int_0^\infty F^{(k)}(x)\,dG(x) = [\int_0^\infty F(x)\,dG(x)]^k$ for any F, let F be degenerate at $x_0 >= 0$. Then by the condition of the Theorem :-

$$\bar{G}(kx_0) = (\bar{G}(x_0))^k.$$

Now let u be a continuity point of G then since :-

$\lim_{x_0 \to 0} x_0[u/x_0] = u$ where $[u/x_0]$ = the integer part of $[u/x_0]$ we have :-

$$\bar{G}(u) = \lim_{x_0 \to 0} \bar{G}(x_0[u/x_0])$$
$$= \lim_{x_0 \to 0} (\bar{G}(x_0))^{[u/x_0]}$$
$$= \lim_{x_0 \to 0} (\bar{G}(x_0))^{1/x_0[u/x_0]x_0}$$
$$= \lim_{x_0 \to 0} ((\bar{G}(x_0)^{1/x_0})^u$$
$$= \lim_{x_0 \to 0} (1 - sx_0 + o(x_0)^{u/x_0}$$
$$\text{for some } 0 < s < \infty$$
$$= e^{-su} \text{ by the exponential limit theorem}$$

i.e. G is exponential.

The next result also due to Esary Marshall and Proschan (1973) provides sufficient condition for $(\bar{P}_k)_{k=0}^\infty$ as defined by (3.2.55) to belong to the IFRA and NBU Classes.

Theorem (3.2C.2)

Let $(\bar{p}_k)_{k=0}^{\infty}$ be defined by (3.2.55) then we have :-

(a) $(\bar{p}_k)_{k=0}^{\infty}$ is discrete IFRA for all F is G is IFR

(b) $(\bar{p}_k)_{k=0}^{\infty}$ is discrete NBU for all F iff G is NBU.

Note that the condition that G must be IFR for $(\bar{p}_k)_{k=0}^{\infty}$ to be discrete IFRA is stronger than we would really like but as yet it has not been shown that in general G IFRA => $(\bar{p}_k)_{k=0}^{\infty}$ discrete IFRA. BlocK and Savits (1978), however, have shown that this is the case if the common damage distribution F is exponential.

We note also that Esary, Marshall and Proschan have also shown that G in (3.2.55) is IFRA whenever $(\bar{p}_k)_{k=0}^{\infty}$ is discrete IFRA.

Theorem (3.2C.2) can be used to establish the class to which the lifetime distribution defined by (3.2.57) belongs by combining it with the appropriate results from Chapter two. Thus, directly from Theorem (3.2C.2) and Corollaries (2.1B.1) and (2.1D.1) and Theorems (2.1A.1) and (2.1C.1) we have :-

Theorem (3.2C.3)

Let H be defined by (3.2.57) so that H is the lifetime distribution of a cumulative damage shock model with a damage threshold which has a distribution G , and damages which are i.i.d. with a common distribution F. If G is IFR we have :-

(a) If $\{N(t)\}$ is a homogeneous Poisson process of rate s then H is IFRA.

(b) If $\{N(t)\}$ is a non-stationary Poisson process with mean-value function :-

$$I(t) = \int_0^t s(x)dx, \text{ such that}$$

$I(t)/t$ is increasing in t then H is IFRA.

(c) If $\{N(t)\}$ is a stationary pure birth process with birth coefficients $(s_k)_{k=0}^{\infty}$ which are increasing in k then H is IFRA.

(d) If $\{N(t)\}$ is a non-stationary pure birth process with transition probabilities given by :-

$$P(\text{shock in } (t, t+d) \mid k \text{ shocks in } (0,t))$$
$$s_k\, s(t)d + o(d) \; ; \quad s_k > 0 \, , \quad s(t) > 0$$

then H is IFRA whenever $(s_k)_{k=0}^{\infty}$ is increasing in k and $\int_0^t s(x)dx/t$ is increasing in t.

By using Theorem (3.2C.2(b)) and Corollaries (2.1B.1) and (2.1D.1) and Theorems (2.1A.1) and (2.1C.1) an exactly similar result could be established in the NBU case.

We turn now to the slightly more general case where the damages X_i i = 1,2,... are no longer i.i.d. but their distribution $F_i(z)$; i = 1,2,... are decreasing in i for all z. As in section ~3.2A we note that this provides a good model of a situation in which successive shocks become increasingly effective at causing damage. Esary, Marshall and Proschan state that Theorem (3.2C.2(b)) holds in this more general case, i.e., if :-

(3.2.58) $\quad P_k = \int_0^{\infty} F_1 * F_2 * \dots F_k(x) dG(x)$

where $\quad (F_i(x))_{i=1}^{\infty}$ is decreasing in i for each x

then $\quad (\bar{P}_k)_{k=0}^{\infty}$ is NBU iff G is NBU.

Using this fact and again applying Corollaries (2.1B.1) and (2.1D.1) and Theorems (2.1A.1) and (2.1C.1) yields :-

<u>Theorem (3.2C.4)</u>

Let H be defined by :-

$\bar{H}(t) = \Sigma_{k=0}^{\infty} \; P(N(t)=k) \; \int_0^{\infty} F_1 * F_2 * \dots F_k(x) dG(x)$ where $(F_j(x))_{j=1}^{\infty}$ is decreasing in j for each x then if G is NBU we have :-

a) If $\{N(t)\}$ is a homogeneous Poisson process of rate s then H is NBU.

b) If $\{N(t)\}$ is a non-stationary Poisson process with mean value function $I(t) = \int_0^t s(x)\,dx/t$ such that : $I(t)$ is increasing in t then H is NBU.

(c) If $\{N(t)\}$ is a non-stationary pure birth process with birth coefficients $(s_k)_{k=0}^{\infty}$ which form an increasing sequence then H is NBU.

(d) If $\{N(t)\}$ is a non-stationary pure birth process with transition probabilities given by :-
P(shock in $(t,t+d)$ | k shocks in $(0,t)$) $= s_k\, s(t)d + o(d)$ for small d, where $(s_k)_{k=0}^{\infty}$ is increasing in k and $\int s(x)dx/t$ is increasing in t then H is NBU.

So far, the case where $\{N(t)\}$, the process governing the arrival of shocks, is a generalised renewal process has not been considered. However, by combining Theorem (3.2C.2) and/or (3.2.58) with Theorems (2.2.1a) and (2.2.2(a) and Corollary (2.2.1) the following result is easily obtained.

<u>Theorem (3.2C.5)</u>

Define H by :-

$\bar{H}(t) = \Sigma_{k=0}^{\infty} P(N(t)=k) \int_0^{\infty} F_1 * F_2 * \ldots F_k(x)dG(x)$ where $\{N(t)\}$ is a generalised renewal process but has independent interarrival times, and either :-

$F_i(x) = F(x) \quad i = 1,2,\ldots, \quad x >= 0$

or $(F_i(x))_{i=1}^{\infty}$ is decreasing in i, $x >= 0$ Then we have :-

(a) H is NBU if the shock interarrival times J_i, $i = 1,2,\ldots$ are NBU and decreasing in i.

(b) H is NBUE if the shock interarrival times, Y_i , say, have NBUE distributions and $E(Y_i) = A_i$ is decreasing in i.

(c) H is HNBUE if the shock interarrival times have distributions which are HNBUE and $E(Y_i) \equiv A$; $i = 1,2,\ldots$

(d) H belongs to L if the shock interarrival times have distributions which belong to L and $E(Y_i) = A$, $i = 1,2,...$

The results presented so far have assumed only fairly mild conditions on the damage distribution F_i $i = 1,2,...$ If stronger conditions are imposed on the F_i it might be expected that, under appropriate conditions on the threshold distribution G, stronger conclusions could be drawn. It will be shown that This is in fact the case. A-Hameed and Proschan (1975) proposed a model in which the damages have gamma distributions F_i $i = 1,2,...$ with densities :-

$$f_i(x) = \frac{b^{ci} x^{(ci)-1} e^{-bx}}{\Gamma(ci)} ; \quad b >= 0, \ (ci) > 0, \ i = 1,2,...$$

In this case, the probability of surviving k shocks is given by :-

$$\bar{P}_k = \int_0^\infty F_1 * ... * F_k(x) dG(x)$$

$$= \int_0^\infty \bar{G}(x) \, d(F_1 * ... F_k(x))$$

(3.2.59) $$= b \int_0^\infty \bar{G}(x) \, (e^{-bx}(bx)(ck)-1)/\Gamma(S_k) \, dx$$

where $S_k = \Sigma_{i=1}(ci)$

Note that if $S_k = k$ the $\bar{P}_k$ of (3.2.59) are just the $a_k(b)$ used in Chapter one to provide alternative characterisation of classes of lifetime distribution. A-Hameed and Proschan (1975) established conditions under which the $(\bar{P}_k)_{k=0}^{\infty}$ defined by (3.2.58) belong to the discrete IFR, discrete IFRA, discrete NBU, discrete NBUE and discrete DMRL Classes. These results were extended to the HNBUE and L Classes by Klefsjo (1980, 1983). Combining these results yields the following theorem :-

Theorem (3.2C.6)

Define $\bar{P}_k$ as in (3.2.59); $k = 0,1,2,...$ where $\bar{G}$ is a survivor function such that $\bar{G}(0) = 1$. Then we have :-

(a) $(\bar{P}_k)_{k=0}^{\infty}$ is discrete IFR for every $b > 0$ if $(S_k)_{k=1}$ is convex i.e., (ck) increasing in k, and G is IFR.

(b) $(\bar{p}_k)_{k=0}^{\infty}$ is discrete IFRA for every $b > 0$ if $(S_k/k)_k$ is increasing and G is IFRA.

(c) $(\bar{p}_k)_{k=0}^{\infty}$ is discrete NBU for every $b > 0$ if $(S_k)_{k=0}^{\infty}$ is superadditive and G is NBU.

(d) $(\bar{p}_k)_{k=0}^{\infty}$ is discrete DMRL for every $b > 0$ if $S_k = k$, $k = 1,2,...$ and G is DMRL.

(e) $(\bar{p}_k)_{k=0}^{\infty}$ is discrete NBUE for every $b > 0$ if $S_k = k$, $k = 1,2,...$ and G is NBUE.

(f) $(\bar{p}_k)_{k=0}^{\infty}$ is discrete HNBUE for every $b >= 0$ if $S_k = k$, $k = 1,2,...$ and G is HNBUE.

(g) $(P(_k)_{k=0}^{\infty}$ belongs to G for every $b > 0$ if $S_k = k$, $k = 1,2,...$ and G belongs to L.

Note that in the special case that $S_k = k$ it was shown in Chapter one that the conditions of this Theorem are both necessary and sufficient. Note also that in Theorem (3.2C.6) the discrete sequence of survival probabilities $(\bar{p}_k)_{k=0}^{\infty}$ inherits its class from the distribution G in much the same way as the lifetime distribtion H of Chapter two inherited its' class from the discrete sequence of survival probabilities.

A companion result to Theorem (3.2C.6) holds for the dual Classes DFR, DFRA, IMRL, NWU, NWUE, HNWUE and $\bar{L}$. Similarly the results to be presented below show corresponding results for the dual class also. The next result is an obvious consequence of Theorem (3.2C.6) above and Corollary (2.10.1).

Theorem (3.2C.7)

Let $H(t) = 1 - \bar{H}(t)$ be defined by :-

(3.2.60) $\bar{H}(t) = \sum_{k=0}^{\infty} P(N(t)=k) \int_0^{\infty} b\bar{G}(x)e^{-bx} \frac{bx^{S_k-1}}{\Gamma(S_k)}dx$

where $b > 0$, $S_k >= 0$, and $\bar{G}$ is a survivor function satisfying $\bar{G}(0) = 1$.

If $\{N(t)\}$ is a non-stationary Poisson process with mean value $I(t) = \int_0^t s(u)dx$ then :-

(a) H is IFR whenever G is IFR, $(S_k)_{k=1}^{\infty}$ is convex and $I(t)$ is convex

(b) H is IFR wherever G is IFRA, (S_k/k) is increasing in k and $I(t)/t$ is increasing in t.

(c) H is NBU whenever G is NBU $(S_k)_{k=1}^{\infty}$ is superadditive and $I(t)$ is superadditive.

(d) H is DMRL whenever G is DMRL $S_k = k$ $k = 1,2,...$ and $I(t)$ is convex.

(e) H is NBUE whenever G is NBUE $S_k = k$, $k = 1,2,...$ and $I(t)/t$ is increasing in t.

(f) H is HNBUE whenever G is HNBUE $S_k = k$; $k = 1,2,...$ and $I(t)/t$ is starshaped.

(g) H belongs to L whenever G belongs to L, $S_k = k$, $k = 1,2,...$ and $I(t)/t$ is increasing in t.

Of course, this result includes the case where $\{N(t)\}$ is a homogeneous Poisson process. One need only put $s(t) = s > 0$ in the above, so that $I(t) = t$ and all the conditions required of $I(t)$ are automatically satisfied. Thus only the condition on $(S_k)_{k=1}^{\infty}$ and G are required for the homogeneous Poisson Model.

A similar result to Theorem (3.2C.7) can be established when $\{N(t)\}$ is a non-stationary pure birth process. This result follows from Theorem (3.2C.6) and (2.1C.1)

Theorem (3.2C.8)

Let H(t) be defined as in (3.2.60) and let {N(t)} be a non-stationary pure-birth process with transition probabilities given by :-

$$P(\text{shock in } (t, t+d) \mid k \text{ shocks in } (0,t)) = s_k\, s(t) d + o(d)\ ; \quad \text{where } d \text{ is small}, \quad s_k > 0,\ s(t) > 0.$$

Then we have :-

(a) H is IFR whenever S_k as defined in (3.2.59) is convex, $(s_k)_{k=1}^{\infty}$ is increasing, $I(t) = \int_0^t s(x)dx$ is convex and G is IFR.

(b) H is IFRA whenever G is IFRA, $(S_k/k)_{k=0}^{\infty}$ is increasing in k, $I(t)/t$ is increasing in t and $(s_k)_{k=0}^{\infty}$ is increasing in k.

(c) H is NBU whenever G is NBU, $(S_k)_{k=0}^{\infty}$ and I(t) are superadditive and $(s_k)_{k=0}^{\infty}$ is increasing in k.

(d) H is DMRL whenever G is DMRL, $S_k = k$, $k = 1,2,...,$ I(t) is convex and $(s_k)_{k=0}^{\infty}$ is increasing in k.

(e) H is NBUE whenever G is NBUE $S_k = k$, $k = 1,2,...$ I(t) is starshaped and $(s_k)_{k=0}^{\infty}$ is increasing in k.

By putting $s(t) \equiv 1$ in the above Theorem, the corresponding result for the stationary pure birth model is obtained.

Unfortunately, in the case of the non-stationary and stationary pure birth models, straight forward results for the HNBUE and L classes are not available. This is because the required conditions on the survival probabilities $(\bar{P}_n)$ and on the birth coefficient, $(s_k)_{k=0}^{\infty}$ are stronger than (s_k) increasing and $(\bar{P}_k)$ discrete HNBUE or $(\bar{P}_k)$ belongs to G (C.f Theorems (2.1A.1) and (2.1C.1))

The next result deals with the case where shocks arrive according to a generalised renewal process. The result is a direct consequence of Theorem (3.2C.6) and Theorems (2.2.1(a)), (2.2.2(a)) and Corollary (2.2.1).

<u>Theorem (3.2C.9)</u>

Let H be defined as in (3.2.60) and let $\{N(t)\}$ be a generalised renewal process :-

(a) H is NBU whenever the shock inter-arrival times have distributions J_i, which are NBU and decreasing in i, G is NBUE and (S_k) superadditive.

(b) H is NBUE whenever the shock interarrival times, Y_i, have NBUE distributions of $E(Y_i) = A_i$ is decreasing in i and G is NBUE.

(c) H is HNBUE whenever the shock inter-arrival times (Y_i) have HNBUE distributions and $E(Y_i) = A$, i = 1,2,.... and G is HNBUE.

(d) H belongs to L whenever the shock inter-arrival times have distributions belonging to L, $E(X_i) = A$; i = 1,2,.... and G belongs to L.

It is clear from the above results that when the arrival of shocks is governed by a homogeneous or non-stationary Poisson process or by a stationary or non-stationary pure birth process, the lifetime distribution of the associated random threshold Cumulative Damage model, in which the damages are assumed to be independent Gamma random variables, inherits its class from the threshold distribution.

In the NBU case the assumption of independent shock interarrival times may be relaxed somewhat. If the assumption of independence is replaced by :-

$\{N(t)\}$ is a stochastic counting process such that :-

(3.2.61) $P(N(x+y) > j+k \mid N(x) = k) >= P(N(y)>j)$
for any $x>0$, $y>0$, $j>0$ $k>0$.

The following result is a direct consequence of Theorems (2.2.3) and (3.2C.6).

Theorem (3.2C.10)

Define H by (3.2.60) where {N(t)} satisfies (3.2.61) then H is NBU whenever (S_{k-0}^{∞}) is superadditive and G is NBU.

Once again we note that the life distribution H inherits its class (NBU in this case) from the threshold distribution.

A random threshold cumulative damage model in which the shock interarrival times are constantly correlated has been studied by Sathiyamoorthi (1980). In this model, no distribution is assumed for the damages but the threshold is assumed to have an exponential distribution and as well as being constantly correlated, it is assumed that the interarrival times are exchangeable exponential random variables. (A sequence (Y_n) of random variables is said to be exchangeable if the joint distribution, $J_n(y_1,\ldots.y_n)$ of any n of the variables can be written as :-

$$J_n(y_1,\ldots.y_n) = \int_{\Omega} K_w(y_1)\ldots K_w(y_n)dP(w)$$

where Ω is the space of w and for each w in Ω $K_w(y)$ is a conditional distribution function in y, and for a given y $K_w(y)$ is a random variable in w.

Using an earlier result of Gurland (1955) Sathiyamoorthi (1980) derives the Laplace-Stieltjes transform of the life distribution for such a model and gives the first and second moments of the distribution.

Another cumulative damage model with an exponentially distributed threshold is proposed by Ramanarayanan (1976). In this model it is assumed that a shock causes damage only if the worker in charge of the device is not alert. Periods of alertness are assumed to be i:i.d random variables and initially the worker is assumed to be alert. After the occurrence of a shock the worker is re-alerted with probability P.

Ramanarayanan obtains the Laplace-Streltges transform of the lifetime distribution for such a model in terms of the Laplace-Stieltges transform of the distribution of the interval between successive damages.

To conclude this section, we remark that a random threshold model in which wear or recovery is allowed between shocks as in section 3.2B would be of some interest. As yet, however, such a model appears not to have been considered in the literature. The NBU results of section 3.2B do not carry over to the random threshold case in any obvious way since if G is the Threshold distribution and $T_z(z)$ is the first passage time of the damage process $\{Z(t)\}$ to the level z then the survivor function of interest is :-

$H(t) = \int_0^\infty P(T_z(Z)>t)dG(z)$ and the NBU class is not closed under the mixture operation.

Conclusions and Suggestions

The common theme running through the models presented in this thesis is that the lifetime distributions associated with the models inherit their class from the shock survival probabilities and, in the case of more general models, the shock interarrival times.

This prompts the question :-

> Is it any easier to establish the class to which the shock survival probabilities and the shock interarrival times belong than it is to determine the class of the lifetime distribution directly ?

For an answer to this question, we will have to wait until such time as the theory of shock models is applied more frequently to practical problems.

To date, the literature on applied Shock Models is extremely sparse. However, with some of the more recent theoretical developments, e.g., the models with wear and recovery of section 3.2B, seemingly quite plausible and well-suited to practical application, this situation may be rectified soon. Of course, even if establishing the Class of the survival probabilities and shock interarrival times proves no easier than establishing the Class of the lifetime distribution directly, the results presented in this thesis may still be of some interest, provided that the physical properties of the system being modelled lend themselves to the assumption of a particular class for the survival probabilities and/or the interarrival time distributions.

While the application of the theory of Shock Models to practical problems would seem to be an obvious step in the development of the subject, further development of the theory itself is also required. In particular, models with correlated shock interarrival times have not yet received much attention, although the paper by Sathiyamoothi (1980) is a step in this direction and we also noted, in passing, that the interarrival times between shocks which occur according to a non-stationary Poisson Process are not independent. In adopting a correlation scheme for the interarrival times, it may well be that an Exponential Autoregressive process (EAR process) is appropriate, since it is a quite mathematically tractable generalisation of the Poisson Process and is easy to simulate on a computer (see Gower and Lewis (1980), Jacobs and Lewis (1977) and also Cox and Isham (1980) pp. 62-65.

In this thesis, a "Black-box" approach has been taken regarding the device which is subject to shocks, i.e., the structure of the device has been ignored. It may be, however, that the device in question is made up of several components, each of which has a different tolerance to shocks, hence the internal structure of the device may be of some importance in determining the lifetime distribution of the device. Considerations such as these lead to Multivariate Shock Models and require the definition of classes of multi-variate distributions. Some work has been done in this area, notably by Ghosh and Ebrahimi (1982), Ghuyre and Marshall (1984) and Marshall and Shaked (1979).

The theory of multivariate models, however, is not as well advanced as it is in the univariate case. In fact, it may often be the case that a univariate model can be applied successfully to a multi-component device. Suppose, for example, that the components of a device are connected in series and all have different shock survival probabilities, then provided we assume that each component is equally prone to shocks, the shock survival probability of the device is equal to the smallest survival probability of the components. If the components are connected in parallel, the shock survival probability of the device is equal to the largest survival probability of the components.

Another type of multivariate Shock Model is one in which there is more than one source of shocks. Such a model has been studied by Barlow and Proschan (1975) although their main motivation for introducing such a model appears to have been to generate a multivariate exponential distribution. Multivariate models of this form do not appear to have received much subsequent attention. Consequently, this is another area which could provide scope for further developments in the theory and application of Shock Models. It is likely that the theory of superposition of Point Processes could prove useful in this context.

In conclusion, then, it appears that while the theory of shock models, especially in the univariate case, is quite well developed and encompasses some apparently quite plausible situations, it has not often been applied in a meaningful way. Thus there is a need for further development in the area of applied shock models and it is likely that this will generate a need for further theoretical developments.

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