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**Two Generator Discrete Groups of Isometries
and Their Representation**

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Abstract

Let M_ϕ and M_ψ be elements of $PSL(2, \mathbb{C})$ representing orientation preserving isometries on the upper half-space model of hyperbolic 3-space ϕ and ψ respectively. The parameters

$$\beta = \text{tr}^2(M_\phi) - 4, \quad \beta' = \text{tr}^2(M_\psi) - 4, \quad \gamma = \text{tr}[M_\phi, M_\psi] - 2,$$

determine the discrete group $\langle \phi, \psi \rangle$ uniquely up to conjugacy whenever $\gamma \neq 0$. This thesis is concerned with explicitly lifting this parameterisation of $\langle \phi, \psi \rangle$ to $PSO(1, 3)$ realised as a discrete 2 generator subgroup of orientation preserving isometries on the hyperboloid model of hyperbolic 3-space. We particularly focus on the case where both ϕ and ψ are elliptic.

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