Holomorphic Solutions to Functional Differential Equations

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Abstract

Functional differential equations with an entire functional argument g are examined and theory regarding the presence of holomorphic solutions to these equations presented. There are two main problems analysed, each related to the other. The first is the existence of local holomorphic solutions about a point fixed under g, and the second is the analytic continuation of such a solution throughout the complex plane. The local behaviour of g about its fixed points determines whether holomorphic solutions exist about such points, whilst the global behaviour of g under iteration determines the analytic continuation of these solutions. The dynamics of the functional argument g, therefore, is the driving force in both problems.

Both a local and global theory is developed for the existence of solutions, and for defining where such solutions are holomorphic. The case where g is a polynomial is considered in detail, although much of the theory applies equally well to the general case where g is entire.

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