Differential Geometry of Projectively Related Finsler Spaces

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This Thesis is Dedicated to the Memory of

My Loving Mother

and

My Loving Father

Abstract

The aim of this thesis is to study the theory of Finsler spaces by considering the following main objectives.

(i) To present the basic concepts of Finsler geometry including connections, flag curvature, projective changes, Randers spaces and Finsler spaces with other types of (α, β)-metric, where α is a Riemannian metric and β is a one-form.

(ii) To introduce a Riemannian space of non-zero constant sectional curvature by considering a locally projectively flat Finsler space. The requirement for the Riemannian connection to be metric compatible gives a system of partial differential equations. Further, we compute two standard Riemannian metrics of non-zero constant sectional curvature by choosing two solutions of this system of partial differential equations.

(iii) To give two examples of locally projectively flat Randers metrics of scalar curvature by using a Riemannian metric computed in (ii) to illustrate the fact that some locally projectively flat Randers metrics of scalar curvature do not have isotropic S-curvature. We also prove that the scalar curvature of a Randers metric is not necessarily a constant if the metric has isotropic S-curvature and closed one-form by using an example.

(iv) To find necessary and sufficient conditions for Finsler spaces with various types of (α,β)-metric to be locally projectively flat and determine whether the conditions, a Riemannian metric (α) is locally projectively flat and a one-form (β) is closed, can occur at the same time in the locally projectively flat Finsler spaces with various types of (α,β)-metric.

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Notation

Throughout this dissertation the following notation is used.

M : - Differentiable manifold of dimension *n*.

p : - A point on M.

$(x^1,...,x^n)$: - Local coordinates of p denoted by x.

T_pM : - Tangent space of M at p.

y : - A tangent vector at p.

$(y^1,...,y^n)$: - Vector components of y.

$\left\{\frac{\partial}{\partial x^1},...,\frac{\partial}{\partial x^n}\right\}$: - A basis in T_pM.

$y = y^1\frac{\partial}{\partial x^1} + ... + y^n\frac{\partial}{\partial x^n} = y^i\frac{\partial}{\partial x^i}$: - The Einstein summation convention.

$TM = \bigcup_{p\in M} T_pM$: - Tangent bundle of M.

(x, y) : - Local coordinates of a point in TM.

$\mathfrak{I}^n = (M, F)$: - A Finsler space of dimension *n*.

$F = F(x, y)$: - Finsler metric on M.

$l = (l^1,...,l^n)$: - Normalized supporting element of F, where $l^i = y^i / F$.

$l_i = \frac{\partial F}{\partial y^i}$: - Partial derivative of F with respect to y^i.

$g_{ij} = g_{ij}(x, y)$: - Finsler metric tensor of F (page 9, chapter 2).

$h_{ij} = h_{ij}(x, y)$: - Angular metric tensor of F (page 15, chapter 2).

$G^i = G^i(x, y)$: - Geodesic coefficients of F (page 19, chapter 2).

$G^i_{jk} = G^i_{jk}(x, y) = \dfrac{\partial^2 G^i}{\partial y^j \partial y^k}$: - Coefficients of the Berwald connection (page 19, chapter 2).

$\alpha = \alpha(x, y)$: - Riemannian metric defined on M.

$\Gamma^i_{jk} = \Gamma^i_{jk}(x)$: - Christoffel symbols of the Riemannian connection (page 17, chapter 2).

$\beta = \beta(x, y)$: - Differentiable one-form defined on M.

R : - Riemann curvature (page 31, chapter 3).

K : - Scalar curvature (page 36, chapter 3).

τ : - Distortion (page 40, chapter 3).

C : - Cartan torsion (page 40, chapter 3).

I : - Mean Cartan torsion (page 41, chapter 3).

S : - S-curvature (page 42, chapter 3).

E : - E-curvature (page 43, chapter 3).

L : - Landsberg curvature (page 48, chapter 3).

J : - Mean Landsberg curvature (page 49, chapter 3).

R^n : - *n*-dimensional real vector space.

All lower case Latin letters of the Einstein summation run from 1 to *n*.

That is, $i, j, k, r, s... = 1,...,n$.