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On the number of representations of certain integers as sums of eleven or thirteen squares

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Dedicated to Sri Chinmoy on the occasion of his 70th birthday.

Abstract
Let $r_k(n)$ denote the number of representations of an integer n as a sum of k squares. We prove that for odd primes p,

$$r_{11}(p^2) = \frac{330}{31}(p^9+1) - 22(-1)^{(p-1)/2}p^4 + \frac{352}{31}H(p),$$

where $H(p)$ is the coefficient of q^p in the expansion of

$$q\prod_{j=1}^{\infty}(1-(-q)^j)^{16}(1-q^{2j})^4 + 32q^2\prod_{j=1}^{\infty}\frac{(1-q^{2j})^{28}}{(1-(-q)^j)^8}.$$

This result, together with the theory of modular forms of half integer weight is used to prove that

$$r_{11}(n) = r_{11}(n')\frac{2^{9\lfloor\lambda/2\rfloor+9}-1}{2^9-1}\prod_p\left[\frac{p^{9\lfloor\lambda_p/2\rfloor+9}-1}{p^9-1} - p^4\left(\frac{-n'}{p}\right)\frac{p^{9\lfloor\lambda_p/2\rfloor}-1}{p^9-1}\right],$$

where $n = 2^\lambda\prod_p p^{\lambda_p}$ is the prime factorisation of n and n' is the square-free part of n, in the case that n' is of the form $8k+7$. The products here are taken over the odd primes p, and $\left(\frac{n}{p}\right)$ is the Legendre symbol.

We also prove that for odd primes p,

$$r_{13}(p^2) = \frac{4030}{691}(p^{11}+1) - 26p^5 + \frac{13936}{691}\tau(p),$$

where $\tau(n)$ is Ramanujan's τ function, defined by $q\prod_{j=1}^{\infty}(1-q^j)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n$. A conjectured formula for $r_{2k+1}(p^2)$ is given, for general k and general odd primes p.

1 Introduction

Let $r_k(n)$ denote the number of representations of n as a sum of k squares. The generating function for $r_k(n)$ is

$$\left(\sum_{j=-\infty}^{\infty} q^{j^2}\right)^k = \sum_{n=0}^{\infty} r_k(n)q^n.$$

The study of $r_k(n)$ has a long and interesting history. Generating functions which yield the value of $r_k(n)$ for $k = 2,\ 4,\ 6$ and 8 were found by Jacobi [18]. Glaisher [12] proved formulas for $k = 10,\ 12,\ 14,\ 16$ and 18, and Ramanujan [25, eqs. (145)–(147)] stated a general formula for

$r_k(n)$ for arbitrary even values of k. Ramanujan's formula was proved by Mordell [23] and a simple proof was given by Cooper [4].

The problem of determining $r_k(n)$ for odd values of k is more difficult, and not so well known. The value of $r_3(n)$ was found by Gauss [11, Section 291]. Formulas for $r_5(n)$ and $r_7(n)$ for square-free values of n were stated without proof by Eisenstein [9], [10]. These formulas were extended to all non-negative integers n (again without proof) by Smith [30]. The Paris Academy of Sciences, apparently unaware of Smith's work [30], proposed as its Grand Prix des Sciences Mathématiques competition for 1882 the problem of completely determining the value of $r_5(n)$. The prize was awarded jointly to Smith [32] and Minkowski [22], who both gave formulas as well as proofs. An interesting account of this competition and the controversy surrounding it has been given by Serre [29]. Others to have worked on this problem are (chronologically) Stieltjes [33], Hurwitz [16], Hardy [13], [14], Lomadze [20], [21], Sandham [27], [28] and Hirschhorn and Sellers [15]. More information can be found in [7, Chapters VI–IX].

Eisenstein [8], [31] stated that the sequence of formulas for $r_k(n)$ ceases for $k \geq 9$. As a result of computer investigations [5], I found that there are in fact simple, closed form formulas for $r_9(n)$, but only for certain values of n. The purpose of this article is to state and prove the corresponding results for $r_{11}(n)$. These results were also initially discovered as a result of computer investigations.

There appear to be no similar formulas for $r_k(n)$ for $k = 13,\ 15,\ 17, \cdots$. Formulae can be given, but they all involve more complicated number theoretic functions.

2 Summary of results

Throughout this article p will always denote an odd prime, and $\prod_p$ will always denote a product over all odd primes p. Accordingly, let us denote the prime factorisation of n by

$$n = 2^{\lambda} \prod_p p^{\lambda_p},$$

where λ and λ_p are all nonnegative integers, only finitely many of which are non-zero. Let n' denote the square-free part of n.

Hirschhorn and Sellers [15] proved that

$$r_3(n) = r_3(n') \prod_p \left[\frac{p^{\lfloor \lambda_p/2 \rfloor + 1} - 1}{p - 1} - \left(\frac{-n'}{p} \right) \frac{p^{\lfloor \lambda_p/2 \rfloor} - 1}{p - 1} \right] \tag{2.1}$$

where n' is the square-free part of n. The analogous results for $r_5(n)$, $r_7(n)$ and $r_9(n)$ were proved by Cooper [5]. For sums of five squares we have

$$\begin{aligned} r_5(n) &= r_5(n') \left[\frac{2^{3\lfloor \lambda/2 \rfloor + 3} - 1}{2^3 - 1} + \epsilon \, \frac{2^{3\lfloor \lambda/2 \rfloor} - 1}{2^3 - 1} \right] \\ &\quad \times \prod_p \left[\frac{p^{3\lfloor \lambda_p/2 \rfloor + 3} - 1}{p^3 - 1} - p \left(\frac{n'}{p} \right) \frac{p^{3\lfloor \lambda_p/2 \rfloor} - 1}{p^3 - 1} \right], \end{aligned} \tag{2.2}$$

where

$$\epsilon = \begin{cases} 0 & \text{if } n' \equiv 1 \pmod 8 \\ -4 & \text{if } n' \equiv 2 \text{ or } 3 \pmod 4 \\ -16/7 & \text{if } n' \equiv 5 \pmod 8. \end{cases} \tag{2.3}$$

The result for sums of seven squares is

$$\begin{aligned} r_7(n) &= r_7(n') \left[\frac{2^{5\lfloor \lambda/2 \rfloor + 5} - 1}{2^5 - 1} + \epsilon \, \frac{2^{5\lfloor \lambda/2 \rfloor} - 1}{2^5 - 1} \right] \\ &\quad \times \prod_p \left[\frac{p^{5\lfloor \lambda_p/2 \rfloor + 5} - 1}{p^5 - 1} - p^2 \left(\frac{-n'}{p} \right) \frac{p^{5\lfloor \lambda_p/2 \rfloor} - 1}{p^5 - 1} \right], \end{aligned} \tag{2.4}$$

where

$$\epsilon = \begin{cases} 8 & \text{if } n' \equiv 1 \text{ or } 2 \pmod 4 \\ 0 & \text{if } n' \equiv 3 \pmod 8 \\ -64/37 & \text{if } n' \equiv 7 \pmod 8). \end{cases} \tag{2.5}$$

For squares, these formulas reduce to

$$r_3(n^2) = 6\prod_p \left[\frac{p^{\lambda_p+1}-1}{p-1} - (-1)^{(p-1)/2}\frac{p^{\lambda_p}-1}{p-1}\right], \tag{2.6}$$

$$r_5(n^2) = 10\left[\frac{2^{3\lambda+3}-1}{2^3-1}\right]\prod_p \left[\frac{p^{3\lambda_p+3}-1}{p^3-1} - p\frac{p^{3\lambda_p}-1}{p^3-1}\right], \tag{2.7}$$

$$r_7(n^2) = 14\left[\frac{2^{5\lambda+5}-1}{2^5-1} + 8\frac{2^{5\lambda}-1}{2^5-1}\right] \times \prod_p \left[\frac{p^{5\lambda_p+5}-1}{p^5-1} - (-1)^{(p-1)/2}p^2\frac{p^{5\lambda_p}-1}{p^5-1}\right]. \tag{2.8}$$

Equations (2.6) and (2.7) seem to have been first been explicitly stated by Hurwitz [17] and [16], respectively, and (2.8) is due to Sandham [27].

For sums of nine squares we have

$$r_9(n) = r_9(n')\frac{2^{7\lfloor\lambda/2\rfloor+7}-1}{2^7-1}\prod_p \left[\frac{p^{7\lfloor\lambda_p/2\rfloor+7}-1}{p^7-1} - p^3\left(\frac{n'}{p}\right)\frac{p^{7\lfloor\lambda_p/2\rfloor}-1}{p^7-1}\right], \tag{2.9}$$

in the case that $n' \equiv 5 \pmod 8$. Results for when $n' \not\equiv 5 \pmod 8$ were also given in [5], but these involve additional and more complicated number theoretic functions.

The first purpose of this article is to prove that for odd primes p,

$$r_{11}(p^2) = \frac{330}{31}(p^9+1) - 22(-1)^{(p-1)/2}p^4 + \frac{352}{31}H(p),$$

where $H(p)$ is the coefficient of q^p in the expansion of

$$q\prod_{j=1}^{\infty}(1-(-q)^j)^{16}(1-q^{2j})^4 + 32q^2\prod_{j=1}^{\infty}\frac{(1-q^{2j})^{28}}{(1-(-q)^j)^8}.$$

This result, together with the theory of modular forms of half integer weight, is then used to prove

$$r_{11}(n) = r_{11}(n')\frac{2^{9\lfloor\lambda/2\rfloor+9}-1}{2^9-1}\prod_p \left[\frac{p^{9\lfloor\lambda_p/2\rfloor+9}-1}{p^9-1} - p^4\left(\frac{-n'}{p}\right)\frac{p^{9\lfloor\lambda_p/2\rfloor}-1}{p^9-1}\right]. \tag{2.10}$$

in the case that $n' \equiv 7 \pmod 8$. Some results for when $n' \not\equiv 7 \pmod 8$ will also be given, but just as for sums of nine squares, these involve additional and more complicated number theoretic functions.

The value of $r_{11}(n')$ for square-free n' is given by

$$r_{11}(n') = \frac{31680}{31}\sum_{j=1}^{(n'-1)/2}\left(\frac{j}{n'}\right)j^3(j-n'), \qquad \text{if } n' \equiv 7 \pmod 8). \tag{2.11}$$

The second purpose of this article is to prove that for odd primes p,

$$r_{13}(p^2) = \frac{4030}{691}(p^{11}+1) - 26p^5 + \frac{13936}{691}\tau(p),$$

where $\tau(n)$ is Ramanujan's τ function, defined by

$$q\prod_{j=1}^{\infty}(1-q^j)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n. \tag{2.12}$$

We also compute the eigenvalues and eigenfunctions of the Hecke operators T_{p^2}. There does not appear to be a formula like (2.1), (2.2), (2.4), (2.9) or (2.10) that expresses $r_{13}(n)$ in terms of just $r_{13}(n')$ and the prime factorisation of n, for any special cases of n'.

Finally, we offer a conjectured formula for the value of $r_{2k+1}(p^2)$, for general k and a general odd prime p.

3 Definitions

Let $|q| < 1$ and set

$$(a;q)_\infty = \prod_{j=1}^{\infty}(1-aq^{j-1}).$$

Let

$$\begin{aligned}\phi(q) &= \sum_{j=-\infty}^{\infty} q^{j^2},\\ \psi(q) &= \sum_{j=0}^{\infty} q^{j(j+1)/2}.\end{aligned}$$

Let s and t be non-negative integers. We will always assume t is a multiple of 4. Put

$$\phi_{s,t}(q) = \phi(q)^s\left(2q^{1/4}\psi(q^2)\right)^t,$$

and let us write

$$\phi_{s,t}(q) = \sum_{n=0}^{\infty}\phi_{s,t}(n)q^n,$$

so that $\phi_{s,t}(n)$ is the coefficient of q^n in the series expansion of $\phi_{s,t}(q)$. Geometrically, $\phi_{s,t}(n)$ counts the number of lattice points, that is, points with integer coordinates, on the sphere

$$\sum_{j=1}^{s} x_j^2 + \sum_{j=s+1}^{s+t}(x_j - 1/2)^2 = n.$$

Let $r_{s,t}(n)$ denotes the number of representations of n as a sum of $s+t$ squares, of which s are even and t are odd. Then $r_{s,t}(n)$ has the generating function

$$\sum_{n=0}^{\infty} r_{s,t}(n)q^n = \binom{s+t}{t}(2q)^t\phi(q^4)^s\psi(q^8)^t = \binom{s+t}{t}\phi_{s,t}(q^4).$$

Let

$$\begin{aligned}z &= \phi(q)^2\\ x &= 16q\frac{\psi(q^2)^4}{\phi(q)^4}.\end{aligned}$$

Then [3] or [24, Ch. 16, Entry 25 (vii)]

$$\begin{aligned} z^{(s+t)/2}x^{t/4} &= \phi_{s,t}(q) \\ 1-x &= \frac{\phi(-q)^4}{\phi(q)^4}. \end{aligned}$$

By Jacobi's triple product identity,

$$z = \frac{(q^2;q^2)_\infty^{10}}{(q;q)_\infty^4(q^4;q^4)_\infty^4} = \frac{(-q;-q)_\infty^4}{(q^2;q^2)_\infty^2}$$

$$x(1-x) = 16q\frac{(q^2;q^2)_\infty^{24}}{(-q;-q)_\infty^{24}}.$$

Therefore

$$z^5x(1-x) = 16q\frac{(q^2;q^2)_\infty^{14}}{(-q;-q)_\infty^4} \tag{3.1}$$

$$z^6x(1-x) = 16q(q^2;q^2)_\infty \tag{3.2}$$

$$z^{10}x^2(1-x)^2 = 256q^2\frac{(q^2;q^2)_\infty^{28}}{(-q;-q)_\infty^8} \tag{3.3}$$

$$z^{12}x(1-x) = 16q(-q;-q)_\infty^{24} = 16\sum_{n=1}^{\infty}(-1)^{n+1}\tau(n)q^n, \tag{3.4}$$

where $\tau(n)$ is Ramanujan's τ-function defined in equation (2.12).

Let $H(q)$ and $H(n)$ be defined by

$$H(q) = q(-q;-q)_\infty^{16}(q^2;q^2)_\infty^4 + 32q^2\frac{(q^2;q^2)_\infty^{28}}{(-q;-q)_\infty^8} = \sum_{n=1}^{\infty}H(n)q^n. \tag{3.5}$$

Then

$$H(q) = \frac{1}{16}z^{10}x(1-x)(1+2x-2x^2). \tag{3.6}$$

We will require some facts about Hecke operators on modular forms of half integer weight. All of the facts below can be found in [19].

Fact 1

$M_{(2k+1)/2}(\tilde\Gamma_0(4))$ is the vector space consisting of all linear combinations of $\phi_{2k+1-4j,4j}(q)$, $j = 0, 1, \cdots, \lfloor k/2\rfloor$ [19, p. 184, Prop. 4].

Fact 2

The Hecke operators T_{p^2}, where p is any prime, map $M_{(2k+1)/2}(\tilde\Gamma_0(4))$ into itself [19, p. 206]. That is,

$$f \in M_{(2k+1)/2}(\tilde\Gamma_0(4)) \Rightarrow T_{p^2}f \in M_{(2k+1)/2}(\tilde\Gamma_0(4)).$$

Fact 3

Suppose

$$f(q)=\sum_{n=0}^{\infty}a(n)q^n\in M_{(2k+1)/2}(\tilde{\Gamma}_0(4)).$$

If p is an odd prime, then [19, p. 207, Prop. 13] implies

$$T_{p^2}f=\sum_{n=0}^{\infty}a(p^2n)q^n+p^{k-1}\sum_{n=0}^{\infty}\left(\frac{(-1)^kn}{p}\right)a(n)q^n+p^{2k-1}\sum_{n=0}^{\infty}a(n/p^2)q^n,$$

while if $p=2$, the note in [19, page 210] implies

$$T_{2^2}f=\sum_{n=0}^{\infty}a(4n)q^n.$$

4 Some preliminary lemmas

Lemma 4.1 *Let*

$$\begin{aligned}
\frac{1}{4}z^5(5-x)(1-x) &= \sum_{n=0}^{\infty}a(n)q^n\\
\frac{1}{64}z^5(4x+x^2) &= \sum_{n=0}^{\infty}b(n)q^n\\
\frac{1}{16}z^5x(1-x) &= \sum_{n=0}^{\infty}c(n)q^n\\
-\frac{1}{8}z^6(1-x)(1-x+x^2) &= \sum_{n=0}^{\infty}d(n)q^n\\
\frac{1}{8}z^6(1-x)(1-x^2) &= \sum_{n=0}^{\infty}e(n)q^n.
\end{aligned}$$

Let $n=2^{\lambda}\prod_p p^{\lambda_p}$ *be the prime factorisation of* n. *For any prime* $p\equiv 1 \pmod 4$, *let* x_p *and* y_p *be the unique pair of positive integers satisfying*

$$x_p^2+y_p^2=p,\quad 2|x_p,$$

and define θ_p *by*

$$\tan\theta_p=\frac{y_p}{x_p},\quad 0<\theta<\frac{\pi}{2}.$$

Then

$$a(0)=5/4,\quad b(0)=0,\quad c(0)=0,\quad d(0)=-1/8,\quad e(0)=1/8$$

and for $n\geq 1$,

$$\begin{aligned}
a(n) &= \prod_p(-1)^{\lambda_p(p-1)/2}\left[\frac{p^{4(\lambda_p+1)}-(-1)^{(\lambda_p+1)(p-1)/2}}{p^4-(-1)^{(p-1)/2}}\right]\\
b(n) &= 2^{4\lambda}\prod_p\frac{p^{4(\lambda_p+1)}-(-1)^{(\lambda_p+1)(p-1)/2}}{p^4-(-1)^{(p-1)/2}}
\end{aligned}$$

$$c(n) = \begin{cases} 0, & \textit{if } \lambda_p \textit{ is odd for any } p \equiv 3 \pmod 4 \\ n^2(-1)^{\lambda} \prod\limits_{p\equiv 1 \pmod 4} \dfrac{\sin 4(1+\lambda_p)\theta_p}{\sin 4\theta_p}, & \textit{otherwise} \end{cases}$$

$$d(n) = \begin{cases} \prod\limits_{p} \dfrac{p^{5\lambda_p+5}-1}{p^5-1} & \textit{if } n \textit{ is odd} \\ -\left(\dfrac{30\times 2^{5\lambda}+63}{2^5-1}\right)\prod\limits_{p}\dfrac{p^{5\lambda_p+5}-1}{p^5-1} & \textit{if } n \textit{ is even} \end{cases}$$

$$e(n) = \begin{cases} \prod\limits_{p} \dfrac{p^{5\lambda_p+5}-1}{p^5-1} & \textit{if } n \textit{ is odd} \\ -\left(\dfrac{2^{5\lambda+5}-63}{2^5-1}\right)\prod\limits_{p}\dfrac{p^{5\lambda_p+5}-1}{p^5-1} & \textit{if } n \textit{ is even.} \end{cases}$$

Proof.
From [3] or [24, Ch. 17, Entry 17 (viii)] we have

$$\frac{1}{4}z^5(5-x)(1-x) = \frac{5}{4} + \sum_{k=0}^{\infty} \frac{(-1)^k(2k+1)^4 q^{2k+1}}{1-q^{2k+1}}.$$

Therefore $a(0) = 5/4$ and

$$\begin{aligned} a(n) &= \sum_{\substack{d\mid n \\ d \text{ odd}}} (-1)^{(d-1)/2} d^4 \\ &= \prod_{p} (-1)^{\lambda_p(p-1)/2} \left[\frac{p^{4(\lambda_p+1)} - (-1)^{(\lambda_p+1)(p-1)/2}}{p^4 - (-1)^{(p-1)/2}}\right]. \end{aligned}$$

Next, from [3] or [24, Ch. 17, Entry17 (iii)] we have

$$\frac{1}{64}z^5(4x+x^2) = \sum_{k=1}^{\infty} \frac{k^4 q^k}{1+q^{2k}}.$$

Therefore $b(0) = 0$ and

$$\begin{aligned} b(n) &= \sum_{\substack{d\mid n \\ d \text{ odd}}} (-1)^{(d-1)/2} \left(\frac{n}{d}\right)^4 \\ &= 2^{4\lambda} \prod_{p} \frac{p^{4(\lambda_p+1)} - (-1)^{(\lambda_p+1)(p-1)/2}}{p^4 - (-1)^{(p-1)/2}}. \end{aligned}$$

For the coefficients $c(n)$, by (3.1) and one of the Macdonald identities for BC_2 (for example, see [6]) we have

$$\frac{1}{16}z^5 x(1-x) = q\frac{(q^2;q^2)_\infty^{14}}{(-q;-q)_\infty^4} = \frac{1}{6} \sum_{\substack{\alpha\equiv 2 \pmod 5 \\ \beta\equiv 1 \pmod 5}} \alpha\beta(\alpha^2-\beta^2) q^{(\alpha^2+\beta^2)/5}.$$

Therefore $c(0) = 0$ and

$$c(n) = \frac{1}{6} \sum_{\substack{\alpha^2+\beta^2=5n \\ \alpha\equiv 2,\beta\equiv 1 \pmod 5}} \alpha\beta(\alpha^2-\beta^2)$$

$$= \begin{cases} 0, & \text{if } \lambda_p \text{ is odd for any } p \equiv 3 \pmod 4 \\ n^2(-1)^{\lambda} \displaystyle\prod_{p\equiv 1 \pmod 4} \frac{\sin 4(1+\lambda_p)\theta_p}{\sin 4\theta_p}, & \text{otherwise.} \end{cases}$$

The second part of this formula expressing $c(n)$ in terms of θ_p and λ_p was stated (without proof) by Ramanujan [25, eq. (156)].

Lastly, from [3] or [24, Ch. 17, Entries 14 (iii) & (vi)] we have

$$\begin{aligned} -\frac{1}{8}z^6(1-x)(1-x+x^2) &= -\frac{1}{8} - \sum_{k=1}^{\infty} \frac{(-1)^k k^5 q^k}{1+q^k} \\ \frac{1}{8}z^6(1-x)(1-x^2) &= \frac{1}{8} - \sum_{k=1}^{\infty} \frac{(-1)^k k^5 q^k}{1-q^k}. \end{aligned}$$

Therefore

$$\begin{aligned} d(n) &= \sum_{d|n} (-1)^{(d+n/d)} d^5 \\ &= \begin{cases} \displaystyle\prod_p \frac{p^{5\lambda_p+5}-1}{p^5-1} & \text{if } n \text{ is odd} \\ -\left(\dfrac{30\times 2^{5\lambda}+63}{2^5-1}\right) \displaystyle\prod_p \frac{p^{5\lambda_p+5}-1}{p^5-1} & \text{if } n \text{ is even} \end{cases} \\ e(n) &= -\sum_{d|n} (-1)^d d^5 \\ &= \begin{cases} \displaystyle\prod_p \frac{p^{5\lambda_p+5}-1}{p^5-1} & \text{if } n \text{ is odd} \\ -\left(\dfrac{2^{5\lambda+5}-63}{2^5-1}\right) \displaystyle\prod_p \frac{p^{5\lambda_p+5}-1}{p^5-1} & \text{if } n \text{ is even.} \end{cases} \end{aligned}$$

□

Lemma 4.2 *If p is an odd prime then*

$$\begin{aligned} a(p) &= (-1)^{(p-1)/2}p^4 + 1 \\ d(p) &= p^5 + 1 \\ e(p) &= p^5 + 1. \end{aligned}$$

Proof.
These follow right away from Lemma 4.1. □

Lemma 4.3 *Each of $a(n)$, $b(n)$, $c(n)$, $d(n)$ and $e(n)$ are multiplicative. That is, $a(mn) = a(m)a(n)$, if m and n are relatively prime; and similarly for b, c, d and e.*

Proof.
These follow right away from Lemma 4.1. □

Lemma 4.4 *Under the change of variables*

$$z \to z(1-x)^{1/2}, \quad x \to \frac{x}{x-1}$$

the functions

$$\begin{array}{c} z^{10}(5x^5+411x^4-2118x^3+4262x^2-4275x+1710) \quad \textit{and} \\ z^{12}(43870208x^6-131830116x^5+162785487x^4-105780950x^3 \\ +162565995x^2-131610624x+43870208) \end{array}$$

are invariant and

$$z^6x(1157-1157x+1008x^2)$$

changes sign.

Proof.
This follows by straightforward calculation. □

Lemma 4.5 *The q-expansions of*

$$\begin{array}{c} z^{10}(5x^5+411x^4-2118x^3+4262x^2-4275x+1710) \quad \textit{and} \\ z^{12}(43870208x^6-131830116x^5+162785487x^4-105780950x^3 \\ +162565995x^2-131610624x+43870208) \end{array}$$

contain only even powers of q, and the q-expansion of

$$z^6x(1157-1157x+1008x^2)$$

contains only odd powers.

Proof.
This follows from the Change of Sign Principle [3, p. 126] or [24, Ch. 17, Entry 14 (xii)]. □

Lemma 4.6

$$\begin{aligned} [q^p]z^{10}(1+x)(1+14x+x^2)(1-34x+x^2) &= -264(p^9+1) \\ [q^p]\, z^{12}\left[441(1+14x+x^2)^3+250(1+x)^2(1-34x+x^2)^2\right] &= 65520(p^{11}+1). \end{aligned}$$

Proof.
From [3] or [24, Ch. 17, Entries 13 (iii) and (iv)], followed by [2] or [24, Ch. 15, Entry 12 (iii)] we have

$$z^{10}(1+x)(1+14x+x^2)(1-34x+x^2) = M(q)N(q) = 1-264\sum_{k=1}^{\infty}\frac{k^9q^k}{1-q^k}.$$

The coefficient of q^p in this is readily seen to be $-264(p^9+1)$.

Also, from [3] or [24, Ch. 17, Entries 13 (iii) and (iv)] followed by [2] or [24, Ch. 15, Entry 13 (i)] we have

$$\begin{aligned} & 441z^{12}(1+14x+x^3)^3+250z^{12}(1+x)^2(1-34x+x^2)^2 \\ = \;& 441M(q)^3+250N(q)^2 = 691+65520\sum_{j=1}^{\infty}\frac{j^{11}q^j}{1-q^j}. \end{aligned}$$

The coefficient of q^p in this is readily seen to be $65520(p^{11}+1)$. □

5 The value of $r_{11}(p^2)$ for odd primes p

Theorem 5.1 *If p is an odd prime and $H(p)$ is defined by equation (3.5), then*

$$r_{11}(p^2) = \frac{330}{31}(p^9+1) - 22(-1)^{(p-1)/2}p^4 + \frac{352}{31}H(p).$$

Proof.
Let us write

$$p^2 = \sum_{i=1}^{11} x_i^2.$$

There are three possibilities: either one, five or nine of the x_i are odd, and the others are even. Accordingly, we have

$$\begin{aligned} r_{11}(p^2) &= 22\sum_{j\text{ odd}} r_{10,0}(p^2-j^2) + \frac{22}{5}\sum_{j\text{ odd}} r_{6,4}(p^2-j^2) + \frac{22}{9}\sum_{j\text{ odd}} r_{2,8}(p^2-j^2) \\ &= \sum_{j\text{ odd}} \left[q^{p^2-j^2}\right]\Big\{22\phi(q^4)^{10} + \frac{22}{5}\times\binom{10}{4}\times 16q^4\phi(q^4)^6\psi(q^8)^4 \\ &\qquad + \frac{22}{9}\times\binom{10}{8}\times 16^2q^8\phi(q^4)^2\psi(q^8)^8\Big\} \\ &= 2\sum_{j\text{ odd}} \left[q^{(p^2-j^2)/4}\right]\Big\{\binom{11}{1}\phi(q)^{10} + \binom{11}{5}\times 16q\phi(q)^6\psi(q^2)^4 \\ &\qquad + \binom{11}{9}\times 16^2q^2\phi(q)^2\psi(q^2)^8\Big\} \\ &= \sum_{j=0}^{p}\left[q^{j(p-j)}\right]z^5\left\{\binom{11}{1} + \binom{11}{5}x + \binom{11}{9}x^2\right\} \\ &= \sum_{j=0}^{p}\left[q^{j(p-j)}\right]z^5(11+462x+55x^2). \end{aligned}$$

In general we have

$$r_{2k+1}(p^2) = \sum_{j=0}^{p}[q^{j(p-j)}]z^k\left(\sum_{i=0}^{\lfloor k/2\rfloor}\binom{2k+1}{4i+1}x^i\right). \tag{5.2}$$

Now

$$z^5(11+462x+55x^2) = \frac{11}{5}z^5(5-x)(1-x) + \frac{528}{5}z^5(4x+x^2) + \frac{264}{5}z^5x(1-x).$$

Using this together with Lemmas 4.1, 4.2 and 4.3 gives

$$\begin{aligned} r_{11}(p^2) &= \sum_{j=0}^{p}\left(\frac{44}{5}a(j(p-j)) + \frac{33792}{5}b(j(p-j)) + \frac{4224}{5}c(j(p-j))\right) \\ &= \frac{88}{5}a(0) - \frac{88}{5}a(0)a(p) + \frac{44}{5}\sum_{j=0}^{p}a(j)a(p-j) \\ &\quad + \frac{33792}{5}\sum_{j=0}^{p}b(j)b(p-j) + \frac{4224}{5}\sum_{j=0}^{p}c(j)c(p-j) \end{aligned}$$

$$\begin{aligned} &= -22(-1)^{(p-1)/2}p^4 + \frac{44}{5}[q^p]\left(\frac{1}{4}z^5(5-x)(1-x)\right)^2 \\ &\quad + \frac{33792}{5}[q^p]\left(\frac{1}{64}z^5(4x+x^2)\right)^2 + \frac{4224}{5}[q^p]\left(\frac{1}{16}z^5x(1-x)\right)^2 \\ &= -22(-1)^{(p-1)/2}p^4 + \frac{11}{4}[q^p]z^{10}(2x^4+20x^2-12x+5). \end{aligned}$$

Now

$$\frac{11}{4}(2x^4+20x^2-12x+5) = -\frac{5}{124}p_1(x) + \frac{22}{31}p_2(x) + p_3(x)$$

where

$$\begin{aligned} p_1(x) &= (1+x)(1+14x+x^2)(1-34x+x^2) \\ p_2(x) &= x(1-x)(1+2x-2x^2) \\ p_3(x) &= \frac{1}{124}(5x^5+411x^4-2118x^3+4262x^2-4275x+1710). \end{aligned}$$

This, together with Lemmas 4.5, 4.6 and Equation (3.6) gives

$$\begin{aligned} r_{11}(p^2) &= -22(-1)^{(p-1)/2}p^4 - \frac{5}{124}\times(-264)(p^9+1) + \frac{22}{31}\times 16H(p) + 0 \\ &= \frac{330}{31}(p^9+1) - 22(-1)^{(p-1)/2}p^4 + \frac{352}{31}H(p). \end{aligned}$$

□

Remark 5.3 *The coefficients $H(n)$ have some interesting properties.*

1. *H is multiplicative. That is, $H(mn) = H(m)H(n)$ for any pair of relatively prime positive integers m and n.*
2. *$|H(n)| \le n^{9/2}d(n)$, where $d(n)$ is the number of divisors of n. In particular,*

$$|H(p)| \le 2p^{9/2} \tag{5.4}$$

for primes p. Ramanujan [25, §28] made some conjectures concerning the orders of some similar functions. See his equations (157), (160) and (163). See also Berndt's commentary [26, p. 367] for references and more information.

6 The Hecke operator T_{p^2} when $p = 2$ for eleven squares

Theorem 6.1

$$\begin{aligned} T_4\phi_{11,0} &= \phi_{11,0} + 330\phi_{7,4} + 165\phi_{3,8} \\ T_4\phi_{7,4} &= 336\phi_{7,4} + 176\phi_{3,8} \\ T_4\phi_{3,8} &= 320\phi_{7,4} + 192\phi_{3,8}. \end{aligned}$$

That is,

$$\begin{aligned} \phi_{11,0}(4n) &= \phi_{11,0}(n) + 330\phi_{7,4}(n) + 165\phi_{3,8}(n) \\ \phi_{7,4}(4n) &= 336\phi_{7,4}(n) + 176\phi_{3,8}(n) \\ \phi_{3,8}(4n) &= 320\phi_{7,4}(n) \quad 192\phi_{3,8}(n). \end{aligned}$$

Proof.
Let us consider $\phi_{11,0}$ first. By Facts 1 and 2, we have

$$T_4\phi_{11,0} = \alpha_1\phi_{11,0} + \alpha_2\phi_{7,4} + \alpha_3\phi_{3,8},$$

for some constants α_1, α_2, α_3. By Fact 3, we have

$$T_4\phi_{11,0} = \sum_{n=0}^{\infty} \phi_{11,0}(4n)q^n.$$

Therefore

$$\sum_{n=0}^{\infty} \phi_{11,0}(4n)q^n = \alpha_1 \sum_{n=0}^{\infty} \phi_{11,0}(n)q^n + \alpha_2 \sum_{n=0}^{\infty} \phi_{7,4}(n)q^n + \alpha_3 \sum_{n=0}^{\infty} \phi_{3,8}(n)q^n.$$

Equating coefficients of 1, q and q^2 gives

$$\begin{aligned} \alpha_1 &= 1 \\ 22\alpha_1 + 16\alpha_2 &= 5302 \\ 220\alpha_1 + 224\alpha_2 + 256\alpha_3 &= 116380 \end{aligned}$$

and so

$$\alpha_1 = 1, \quad \alpha_2 = 330, \quad \alpha_3 = 165.$$

This proves the first part of the Theorem. The results for $T_4\phi_{7,4}$ and $T_4\phi_{3,8}$ follow similarly. □

Lemma 6.2 *The eigenfunctions and eigenvalues of T_4 are*

$$\begin{array}{lllllll} \chi_{11,1} & := & 511\phi_{11,0} & - & 682\phi_{7,4} & + & 187\phi_{3,8}, & \lambda_1 = 1, \\ \chi_{11,2} & := & & & 20\phi_{7,4} & + & 11\phi_{3,8}, & \lambda_2 = 512, \\ \chi_{11,3} & := & & & \phi_{7,4} & - & \phi_{3,8}, & \lambda_3 = 16. \end{array}$$

These coefficients satisfy many interesting properties. Some of these are summarised in

Theorem 6.3

$$\begin{aligned} 32\chi_{11,1}(n) - 33\chi_{11,2}(n) &= 0 \ \textit{if } n \equiv 1 \textit{ or } 2 \pmod 4 \\ 512\chi_{11,1}(n) - 17\chi_{11,2}(n) &= 0 \ \textit{if } n \equiv 3 \pmod 8 \\ 15872\chi_{11,1}(n) + 495\chi_{11,2}(n) &= 0 \ \textit{if } n \equiv 7 \pmod 8 \\ \chi_{11,3}(n) &= 0 \ \textit{if } n \equiv 7 \pmod 8. \end{aligned}$$

Proof.
A proof of Theorem 6.3 has been given by Barrucand and Hirschhorn [1]. □

As a consequence of the previous results we have

Theorem 6.4

$$r_{11}(4^\lambda(8k+7)) = \frac{2^{9\lambda+9}-1}{2^9-1} r_{11}(8k+7).$$

Proof.

$$\begin{aligned} & r_{11}(4^\lambda(8k+7)) \\ = \ & \phi_{11,0}(4^\lambda(8k+7)) \\ = \ & \frac{1}{31 \times 511}\left(31\chi_{11,1}(4^\lambda(8k+7)) + 495\chi_{11,2}(4^\lambda(8k+7)) + 11242\chi_{11,3}(4^\lambda(8k+7))\right) \\ = \ & \frac{1}{31 \times 511}\left(31\chi_{11,1}(8k+7) + 495 \times 512^\lambda\chi_{11,2}(8k+7) + 11242 \times (16)^\lambda\chi_{11,3}(8k+7)\right). \end{aligned}$$

Applying Theorem 6.3 gives

$$\begin{aligned} r_{11}(4^\lambda(8k+7)) &= \frac{1}{31\times 511}\big(31\chi_{11,1}(8k+7) - 15872\times 512^\lambda\chi_{11,1}(8k+7)+0\big) \\ &= \frac{-1}{511}(512^{\lambda+1}-1)\chi_{11,1}(8k+7). \end{aligned}$$

Taking $\lambda = 0$ in this gives

$$r_{11}(8k+7) = -\chi_{11,1}(8k+7)$$

and so

$$r_{11}(4^\lambda(8k+7)) = \frac{1}{511}(512^{\lambda+1}-1)r_{11}(8k+7) = \frac{2^{9\lambda+9}-1}{2^9-1}r_{11}(8k+7),$$

as required. □

7 The Hecke operator T_{p^2} when p is an odd prime for sums of eleven squares

By Facts 2 and 3 we have

$$\begin{aligned} T_{p^2}\phi_{11,0} &= \sum_{n=0}^{\infty}\phi_{11,0}(p^2n)q^n + p^4\sum_{n=0}^{\infty}\left(\frac{-n}{p}\right)\phi_{11,0}(n)q^n + p^9\sum_{n=0}^{\infty}\phi_{11,0}(n/p^2)q^n \\ &= c_1\phi_{11,0}(q)+c_2\phi_{7,4}(q)+c_3\phi_{3,8}(q), \qquad (7.1)\\ T_{p^2}\phi_{7,4} &= \sum_{n=0}^{\infty}\phi_{7,4}(p^2n)q^n + p^4\sum_{n=0}^{\infty}\left(\frac{-n}{p}\right)\phi_{7,4}(n)q^n + p^9\sum_{n=0}^{\infty}\phi_{7,4}(n/p^2)q^n \\ &= d_1\phi_{11,0}(q)+d_2\phi_{7,4}(q)+d_3\phi_{3,8}(q), \qquad (7.2)\\ T_{p^2}\phi_{3,8} &= \sum_{n=0}^{\infty}\phi_{3,8}(p^2n)q^n + p^4\sum_{n=0}^{\infty}\left(\frac{-n}{p}\right)\phi_{3,8}(n)q^n + p^9\sum_{n=0}^{\infty}\phi_{3,8}(n/p^2)q^n \\ &= e_1\phi_{11,0}(q)+e_2\phi_{7,4}(q)+e_3\phi_{3,8}(q), \qquad (7.3) \end{aligned}$$

for some constants $c_1,\ c_2,\ c_3,\ d_1,\ d_2,\ d_3,\ e_1,\ e_2,\ e_3$. Equating the constant terms in each of the three equations above gives

$$\begin{aligned} c_1 &= p^9+1 \\ d_1 &= 0 \\ e_1 &= 0. \end{aligned}$$

Equating the coefficients of q in (7.1)–(7.3) gives

$$\begin{aligned} 22c_1+16c_2 &= \phi_{11,0}(p^2)+22(-1)^{(p-1)/2}p^4 \\ 22d_1+16d_2 &= \phi_{7,4}(p^2)+16(-1)^{(p-1)/2}p^4 \\ 22e_1+16e_2 &= \phi_{3,8}(p^2). \end{aligned}$$

By Theorem 5.1 we have

$$\phi_{11,0}(p^2) = \frac{330}{31}(p^9+1) - 22(-1)^{(p-1)/2}p^4 + \frac{352}{31}H(p). \qquad (7.4)$$

By the same methods it can be shown that

$$\begin{aligned} \phi_{7,4}(p^2) &= \frac{320}{31}(p^9+1) - 16(-1)^{(p-1)/2}p^4 + \frac{176}{31}H(p) \qquad (7.5)\\ \phi_{3,8}(p^2) &= \frac{320}{31}(p^9+1) - \frac{320}{31}H(p). \qquad (7.6) \end{aligned}$$

It follows that

$$\begin{aligned} c_2 &= -\frac{22}{31}(p^9+1)+\frac{22}{31}H(p) \\ d_2 &= \frac{20}{31}(p^9+1)+\frac{11}{31}H(p) \\ e_2 &= \frac{20}{31}(p^9+1)-\frac{20}{31}H(p). \end{aligned}$$

Next, equating coefficients of q^4 in (7.1)–(7.3) gives

$$\begin{aligned} \phi_{11,0}(4p^2)+5302(-1)^{(p-1)/2}p^4 &= 5302c_1+5376c_2+5120c_3 \\ \phi_{7,4}(4p^2)+5376(-1)^{(p-1)/2}p^4 &= 5302d_1+5376d_2+5120d_3 \\ \phi_{3,8}(4p^2)+5120(-1)^{(p-1)/2}p^4 &= 5302e_1+5376e_2+5120e_3. \end{aligned}$$

By Theorem 6.1 and equations (7.4)–(7.6) we have

$$\begin{aligned} \phi_{11,0}(4p^2) &= \phi_{11,0}(p^2)+330\phi_{7,4}(p^2)+165\phi_{3,8}(p^2) \\ &= \frac{158730}{31}(p^9+1)-5302(-1)^{(p-1)/2}p^4+\frac{5632}{31}H(p). \end{aligned}$$

Similarly,

$$\begin{aligned} \phi_{7,4}(4p^2) &= \frac{163840}{31}(p^9+1)-5376(-1)^{(p-1)/2}p^4+\frac{2816}{17}H(p) \\ \phi_{3,8}(4p^2) &= \frac{163840}{31}(p^9+1)-5120(-1)^{(p-1)/2}p^4-\frac{5120}{17}H(p). \end{aligned}$$

It follows that

$$\begin{aligned} c_3 &= \frac{22}{31}(p^9+1)-\frac{22}{31}H(p) \\ d_3 &= \frac{11}{31}(p^9+1)-\frac{11}{31}H(p) \\ e_3 &= \frac{11}{31}(p^9+1)+\frac{20}{31}H(p). \end{aligned}$$

The above results may be summarised as

Theorem 7.7 *Let*

$$\alpha = p^9+1, \qquad \beta = H(p).$$

Then

$$\begin{pmatrix} T_{p^2}\phi_{11,0} \\ T_{p^2}\phi_{7,4} \\ T_{p^2}\phi_{3,8} \end{pmatrix} = \frac{1}{31}\begin{pmatrix} 31\alpha & -22\alpha+22\beta & 22\alpha-22\beta \\ 0 & 20\alpha+11\beta & 11\alpha-11\beta \\ 0 & 20\alpha-20\beta & 11\alpha+20\beta \end{pmatrix}\begin{pmatrix} \phi_{11,0} \\ \phi_{7,4} \\ \phi_{3,8} \end{pmatrix}.$$

Corollary 7.8 *The eigenfunctions and eigenvalues of T_{p^2} are*

$$\begin{aligned} \zeta_{11,1} &:= \phi_{11,0}-2\phi_{7,4}, & \lambda_1 &= \alpha \\ \zeta_{11,2} &:= 20\phi_{7,4}+11\phi_{3,8} = \chi_{11,2}, & \lambda_2 &= \alpha \\ \zeta_{11,3} &:= \phi_{7,4}-\phi_{3,8} = \chi_{11,3}, & \lambda_3 &= \beta. \end{aligned}$$

That is,

$$\begin{aligned} T_{p^2}\zeta_{11,1} &= (p^9+1)\zeta_{11,1} \\ T_{p^2}\zeta_{11,2} &= (p^9+1)\zeta_{11,2} \\ T_{p^2}\zeta_{11,3} &= H(p)\zeta_{11,3}. \end{aligned}$$

Using Fact 3, together with Corollary 7.8, we have

$$\zeta_{11,1}(p^2n)+p^4\left(\frac{-n}{p}\right)\zeta_{11,1}(n)+p^9\zeta_{11,1}(n/p^2) = (p^9+1)\zeta_{11,1}(n) \tag{7.9}$$

$$\zeta_{11,2}(p^2n)+p^4\left(\frac{-n}{p}\right)\zeta_{11,2}(n)+p^9\zeta_{11,2}(n/p^2) = (p^9+1)\zeta_{11,2}(n) \tag{7.10}$$

$$\zeta_{11,3}(p^2n)+p^4\left(\frac{-n}{p}\right)\zeta_{11,3}(n)+p^9\zeta_{11,3}(n/p^2) = H(p)\zeta_{11,3}(n). \tag{7.11}$$

Iterating (7.9)–(7.11) we obtain

Theorem 7.12 *If $p^2 \nmid n$, then*

$$\begin{aligned}
\zeta_{11,1}(p^{2\mu}n) &= \left[\frac{p^{9\mu+9}-1}{p^9-1}-p^4\left(\frac{-n}{p}\right)\frac{p^{9\mu}-1}{p^9-1}\right]\zeta_{11,1}(n)\\
\zeta_{11,2}(p^{2\mu}n) &= \left[\frac{p^{9\mu+9}-1}{p^9-1}-p^4\left(\frac{-n}{p}\right)\frac{p^{9\mu}-1}{p^9-1}\right]\zeta_{11,2}(n)\\
\zeta_{11,3}(p^{2\mu}n) &= \left[c_1(n)r_1^\mu+c_2(n)r_2^\mu\right]\zeta_{11,3}(n),
\end{aligned}$$

where

$$\begin{aligned}
r_1 &= \frac{H(p)+\sqrt{\Delta(p)}}{2}\\
r_2 &= \frac{H(p)-\sqrt{\Delta(p)}}{2}\\
c_1(n) &= \frac{1}{2}+\frac{H(p)-2p^4\left(\frac{-n}{p}\right)}{2\sqrt{\Delta(p)}}\\
c_2(n) &= \frac{1}{2}-\frac{H(p)-2p^4\left(\frac{-n}{p}\right)}{2\sqrt{\Delta(p)}}\\
\Delta(p) &= H(p)^2-4p^9.
\end{aligned}$$

Remark 7.13 *Equation 5.4 implies $\Delta(p)\le 0$.*

It is possible to express $r_{11}(n)$ in terms of $r_{11}(n^o)$ and $\zeta_{11,3}(n^o)$, where n^o is the odd-square free part of n, that is, n^o is the greatest divisor of n which is not divisible by an odd square. Let

$$\begin{aligned}
g_{p,\mu,m} &= \frac{p^{9\mu+9}-1}{p^9-1}-p^4\left(\frac{-m}{p}\right)\frac{p^{9\mu}-1}{p^9-1}\\
h_{p,\mu,m} &= c_1(m)r_1^\mu+c_2(m)r_2^\mu,
\end{aligned}$$

where $c_1,\ c_2,\ r_1,\ r_2$ are as for Theorem 7.12. If $p^2 \nmid m$ then

$$\begin{aligned}
r_{11}(p^{2\mu}m) &= \frac{1}{31}\left[31\zeta_{11,1}(p^{2\mu}m)+2\zeta_{11,2}(p^{2\mu}m)+22\zeta_{11,3}(p^{2\mu}m)\right]\\
&= \frac{1}{31}\left[31g_{p,\mu,m}\zeta_{11,1}(m)+2g_{p,\mu,m}\zeta_{11,2}(m)+22h_{p,\mu,m}\zeta_{11,3}(m)\right]\\
&= \frac{1}{31}\left[31g_{p,\mu,m}\zeta_{11,1}(m)+2g_{p,\mu,m}\zeta_{11,2}(m)+22g_{p,\mu,m}\zeta_{11,3}(m)\right]\\
&\qquad+\frac{22}{31}\left[h_{p,\mu,m}-g_{p,\mu,m}\right]\zeta_{11,3}(m)\\
&= g_{p,\mu,m}r_{11}(m)+\frac{22}{31}\left[h_{p,\mu,m}-g_{p,\mu,m}\right]\zeta_{11,3}(m).
\end{aligned}$$

By induction on j, it follows that if $p_1, p_2, \cdots, p_j$ are distinct odd primes, and $p_i^2 \not| m$ for $1 \le i \le j$, then

$$\begin{aligned} & r_{11}(p_1^{2\mu_1} p_2^{2\mu_2} \cdots p_j^{2\mu_j} m) \\ = & \left(\prod_{i=1}^{j} g_{p_i,\mu_i,m} \right) r_{11}(m) + \frac{22}{31} \left[\left(\prod_{i=1}^{j} h_{p_i,\mu_i,m} \right) - \left(\prod_{i=1}^{j} g_{p_i,\mu_i,m} \right) \right] \zeta_{11,3}(m). \end{aligned}$$

Consequently, if n^o is the odd-square free part of n and either $p^{2\mu_p} \mid\mid n$ or $p^{2\mu_p+1} \mid\mid n$, then

$$\begin{aligned} r_{11}(n) &= r_{11}(n^o) \prod_p \left[\frac{p^{9\mu_p+9}-1}{p^9-1} - p^4 \left(\frac{-n^o}{p} \right) \frac{p^{9\mu_p}-1}{p^9-1} \right] \\ &\quad + \frac{22}{31} \zeta_{11,3}(n^o) \left[\prod_p h_{p,\mu_p,n^o} - \prod_p \left(\frac{p^{9\mu_p+9}-1}{p^9-1} - p^4 \left(\frac{-n^o}{p} \right) \frac{p^{9\mu_p}-1}{p^9-1} \right) \right]. \end{aligned}$$

If $n^o = 4^\mu(8k+7)$ for some nonnegative integers μ and k, then Theorem 6.3 implies $\zeta_{11,3}(n^o) = 0$. In this case we have

$$r_{11}(n) = r_{11}(n^o) \prod_p \left[\frac{p^{9\mu_p+9}-1}{p^9-1} - p^4 \left(\frac{-n^o}{p} \right) \frac{p^{9\mu_p}-1}{p^9-1} \right].$$

This, together with Theorem 6.4 implies

Theorem 7.14 *Let $n = 2^\lambda \prod_p p^{\lambda_p}$ be the prime factorisation of n and let n' be the square-free part of n. If n' is of the form $8k+7$ then*

$$r_{11}(n) = r_{11}(n') \frac{2^{9\lfloor \lambda/2 \rfloor + 9} - 1}{2^9 - 1} \prod_p \left[\frac{p^{9\lfloor \lambda_p/2 \rfloor + 9} - 1}{p^9 - 1} - p^4 \left(\frac{-n'}{p} \right) \frac{p^{9\lfloor \lambda_p/2 \rfloor} - 1}{p^9 - 1} \right].$$

8 The value of $r_{13}(p^2)$ for odd primes p

Theorem 8.1 *If p is an odd prime and $\tau(p)$ is defined by equation (2.12), then*

$$r_{13}(p^2) = \frac{4030}{691}(p^{11}+1) - 26p^5 + \frac{13936}{691}\tau(p).$$

Proof.
We proceed in a similar way to Section 5. Taking $k = 6$ in (5.2) gives

$$r_{13}(p^2) = \sum_{j=0}^{p} [q^{j(p-j)}] z^6 \left\{ 13 + 1287x + 715x^2 + x^3 \right\}. \tag{8.2}$$

Now

$$\begin{aligned} & z^6(13 + 1287x + 715x^2 + x^3) \\ = \; & 1014z^6(1-x)(1-x+x^2) - 1001z^6(1-x)(1-x^2) \\ & +2z^6 x(1157 - 1157x + 1008x^2). \end{aligned} \tag{8.3}$$

Therefore Lemmas 4.1, 4.2 and 4.3 give

$$\begin{aligned} & r_{13}(p^2) \\ = \; & -8112 \sum_{j=0}^{p} d(j(p-j)) - 8008 \sum_{j=0}^{p} e(j(p-j)) + 0 \end{aligned}$$

$$\begin{aligned}
&= -16224d(0)+16224d(0)d(p)-8112\sum_{j=0}^{p}d(j)d(p-j)\\
&\quad -16016e(0)+16016e(0)e(p)-8008\sum_{j=0}^{p}e(j)e(p-j)\\
&= -26p^5-8112[q^p]\left(-\frac{1}{8}z^6(1-x)(1-x+x^2)\right)^2-8008[q^p]\left(\frac{1}{8}z^6(1-x)(1-x^2)\right)^2\\
&= -26p^5-\frac{1}{8}[q^p]z^{12}\left(1014(1-x)^2(1-x+x^2)^2+1001(1-x)^2(1-x^2)^2\right).
\end{aligned}$$

Now

$$\begin{aligned}
&-\frac{1}{8}\left(1014(1-x)^2(1-x+x^2)^2+1001(1-x)^2(1-x^2)^2\right)\\
&= \frac{31}{504\times 691}p_1(x)+\frac{871}{691}p_2(x)+p_3(x), \qquad (8.4)
\end{aligned}$$

where

$$\begin{aligned}
p_1(x) &= 441(1+14x+x^2)^3+250(1+x)^2(1-34x+x^2)^2,\\
p_2(x) &= x(1-x),\\
p_3(x) &= \frac{-1}{174132}\left(43870208x^6-131830116x^5+162785487x^4-105780950x^3\right.\\
&\qquad \left.+\,162565995x^2-131610624x+43870208\right).
\end{aligned}$$

Therefore Lemmas 4.5, 4.6 and Equation (3.4) give

$$\begin{aligned}
r_{13}(p^2) &= -26p^5+\frac{31}{504\times 691}\times 65520(p^{11}+1)+\frac{871}{691}\times 16\tau(p)+0\\
&= \frac{4030}{691}(p^{11}+1)-26p^5+\frac{13936}{691}\tau(p),
\end{aligned}$$

as required. □

Remark 8.5 *The coefficients $\tau(n)$ satisfy some well-known interesting properties.*

1. *τ is multiplicative. That is, $\tau(mn)=\tau(m)\tau(n)$ for any pair of relatively prime positive integers m and n.*
2. *$|\tau(n)|\le n^{11/2}d(n)$, where $d(n)$ is the number of divisors of n. In particular,*

$$|\tau(p)|\le 2p^{11/2}$$

for primes p.

Both properties were conjectured by Ramanujan [25, Eqs. (103), (104) and (105)]. The first was proved by Mordell and the second by Deligne. See Berndt's commentary [26, p. 367] for references and more information.

9 The eigenvalues and eigenfunctions for the Hecke operators T_{p^2} for thirteen squares

In this section we give the analogues of the results in Sections 6.1 and 7 for the Hecke operator T_{p^2} for thirteen squares. The methods of proof are identical with those in Sections 6.1 and 7, and therefore we omit them.

Theorem 9.1

$$\begin{array}{rcrcrcrcr} T_4\phi_{13,0} & = & \phi_{13,0} & + & 715\phi_{9,4} & + & 1287\phi_{5,8} & + & 13\phi_{1,12} \\ T_4\phi_{9,4} & = & & & 744\phi_{9,4} & + & 1296\phi_{5,8} & + & 8\phi_{1,12} \\ T_4\phi_{5,8} & = & & & 768\phi_{9,4} & + & 1280\phi_{5,8} & & \\ T_4\phi_{1,8} & = & & & 512\phi_{9,4} & + & 1536\phi_{5,8}. & & \end{array}$$

Corollary 9.2 *The eigenfunctions of T_4 are*

$$\begin{array}{rcrcrcrcr} \chi_{13,1} & := & 1414477\phi_{13,0} & - & 2298777\phi_{9,4} & + & 908427\phi_{5,8} & - & 2015\phi_{1,12} \\ \chi_{13,2} & := & & & 256\phi_{9,4} & + & 434\phi_{5,8} & + & \phi_{1,12} \\ \chi_{13,3} & := & & & \omega\phi_{9,4} & - & (\omega+2)\phi_{5,8} & + & 2\phi_{1,12} \\ \chi_{13,4} & := & & & \overline{\omega}\phi_{9,4} & - & (\overline{\omega}+2)\phi_{5,8} & + & 2\phi_{1,12} \end{array}$$

and the corresponding eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 2048$, $\lambda_3 = 4\omega$, $\lambda_4 = 4\overline{\omega}$, where $\omega = -3 + \sqrt{-119}$.

The coefficients in the expansions of $\phi_{13,0}$, $\phi_{9,4}$, $\phi_{5,8}$ and $\phi_{1,12}$ satisfy some interesting properties: Some of these are summarised in

Theorem 9.3
If $n \equiv 2$ or $3 \pmod 4$ then

$$\begin{array}{rcl} 8\phi_{9,4}(n) - 9\phi_{5,8}(n) + \phi_{1,12}(n) & = & 0 \\ 64\phi_{13,0}(n) - 104\phi_{9,4}(n) + 39\phi_{5,8}(n) & = & 0. \end{array}$$

If $n \equiv 1 \pmod 8$ then

$$\begin{array}{rcl} \phi_{5,8}(n) - \phi_{1,12}(n) & = & 0 \\ 2048\phi_{13,0}(n) - 3328\phi_{9,4}(n) + 1313\phi_{5,8}(n) & = & 0. \end{array}$$

If $n \equiv 5 \pmod 8$ then

$$\begin{array}{rcl} 64\phi_{9,4}(n) - 57\phi_{5,8}(n) - 7\phi_{1,12}(n) & = & 0 \\ 14336\phi_{13,0}(n) - 23488\phi_{9,4}(n) + 9369\phi_{5,8}(n) & = & 0. \end{array}$$

Proof.
A proof of Theorem 9.3 has been given by Barrucand and Hirschhorn [1]. □
For odd primes p we have

Theorem 9.4 *Let*

$$\alpha = p^{11} + 1, \qquad \beta = \tau(p).$$

Then

$$\begin{pmatrix} T_{p^2}\phi_{13,0} \\ T_{p^2}\phi_{9,4} \\ T_{p^2}\phi_{5,8} \\ T_{p^2}\phi_{1,12} \end{pmatrix} = \frac{1}{691} \begin{pmatrix} 691\alpha & -871\alpha + 871\beta & 871\alpha - 871\beta & 0 \\ 0 & 256\alpha + 435\beta & 434\alpha - 434\beta & \alpha - \beta \\ 0 & 256\alpha - 256\beta & 434\alpha + 257\beta & \alpha - \beta \\ 0 & 256\alpha - 256\beta & 434\alpha - 434\beta & \alpha + 690\beta \end{pmatrix} \begin{pmatrix} \phi_{13,0} \\ \phi_{9,4} \\ \phi_{5,8} \\ \phi_{1,12} \end{pmatrix}.$$

Corollary 9.5 *The eigenfunctions and eigenvalues of T_{p^2} are*

$$\begin{array}{rll} \zeta_{13,1} := & 691\phi_{13,0} - 871\phi_{9,4} + 871\phi_{5,8}, & \lambda_1 = \alpha \\ \zeta_{13,2} := & 256\phi_{9,4} + 434\phi_{5,8} + \phi_{1,12} = \chi_{13,2}, & \lambda_2 = \alpha \\ \zeta_{13,3} := & \phi_{9,4} - \phi_{5,8} = \chi_{13,3}, & \lambda_3 = \beta \\ \zeta_{13,4} := & \phi_{5,8} - \phi_{1,12} = \chi_{13,4}, & \lambda_4 = \beta. \end{array}$$

That is,

$$\begin{aligned} T_{p^2}\zeta_{13,1} &= (p^{11}+1)\zeta_{13,1} \\ T_{p^2}\zeta_{13,2} &= (p^{11}+1)\zeta_{13,2} \\ T_{p^2}\zeta_{13,3} &= \tau(p)\zeta_{13,3} \\ T_{p^2}\zeta_{13,4} &= \tau(p)\zeta_{13,4}. \end{aligned}$$

10 Sums of an odd number of squares

It is interesting to compare the values of $r_k(p^2)$ for various k. We have:

$$r_1(p^2) = 2 \tag{10.1}$$
$$r_2(p^2) = 4(2+(-1)^{(p-1)/2}) \tag{10.2}$$
$$r_3(p^2) = 6(p+1-(-1)^{(p-1)/2}) \tag{10.3}$$
$$r_4(p^2) = 8(p^2+p+1) \tag{10.4}$$
$$r_5(p^2) = 10(p^3-p+1) \tag{10.5}$$
$$r_6(p^2) = 12(p^4+(-1)^{(p-1)/2}p^2+1) \tag{10.6}$$
$$r_7(p^2) = 14(p^5-(-1)^{(p-1)/2}p^2+1) \tag{10.7}$$
$$r_8(p^2) = 16(p^6+p^3+1) \tag{10.8}$$
$$r_9(p^2) = \frac{274}{17}(p^7+1) - 18p^3 + \frac{32}{17}\theta(p) \tag{10.9}$$
$$r_{10}(p^2) = \frac{68}{5}(p^8+(-1)^{(p-1)/2}p^4+1) + \frac{32}{5}c(p^2) \tag{10.10}$$
$$r_{11}(p^2) = \frac{330}{31}(p^9+1) - 22(-1)^{(p-1)/2}p^4 + \frac{352}{31}H(p) \tag{10.11}$$
$$r_{12}(p^2) = 8(p^{10}+p^5+1) + 16\Omega(p^2) \tag{10.12}$$
$$r_{13}(p^2) = \frac{4030}{691}(p^{11}+1) - 26p^5 + \frac{13936}{691}\tau(p). \tag{10.13}$$

where

$$\begin{aligned} \sum_{n=1}^{\infty}\theta(n)q^n &= q(-q;-q)_\infty^8(q^2;q^2)_\infty^8 = \frac{1}{16}z^8x(1-x) \\ \sum_{n=1}^{\infty}c(n)q^n &= q\frac{(q^2;q^2)_\infty^{14}}{(-q;-q)_\infty^4} = \frac{1}{16}z^5x(1-x) \\ \sum_{n=1}^{\infty}H(n)q^n &= q(-q;-q)_\infty^{16}(q^2;q^2)_\infty^4 + 32q^2\frac{(q^2;q^2)_\infty^{28}}{(-q;-q)_\infty^8} \\ &= \frac{1}{16}z^{10}x(1-x)(1+2x-2x^2) \\ \sum_{n=1}^{\infty}\Omega(n)q^n &= q(q^2;q^2)_\infty^{12} = \frac{1}{16}z^6x(1-x) \\ \sum_{n=1}^{\infty}(-1)^{n+1}\tau(n)q^n &= q(-q;-q)_\infty^{24} = \frac{1}{16}z^{12}x(1-x). \end{aligned}$$

Observe that formulas (10.9)–(10.13) are significantly more complicated in nature than (10.1)–(10.8). This is a vivid illustration of Eisenstein's remark [8], [31].

Equation (10.1) is trivial. Equations (10.2), (10.4), (10.6), (10.8), (10.10) and (10.12) follow readily from the well known formulas for $r_2(n)$, $r_4(n)$, $r_6(n)$, $r_8(n)$, $r_{10}(n)$ and $r_{12}(n)$; see, for example, [4]. Equations (10.3), (10.5) and (10.7) follow right away from (2.6)–(2.8). Direct proofs of (10.5) and (10.7) are given in [5], and it is possible to prove (10.3) in the same way. Equations (10.11) and (10.13) are Theorems 5.1 and 8.1, respectively. Equations (10.7), (10.9), (10.11) and (10.13) are originally due to Sandham [27], [28]. Slightly different proofs of (10.7) and (10.9) were given in [5], and the proofs we have given here of (10.11) and (10.13) are different from Sandham's. Results for $r_{2k}(p^2)$ can be written down immediately from the general formulas for $r_{2k}(n)$ in, for example, [4].

For sums of an odd number of squares, we have

Conjecture 10.14

$$\begin{aligned} r_{2k+1}(p^2) &= A_k(p^{2k-1}+1) - (4k+2)(-1)^{k(p-1)/2}p^{k-1} \\ &\quad +[q^p] \sum_{j=1}^{\lfloor k/2 \rfloor} c_{j,k} q^j \frac{(-q;-q)_\infty^{8k-24j}}{(q^2;q^2)_\infty^{4k-24j}}, \end{aligned}$$

where A_k and $c_{j,k}$ are rational numbers and

$$A_k + c_{1,k} = 4k+2.$$

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