Chapter 3
Contexts, Forms and Outcomes of Mathematics Teacher Collaboration

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3.1 Introduction

The chapter aims to synthesise a comprehensive analysis of papers included in Theme B of ICMI Study 25 that focused on forms and outcomes of mathematics teacher collaboration enacted in different contexts and provided insight into studies on teachers' collaborative learning. By doing so, this chapter helps to elucidate the

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relationships between different forms chosen to design collaborative settings and their effects on the outcomes of collaboration regarding teachers' learning, as well as exploring affordances and limitations of some forms of collaboration.

The main sources used for writing this chapter include the papers, presentations, reactions, and discussions that took place at the ICMI Study 25 conference. To carry out our work, we first identify research gaps related to Theme B by reviewing relevant seminal research findings and presenting research questions for Theme B (Sect. 3.1.1). Then, we provide overarching conceptual frameworks which help us to examine the papers in depth (Sect. 3.1.2). After that, we describe methods for analysing the papers and organising the results (Sect. 3.1.3). Finally, the structure of the rest of this chapter is presented (Sect. 3.1.4).

3.1.1 Backgrounds and Purpose

Teacher learning through collaboration has been a research field over the past two decades. Some related and relevant works are summarised here in order to understand the importance and purposes of Theme B of ICMI Study 25.

3.1.1.1 Early Contributions

It is worth acknowledging the early contribution of Peter-Koop et al. (2003) and Even and Ball (2009) who, considering collaborations mainly within professional development programmes, emphasise the contextualised aspect of the collaborative work and its corresponding outcomes. Peter-Koop et al. (2003) highlight that the *form* and *path* that a collaborative project can take depend on the *context of work*. They also point out that interactions within its context may require some changes in the collaborative work and emphasise that *products or outcomes* of collaboration emerge as the process unfolds, including *undesired outcomes*. Even and Ball (2009) emphasise the transformation of teacher professional learning from acquisition-centred to participation-oriented models. They discuss various forms of collaborative work that involve teachers and their professional practices. In this regard, they mention lesson studies, communities of practice, communities of inquiry and collaborative groups.

3.1.1.2 ICME 13 Survey: Expanding Ideas and Contributions

The survey on "Teachers working and learning through collaboration", presented at ICME 13, expands on previous contributions focusing on collaboration as a process connected to the work of teachers (Jaworski et al., 2017; Robutti et al., 2016). "Teachers' work includes all dimensions of teaching in and beyond face-to-face activity with students in the classroom" (Jaworski et al., 2017, p. 263). They argue that participants within a collaborative context, will address issues that challenge

teachers professionally and promote reflections on the role of teachers in school and society. However, the ways in which collaborative work with teachers can be organised depend on who the partners are, the type of initiative, the foci of the work, and its aims.

Regarding the type of initiative, Robutti et al. (2016) identify five main types: (1) initiatives mandated by ministries and national/regional institutions; (2) collaborations supported by ministries and national/regional institutions; (3) research collaborations initiated by researchers; (4) professional development initiated by researchers/didacticians; (5) school-based collaborations that were both initiated and sustained by the teachers without the direct involvement of others. Yet, they recognised that these categories of initiatives could often overlap.

Concerning *foci*, Robutti et al. recognise two broad categories. The *first category* is related to some aspects of *innovation* about mathematical content, the development of new curricula, different pedagogical approaches and the integration of new tools and resources (mainly digital tools). The *second category* is focused on practices that foster teachers who are able to implement innovative ideas, curriculum and tools in their classes. In terms of the *aims* of the collaboration, the multiplicity of objectives is highlighted. The multiplicity depends on the type of initiative on what the participants want to focus on.

As noted above, the ways in which collaborative work can be organised or shaped may vary depending on the project under consideration. Despite this variability, Robutti et al. identify common characteristics such as: (a) the implementation of tasks or approaches that encourage teachers' willingness to participate; (b) the role played by some experts; (c) the fostering of teacher engagement within the communities; (d) the use of theoretical lenses to introduce a specific subject or a specific topic or to support the analysis and the sharing of reflections of the participants. However, Robutti et al. note that, in a significant number of the papers reviewed, the aims of the collaborations were not specified in detail, nor was the impact of local contexts on the collaboration process or its outcomes.

Building on the existing studies and the ICME 13 survey, Theme B of ICMI Study 25 specifies its guiding questions as below.

3.1.1.3 Guiding Questions of Theme B

As described in the Discussion Document for ICMI Study 25, the studies presented in the Theme B group aimed to address the following questions:

1. *What models of teacher collaboration have been developed? What are the design features, goals, and outcomes of the different models?*
2. *How effective are various models for promoting different outcomes?*
3. *Which forms of collaboration are appropriate in different contexts?*
4. *What are the affordances and limitations of each form of teacher collaboration*?
5. *What are the benefits and the challenges that online teacher collaboration poses to the teachers?*

In this chapter, we provide answers to these questions and rise new issues through synthesising the conference papers and presentations to expand and advance the research field on teachers' collaborative learning.

3.1.2 Conceptual Framework for Synthesising Papers

The complexity of teacher collaboration is highlighted in the Discussion Document for ICMI Study 25 (Borko & Potari, 2020) as follows:

> The goals of teacher collaboration are multi-faceted and might be related to the mathematics content, to the learning experience of students, to the development of mathematics teaching that promotes students' learning [. . .], to the design of resources, [. . .], to the creation of a community in which ongoing professional learning is supported, or even to day-to-day teaching [. . .]. (p. 5)

In line with the findings of the ICME 13 Survey, Borko and Potari also point out the diversity of forms that collaborative work can take, of foci that are chosen, or of the results achieved. It is argued that decisions on the different purposes, forms and focus of the collaborative work not only depend on the collaborative project but also on the context which frames the collaborative work and the level of complexity of such work.

The contextualised aspect of teachers' collaborative work, as described in the Discussion Document for ICMI Study 25, can be clearly demonstrated by the case of adaptation of Japanese Lesson Study (JLS) around the world. Even though key elements of JLS are similar, adapted forms and intended goals of learning vary tremendously (Fujii, 2014). Miyakawa and Winsløw (2019) further argued that, even within Japan, JLS can function in different ways depending on institutional conditions and the motives teachers have in the context of their practice.

To have a common language to synthesise the major ideas across different papers in Theme B, we now delimitate the key terms and concepts, and present our overarching framework.

3.1.2.1 Forms of Collaboration

Although there is a variety of meanings of the noun *form*, we highlight three meanings of form related to teacher collaboration as defined by Oxford Advanced Learner's Dictionary (https://www.oxfordlearnersdictionaries.com/): (1) a type or variety of something; (2) the particular way something is, seems, looks, or is presented; (3) the usual way of doing something. Specifically, we consider that the forms of collaborative work represent the types of professional learning environments for teachers' collaborative work to take place. With each form, the corresponding activities are created and organised according to the goals set by those who initiate and/or participate in the collaboration.

3.1.2.2 Context

Horn et al. (2017) acknowledge that specifying a notion of context is problematic due to the polysemy of the concept; context is a notion that is continuously open to be revised within the social and human sciences. According to Lave (1988), context consists of two components, namely a fixed arena and a setting (or scenario) that is defined as "a *relation* between acting persons and the arenas in relation with which they act" (p. 150). The setting refers to what is created by subjects who develop their activities in interaction with the arena and others. The activities and the experiences are "dialectically constituted in relation to the setting" (p. 151). The setting generates the activities and these, in turn, generate the setting. In this way, descriptions of the contexts of collaboration may reveal arenas, settings, actions, activities, and participants. A detailed presentation of the context would provide elements not only to understand the work carried out by the participants, but also the significance of the outcomes achieved.

3.1.2.3 The Nested Nature of Teachers' Professional Work

Esteley (2014) considers that a teacher's professional work can occur at three levels: the micro didactic level of the classroom (e.g. interactions between content, students, and teachers); the institutional level of the school where he/she performs his/her duties as a teacher (e.g. interactions with colleagues or principals to agree and organise assignments for students, assessments, meetings with parents, etc.), which is called the meso or institutional level; and the macro level, referring to the educational system (ministries or other broad institutions) to which the teacher can make various contributions (see Fig. 3.1). Although the three levels of a teacher's work are interrelated, a teacher's collaborative work with others may focus on one or more of these levels. However, the collaborative work at each level informs one another (as highlighted by the arrows), forming a nested hierarchical system.

3.1.2.4 Teachers' Professional Development as Multi-level System

In accordance with Davis and Sengupta (2020), we understand collaborative work for teachers as a complex phenomenon due to the multiple levels involved in the work of teachers (Jaworski et al., 2017) and to the work being a context-sensitive phenomenon (Dowling, 2020; Mellone et al., 2020).

The complexity of professional development of teachers has been well-recognised and described by several studies (e.g. Prediger, 2020; Prediger et al., 2019; Krainer, 2014; Opfer & Pedder, 2011; Loucks-Horsley et al., 2010). For example, Loucks-Horsley et al. (2010) emphasise that the process of professional learning includes five interconnected phases: goal setting, planning, execution, review of results, and reflection on the entire process. Specifically, from a

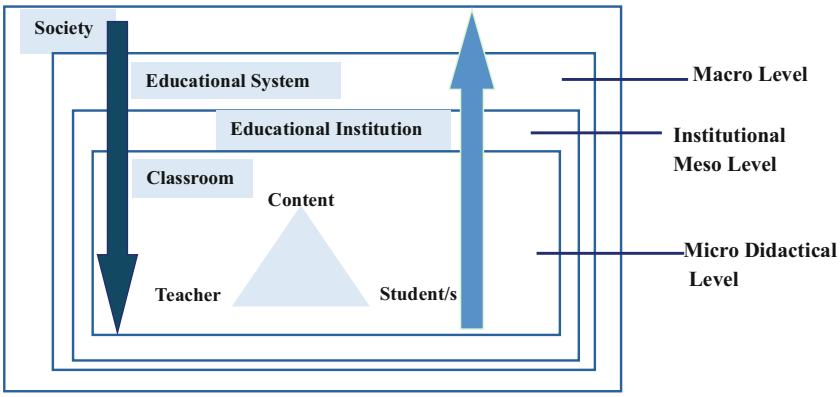

Fig. 3.1 Nested diagram of teachers' professional work. (The diagram in this figure is an English translation of the original Spanish version. The diagram presented here was elaborated with contributions from Esteley's fellow teachers and researchers.) (Esteley, 2014)

complexity theory perspective, Opfer and Pedder (2011) conceptualise professional learning as the dynamics of interactions across three levels of context:[1] micro (individual teachers), meso (school level) and macro (school system).

Krainer (2014) conceptualised the complexity of professional learning in relation to the scale of Professional Development Programs (PDPs), in terms of the number of participants, institutions or communities involved in the PDPs. In her plenary talk, Prediger (2020) presented a nested three-level model to examine the area of teacher development as well as the facilitators' development in collaborative contexts. Each level was represented by a tetrahedron related to classroom teaching, teachers' professional development (TPD) and facilitators' development.

At the *first level* of the model is a tetrahedron representing the teacher's work in the classroom. It is composed of the three elements of the didactic triangle (mathematics, students and teacher) and classroom resources as a fourth element. At the *second level or TPD level*, the tetrahedron has as one of the vertices the teachers, the facilitator is at another vertex, the content (aspects of teaching and learning that are addressed in the program) is placed at a third vertex while the fourth vertex represents the TPD resources (i.e. the materials, the activities of that program). Prediger (2020) uses a similar structure for a *third level* or facilitator professional development (FPD) level. Considering the nature of collaborative work, she also incorporates the concepts of community of practice (Wenger, 1998) and community of inquiry (Jaworski, 2006).

From this brief review, we can see there are different terms applied to levels of professional learning from different perspectives. In this chapter, we use *micro*

[1] It is important to note that, as the teacher's work is the core of the collaboration, the levels considered by these, or other authors are connected to the levels of the teacher's professional work (Esteley, 2014), and in that regard, they can be named in a similar way.

(classroom), *meso* (institution) and *macro* (education system) to represent three levels.

3.1.3 Analytical Approach for Analysing the Papers

The main resources informing this chapter are the 28 conference papers and 25 presentations and reactions, along with the discussion notes generated at the study conference. The 28 papers accepted for Theme B represent a variety of collaborations carried out in different geographic and cultural contexts: 11 from Europe, eight from Latin America,[2] three from North America and six from the Asia–Pacific region. In addition to these primary data resources, additional relevant literature was referenced to help us to frame and sharpen major arguments about the guiding questions for Theme B. To develop this chapter, we have gone through three main phases, which are iterative rather than linear due to the complexity of the five guiding questions.

During the first phase, the participants at the study conference generated four sub-themes (big ideas and major issues) based on the presentations and discussion of all papers. These sub-themes were: (1) the context, needs and goals of collaboration; (2) the forms of collaboration; (3) the outcomes of collaboration; (4) the mathematics content of and/or for collaboration. Participants were then grouped in terms of the sub-themes and developed initial ideas about the major constructs of the sub-theme and a potential structure for organising the sub-theme.

To capture the key features of each sub-theme, each group identified several categorising parameters (key words). For example, for the sub-theme (1) the context, needs and goals for collaborations, the group proposed: needs expressed by teachers; needs expressed by teacher educators; needs expressed by researchers. These parameters were combined into one spreadsheet and shared with all participants who were asked to enter relevant information of their paper into the sheet. Thus, by the end of the Study conference, a spreadsheet was created which contained a short description of each paper along with a brief narrative about each sub-theme.

The second phase entailed the leading authors updating the spreadsheet. Some items were reorganised or combined, and some missing information was added. This sheet then provided a comprehensive and overarching picture of the major features of all papers in Theme B, which created a foundation for each sub-theme leader to develop further sections.

In *the third phase*, the leading authors in each sub-theme identified additional details through reading relevant papers and grouping papers into different categories

[2]Latin America refers to all Central and South American countries as well as Mexico. Although Mexico is geographically and politically part of North America, it is considered part of what it is named as Latin America, because it shares a similar cultural context to the other countries of Central and South America.

based on the framework established in each sub-theme. The reaction presentations and discussion notes were also consulted to ensure the accuracy of our understanding. The papers were compared to synthesise the major ideas of the sub-theme and arguments, which were supported using the original papers.

Within each phase, several dimensions of analysis were identified and chosen based on the relevance to each of the sub-themes. We used the sub-themes and these analyses to structure this chapter.

3.1.4 Layout of the Rest of the Chapter

The rest of this chapter is organised into five main sections. Following this introduction (Sect. 3.1), other sections focus on different sub-themes as follows.

Section 3.2 highlights the institutional or cultural contexts, the origins and the goals of collaborative work. Section 3.3 focuses on the different forms of collaboration. We identify forms as well as specific goals for the joint work, the stakeholders involved and the scale (e.g. number of stakeholders, time) of the collaboration. Section 3.4 presents different outcomes achieved according to the structure and context of the collaborative work. The outcomes of the collaboration are related to products developed, professional learning, knowledges, beliefs and practices. Unexpected results are also identified and dissemination of the results of the collaborative work is discussed. Section 3.5 discusses the role of mathematical content in the forms or results of collaborations that such content can promote, as well as the limitations or difficulties involved.

Although each section has its own major focus, they are interconnected due to the nature of collaborative work. Thus, the same paper may be analysed and cited in more than one section to illustrate purposely the relevant features. At the end of this chapter (Sect. 3.6), we provide answers to the guiding questions, identify further research questions and indicate directions that could be grounds for further studies.

3.2 The Multifaceted Nature of Collaborative Work: Contexts, Origins, Needs and Goals

3.2.1 Introduction

A globalised society offers a complex web of factors that bring people, cultures, beliefs and educational practices into greater proximity with one another. Institutions around the world, pressed by international comparative assessment leagues, are called upon to improve school education through various initiatives such as massive teacher professional development programs. On the other hand, the sense of isolation of teachers who find themselves pressed by constant shifts from ever-changing

educational needs and requests, creates a push from the bottom for teacher professional development.

To understand teachers' collaborative learning in various teacher professional development programs, it is critical to understand the contexts, origins, and goals of the collaborations among teachers and between teachers and researchers. We recognise that all these collaborations are rooted and take place within different contexts (Lave, 1988). Cross-culturally, contexts are culture-bound although the changing and dynamic nature of cultures over time sometimes makes it difficult to recognise the boundaries between different cultures. According to the three-layered model of teacher professional development programs (Opfer & Pedder, 2011; Prediger et al., 2019), as described in Sect. 3.1, we analyse and organise the major points based on the three context levels: micro (classroom level), meso (school or institution level) and macro (school system or multiple-institution level).

3.2.2 Contexts and Origins of the Collaboration

Identifying the contexts and the origins of teacher collaboration in the Theme B papers is not easy. First, the different scales and levels of collaboration, such as *multiple-institution level* (schools, universities, communities) and *single institution level* are related to very different origins, and most of these origins are not explicitly reported in the papers. Second, the origins of teacher collaboration programs reported in research papers, sometimes may be different from those endorsed by the collaborating teachers. As noted in Sect. 3.2.1, we would like to focus on the different visions of collaboration and their relationship with the different cultural (philosophical, political, social, economic) contexts in which these visions developed. Indeed, different cultural contexts and visions generate different forms of collaboration which can be classified at three levels.

Some research studies report the origins of teacher collaboration at multiple-institution or national levels. In these cases, the teachers' collaborations are aimed to improve mathematical teaching practice. An example is the case of Lesson Study involving cycles of study, design, enact and reflect, that was introduced in Iranian schools by the Ministry of Education (Rafiepour, 2020). Another example is the research work of Heck et al. (2020), which represents teacher collaboration organised at a multiple-institution level.

This research was designed to address the lack of professional opportunities for secondary mathematics teachers to learn challenging mathematics in collaboration with colleagues in the United States. In this project, groups of teachers from different cities of the United States have been engaged in PD organised in distance learning mode, using both synchronous and asynchronous approaches. The goals of the PD were to provide participating teachers with experiences of being immersed in mathematical activities which are connected to their teaching, and to establish a blended teacher professional learning community (Heck et al., 2020).

In other research studies, the origins of the collaboration between teachers took place at the *institution level*. These programs focused on professional development courses organised by educators in and for universities or other institutions/providers of PD. Many of these studies were carried out in European countries like France and Spain. For example, Coppé and Roubin (2020) report a French experience of more than 15 years whose origin is linked to the working philosophy of the Institute of Research on Mathematics Education (IREM). In this long tradition, researchers work with teachers in a collaborative research group to develop and disseminate resources on specific mathematical content, like algebra or probability (see, for example, Coppé & Roubin, 2020; Masselin et al., 2020). Similarly, a Spanish experience is reported by Climent et al. (2020), in which reflection on practice (one's own and others) and reciprocal learning represent key features of the collaboration.

These experiences are connected to a community of inquiry (Jaworski & Huang, 2014) where teachers and teacher educators jointly work in the processes of developing mathematics teaching. Thus, these collaborative experiences between teachers and researchers can happen in cultural contexts that allow participants to overcome some stereotypes and beliefs regarding hierarchy in academia. For example, the assumption that researchers who work in the academy have superiority in comparison with teachers who work in school. On the contrary, these research works present an authentic and collaborative dynamic between researchers and teachers (Radford, 2019), that creates important connections between research and practice.

Another interesting study is about a particular experience of collaboration among teachers that took place in Nordic and Baltic countries (Hreinsdóttir, 2020). This project was born and developed through a series of conferences and several smaller meetings regarding the use of *GeoGebra* at school. A group of teachers regularly participated in these conferences and meetings over the past 10 years.

A few studies refer to the origins of teacher collaboration *at the classroom level*. Among these studies, the experience of co-teaching in New Zealand was reported by Eden (2020). The research work describes the joint activity of a group of teachers in a New Zealand primary (elementary) school as they collaboratively inquired into their practice. The aim of their collaboration focused on experiences of *co-teaching* was to improve mathematics learning for low-achieving students.

3.2.3 Needs and Concerns at the Origin of Collaborative Work

There are different needs and requirements that give life to the experiences of collaboration among teachers in different cultural contexts and countries. At the institution level, we can recognise different needs of researchers and teachers with regard to reflecting on and improving teaching practices, but also needs expressed by teachers or teacher educators in responding to changes and demands from the institution. For example, the researcher and teacher partnerships reported by

Coppé and Roubin (2020) and Climent et al. (2020) demonstrate how collaborative work between teachers and researchers can change teachers' beliefs and develop their mathematics knowledge for teaching (e.g. MKT, as defined by Ball et al., 2008, or Mathematics Teacher Specialized Knowledge—MTSK—by Carrillo-Yañez et al., 2018) and get insights into students' mathematical thinking as well. The need to develop teachers' MKT is presented in several studies (see, for example, Pacelli et al., 2020; Collura & Di Paola, 2020).

These studies emphasise the need to support mathematics teachers' development of MKT effectively to orchestrate students' discussions and eventually to promote their mathematics learning. Furthermore, these studies focusing on collaboration between researchers and teachers highlight how this type of experience is useful for researchers to test their research hypotheses, to develop teaching resources for teachers and to explore effective ways of collaborating with teachers.

In contrast, there are cultural contexts in which, for historical and socio-political reasons, collaboration among teachers is made mandatory through teacher professional development programs. This is the case for the practice of teacher professional development through Lesson Study (LS) in Asia, which in recent years has become a model adapted in many countries (Huang et al., 2019). In Japan and China, LS is a job-embedded and system-wide practice with a long history. Particularly in China, the development of LS is the key component of the Teaching Research Group system, which has been promoted throughout China as a powerful form of school-based collaborative professional development for implementing curriculum reform (Li & Huang, 2013).

The growing popularity of LS around the world engenders research needs such as those to establish the potential and limits for adapting LS in other cultural contexts (see, for instance, Rafiepour's (2020) study in the Iranian context); to understand the characteristics of LS, for example by experimenting with LS within different implementation phases (Richit & Tomkelski, 2020); to investigate the role of LS in establishing collaborative and reflective professional learning.

At the classroom level, the need to improve mathematics learning for students identified as at risk of underachievement is the starting point for the experience of co-teaching in New Zealand reported by Eden (2020). Even if there are many self-managed experiences of teachers' collaboration and training around the world (see also Hargreaves & O'Connor, 2018, where the case of a self-managed collaborative group of teachers from one institution is analysed), there are few studies at this level. This is in part because there is restricted access to some institutions, and it can be difficult to access cases.

3.2.4 Main Goals of the Teacher Collaboration

The goals of teacher collaboration are related to the context of the collaboration and the needs and concerns of the people who design and initiate the collaboration. Beyond the two broad categories (innovation and practices) by Robutti et al. (2016),

we identified the following goals of collaborative work: (1) development of mathematics teachers' knowledge needed for teaching (e.g. MKT as defined by Ball et al. (2008), designing lessons, and noticing student thinking) and disposition toward mathematics teaching and learning; (2) development of interdisciplinary knowledge needed for STEM education; (3) improvement of teaching practice that promotes student learning; (4) development of teachers' resources (curricula, tasks, lesson plans, and so on); (5) development of teacher professional learning community. Although a collaborative program was often designed to achieve multiple goals, the intended goals of projects reported are organised in terms of the three context levels.

A few of projects reported are multiple-institution level (or national level) which targeted changes of teachers' knowledge and their teaching practice (e.g. Asgari et al., 2020; Canavarro & Serrazina, 2020; Ekici et al., 2020; Heck et al., 2020). For example, Canavarro and Serrazina (2020) reported a national teacher professional development program, which was grounded in teachers' classrooms with a focus on reflecting on students' mathematical production. In their project, Heck et al. (2020) examined how to develop practicing teachers' MKT and their teaching practice through collaboratively working on mathematics problems and sharing their teaching of the problems through a virtual environment. Ekici et al. (2020) presented a STEM education project which was aimed at developing culturally responsive teaching using mathematical and scientific modelling practices and developing teachers' interdisciplinary knowledge as well.

Most of the projects reported are at institution level with various intended goals (e.g. Acevedo-Rincón, 2020; Coppé & Roubin, 2020; Jahn et al., 2020; Masselin et al., 2020; Soto et al., 2020). Some studies focused on specific goals: for example, Pacelli et al. (2020) examined how to develop teacher knowledge about interpreting student work through a course for a master's degree in mathematics education. Kooloos et al. (2020) examined how to develop teachers' noticing of students' thinking and increase linking of teacher actions to students' thinking through teachers' collaborative reflection on their own videos of classroom practice which was designed to "foster students to think, to articulate their thinking, and to discuss each other's ideas" (p. 372).

Yet other studies focused on broad or interdisciplinary goals. For example, Acevedo-Rincón (2020) investigated how a prospective teacher course could develop participants' interdisciplinary knowledge through bringing together trainees from different degree programs to understand and confront the reality of the classroom in a school setting. In Pinzón and Gómez's (2020) study, they reported a master's degree program through which the practicing teachers collaboratively worked together in studying mathematics topics, designing and implementing lessons for the topics, and reflecting on their implementation, with the support of researchers and mentors in a blended manner (online and on-site). The goals of the program were to develop practicing teachers' PCK and ability to design lessons by using curricular materials and predicting students' solutions and errors.

At the institution level, there are two big sub-groups of studies. One is about the partnership between researchers and teachers and the other is about LS. There is a long tradition of establishing researcher–teacher partnership in Europe (e.g. IREMs

in France). Several studies examine how such traditional partnerships could: (1) develop mathematical tasks and teaching resources in algebra, and train teachers how to use these materials in their classrooms (Coppé & Roubin, 2020; Horoks et al., 2020); (2) develop good tasks in probability using simulation and train teachers to use these in their classes, and subsequently change teaching practice and teachers' specific knowledge for teaching (Masselin et al., 2020); (3) change teachers' teaching practice and develop their profession in general (Modeste & Yvain-Prébiski, 2020).

Climent et al. (2020) reported a longitudinal collaboration between a group of practicing teachers and university faculty in Spain, which aimed to develop teachers' collaborative and reciprocal learning community and reflection on their practice through adopting a problem-solving based approach. In contrast, some studies examined initiatives of researcher–teacher partnerships with different goals of collaboration. For example, Soto et al. (2020) reported a study on a community of inquiry, where researchers created problems for teachers to solve collaboratively, implement in their own classrooms and then come back to collectively reflect on their implementation. The intended goals were to "create a database of good transition problems that can be used by peers" (p. 424) and develop teachers' MKT and positive dispositions needed for helping their students to transition from "primary to secondary school and to help teachers and educators to reflect on professional tensions and practices" (p. 420).

Koichu et al. (2020) revealed the nature of teachers' voice in a teachers' and researchers' community of inquiry while Castro Superfine and Pitvorec (2020) examined how a teacher and researcher collaborative inquiry approach supported teachers' understanding and use of learning trajectory-based formative assessment. Some international researcher-teacher collaborative projects are reported. Hernández et al. (2020) focused on The Seminar on Re-thinking Mathematics (SRM) as an environment of collaboration, which offers resources to enhance different mathematics teaching approaches for teachers and researchers through the process of the dialogue between researchers and teachers. Hreinsdóttir (2020) reported an international collaboration using *GeoGebra* Network to facilitate collaboration and sharing, among teachers and researchers, on the use of technology in the teaching and learning of mathematics.

Several studies examined how an Asian PD tradition of LS could be adopted in other cultures. For example, Richit and Tomkelski (2020) focused on developing professional collaboration through different phases of LS, facilitated by a teacher educator. Rafiepour (2020) examined the affordances and constraints of adapting lesson LS in Iran. Trevisan and Elias (2020) explored how LS could contribute to establishing a collaborative and reflective professional learning community and identified many challenges.

Very few papers focused on the classroom level (e.g. Collura & Di Paola, 2020; Ell, 2020). Collura and Di Paola reported how high-school teachers from different subjects in an Italian school co-design and co-teach mathematics lessons to develop their interdisciplinary knowledge linking scientific and humanistic knowledge. Research by Ell explored how schools could make substantial changes to

organisational, leadership and teacher practices, and to student learning in mathematics, through utilising a Spiral of Inquiry approach to teaching and learning in mathematics.

3.2.5 Summary and Comments

This section of the chapter presents an analysis of contexts, origins, needs and goals of teacher collaboration projects. A three-tiered model comprising a multiple-institution level (e.g. national and/or multiple sites, facilitated by multiple facilitators), an institution level (e.g. organised by a single PD provider) and a classroom level (e.g. within a school) is used to frame the discussion. The national or multiple-site teacher collaboration programs are typically developed to address national needs and concerns, such as curriculum reforms and innovative initiatives.

At the institution level, the major concerns are about developing teachers' MKT or interdisciplinary knowledge, changing their teaching practice, or building partnerships between teachers and researchers to promote teachers' growth, and linking research to classroom practice. At the classroom level, the teachers' collaboration mainly focuses on improving specific teaching strategies which promote innovative ideas and address practical issues. In general, teacher collaborations have multi-dimensional goals. These goals focus on some of the following aspects: developing interdisciplinary knowledge or MKT and dispositions; developing curriculum and teaching materials; improving teaching practice; strengthening professional learning communities.

However, it is hard to identify the origins of teacher collaboration projects due to a lack of explicit explanation in research reports and/or the disparity in explanations from teacher professional development providers and participating teachers. Moreover, there are few projects focusing on teacher collaboration at the classroom or school level. These findings suggest that more attention needs to be paid to teachers' collaboration within their own schools and classrooms, which directly impact on teachers' teaching practice and eventually impact on student learning outcomes. In addition, when projects are reported, the context and origin should be presented explicitly.

3.3 Exploring Forms, Settings and Conditions Related to Teachers' Collaboration

3.3.1 Introduction

In this section, we analyse and report on settings that are implemented to support participants' collaborative work, learning and professional practices. For analytical purposes, we define a setting for collaboration as consisting of four main

components: a form of collaboration; the participants; the topic they will focus on; the scale of the collaboration. It is emphasised that, as underlined in Sect. 3.1, the setting and its components are contextualised in the cultural environments in which they are embedded (Lave, 1988). The implementations of a collaborative setting are the result of a process that involves three interconnected phases: setting goals, planning and execution (in the sense of Loucks-Horsley et al., 2010). Once the goals are agreed upon, as reported in Sect. 3.2, a set of decisions are made during the planning phase for achieving those goals within the context that frames them. For instance, the forms of the implemented settings reported in this section are the result of this process which is itself complex.

Studies on the collaborative work of teachers have highlighted the diversity of forms of collaboration or professional work, and settings implemented. The forms identified are grouped into broad categories such as professional learning communities, communities of inquiry, LS, open classes, and collaborative planning networks (Borko & Potari, 2020; Hargreaves & O'Connor, 2018; Robutti et al., 2016). Rodrigues et al. (2016) analysed various implementations of a form of collaborative work known as "collaborative groups". Considering the case of the Brazilian groups, they point out that, although they share certain characteristics, it is also possible to identify variations among them. Similarities and differences among the forms of collaboration, as pointed out by Rodrigues et al., are facets of the forms of joint work that could make it difficult to categorise them.

Regardless of the forms of collaboration, the main participants and stakeholders of the joint work are teachers and researchers (Jaworski & Huang, 2014; Krainer, 2014); however, other participants may be involved as the collaboration unfolds. Participants and their relationships characterise their communities (Fiorentini, 2013). Some studies discuss the scale of collaboration (size and time). Regarding size, several studies (e.g. Krainer, 2014; Krainer & Spreitzer, 2020; Robutti et al., 2016) emphasise the value of small-scale collaboration; however, the need to promote larger size interventions and studies is also noted. The duration of the collaboration is an aspect that has been discussed mainly from the perspective of the sustainability of collaboration and its effects on long-term outcome analysis (e.g. Hargreaves & Fink, 2003). Regarding the topic on which the collaborations are focused, as discussed in Sect. 3.1, Robutti et al. (2016) provide an extensive discussion. Prediger (2020) offers a valuable insight by focusing on the classroom-level tetrahedron as the content of collaboration, revealing the complexity of such content.

In accordance with what has been previously presented, we report next our analysis of the settings related to the papers presented for Theme B and discussed by the participants in the Lisbon Study conference. The themes[3] for this section, selected by drawing insights from ICMI Study 25 Conference, are: (1) the settings, forms and contexts of collaboration (Sect. 3.3.2); (2) the foci, scale, and participants

[3] During the work in Lisbon, Ana Canavarro, Núria Climent, Alf Coles, Sylvie Coppé, Matthew McLeod, Adriana Richit and Gabriel Soto contributed to selecting the themes considered in this section.

of collaborative settings (Sect. 3.3.3); (3) conditions for promoting or inhibiting collaboration (Sect. 3.3.4). In Sect. 3.3.5, we discuss the limitations of the proposed categories of collaborative forms and synthesise them in terms of models, revealing their essential aspects. Finally, we draw conclusions related to the themes of this section.

3.3.2 Settings, Forms and Contexts of Collaboration

In the Theme B studies, it is possible to recognise diverse forms framing collaborative works. Despite the diversity, some similar aspects can still be recognised. In all cases, communities are established to sustain the joint work. In most of the works, the communities were developed with a declared objective of collaboration, while in others, communities emerge within the framework of a professional development program. In all cases, the participants are mainly teachers and researchers.

In several of the studies, researchers play the role of main facilitator for collaboration. Differences emerge when considering details about the contexts that frame the joint work and consequently differences in the ways in which collaboration is designed and organised are revealed. One aspect that differentiates the papers is the way in which interactions between participants are mediated. In that sense, two main categories are distinguished: face-to-face and blended settings. 25 out of 28 studies described a face-to-face setting while three involved a blended setting.

3.3.2.1 Face-to-Face Settings and Forms

Among the studies mediated by a face-to-face modality, we were able to distinguish three categories of collaborative form identified by the authors themselves: *JLS adapted* and implemented outside Japan; *researchers–teachers partnership*; *workshops.* Another group of studies did not identify their joint work with a particular form, but did so from a broader perspective, referring to the type of community or communities involved. Among the studies that were identified only in terms of the type of community, aspects that especially distinguished them were considered. For example, the type of community and/or number of communities involved the impact of goals on the form and environment of collaboration, and the impact of the context in which the joint work is framed.

Based on this analysis, two categories are proposed: *networks* and *forms related to specific purposes*. Finally, we observed that although several papers used workshops, in only one study (large-scale), the collaboration was entirely mediated by a workshop form. Given that this study involved a network of institutions, it was included in the network category. Thus, four categories of forms are considered: adaptation of *JLS*, *researchers–teachers partnership*, *networks* and *forms related to specific purposes*. We acknowledge that the proposed categories may not be totally discrete which could be recognised as a limitation of our work.

Adaptation of LS As highlighted in Sect. 3.2, four studies report work on LS adaptations. For instance, Richit and Tomkelski (2020) and Rafiepour (2020) focused on JLS adaptations. Richit and Tomkelski report on a study about the development of a professional collaboration which involved six Brazilian teachers from a public primary school and concerned the measurement of length at a fourth-grade level (students 8 or 9 years old). Participating teachers focused on the planning of a research lesson, its enactment and the reflection of the observed classes. Specifically, the paper focuses essentially on the process of the planning of the research lesson.

In that process, all teachers first analysed the contexts of the students and their families, in order to plan the initial activity of the research lesson. They focused on a study of the measurement strategies and instruments used by their families. Then, they conducted a detailed study of the school's interior and exterior space, in order to plan an activity for which the students would use units of length measurement. Throughout the research lesson work process, the group discussed and reflected on the progress made. All the information collected plus the interactions and reflections made within the group were fundamental inputs for the design of the lesson.

Richit and Tomkelski reported that the gradual growth in the level of confidence within the group provided a framework of security for the teacher who finally implemented the collectively designed lesson. In a similar way, Rafiepour (2020) reports work carried out developing a newly modified research lesson on mathematical trigonometric ratios. In this case, before or during engaging in the classical LS cycle, the researcher-facilitator invited the teachers to start reflecting on their trigonometry teaching experiences. After this first step, the refined lesson was taught and observed, followed by a reflection on the whole process.

In both studies, collective work is developed to collect and systematise information about local contexts and teachers' knowledge and experiences. This occurs both at the beginning of the LS process and in moments of lesson design, lesson redesign or reflection (Lewis et al., 2009). By comparing these studies with relevant aspects of the JLS, we observe that both cases are practice-embedded collaborations in which a group of teachers come together to study ways of teaching particular mathematical content (Shimizu, 2014). However, the teachers who are involved in these cases do not have previous experience in working with LS, as is the case with Japanese teachers (Lewis et al., 2009).

Both include cycles of collaborative activities such as lesson planning and delivering planned lessons along with team observation, post-lesson debriefing, and reflection. In both cases, however, collective activity is added to contextualise the content to be taught and the modes of support offered by the mathematics educators. The role of knowledgeable others (Huang & Shimizu, 2016; Shimizu, 2014) who adapt LS outside Japan becomes especially relevant in the whole process of the joint work. For both studies, the adaptations showed affordance in their local contexts as literature indicated (Willems & Van den Bossche, 2019; Cheung & Wong, 2014; Lewis et al., 2009).

Researchers–Teachers Partnership Several of the studies presented in Theme B reported collaborations designated as researchers-teachers partnership. In some of them the group was referred to as a community made up of researchers and teachers. In other cases, the authors referred to the group in terms of some type of community (e.g. community of practice, community of inquiry). Some examples of this form of work are the studies by Castro Superfine and Pitvorec (2020), Climent et al. (2020), Soto et al. (2020), Coppé and Roubin (2020), and Horoks et al. (2020).

For instance, Soto et al. worked with Argentinean teachers with the support of the provincial Ministry of Education and the university where the researchers work. In this case, the continuity of the collaboration depends on constant changes in the policies of the Ministry of Education or the public universities from Argentina. We refer to these examples of collaboration, as well as to the cases of Castro Superfine and Pitvorec and of Climents et al., as *project-based forms,* since the continuity or progress of the collaboration depends on financial or academic support for the project or group fostering the collaboration.

As noted in Sect. 3.2, the study of Coppé and Roubin (2020), and the one by Horoks et al. (2020) are both developed in France. While Coppé and Roubin developed their study in the framework of the IREM tradition, Horoks et al. report on a study developed at a LéA (LéA stands for *lieux d'éducation associés*—associated places of education), which were created in 2010 by the initiative of the French Institute for Education as places for promoting the development of collaborative projects among teachers and researchers interested in relevant issues.

Horoks and colleagues have worked in the LéA RMG,[4] a Parisian school with students aged 12–15, focused on the teaching of algebra, since 2014. During the collaborative work, questions that concern the teachers are collectively identified and transformed into research issues for the researcher and for the whole group. By considering the daily school practices of teachers, researchers do not offer methods to be applied, but rather offer teaching alternatives that could be used to enrich practices, to support collective work or to share and analyse the classes. For the design of resources, an iterative design process is adopted that includes the testing of resources that are appropriate to the work context.

All participants are involved and take responsibility for various tasks. For example, management tasks are divided between the two LéA leaders (a teacher and a researcher). Everyone collects and analyses data that is then shared, for example for use in the resource development cycle. This form of collaborative work is sustained by the co-construction of a common issue, the consideration of teachers' practices and the context that frames them, the design and implementation of resources, and the sharing of tasks and responsibilities.

A characteristic of the work under the IREM tradition or the collaborative culture (Hargreaves & O'Connor, 2018) of the LéA is that they are settings that, with some

[4]RMG is the acronym of the name of the secondary school where the project is being developed. The authors inform that that school is considered as rather disadvantaged and the students are usually assessed as low achieved.

variations, can go beyond the implementation of the named studies. We could point out that they are institutionalised collaborative works, as they are accessible to teachers and researchers and have resources of time, human resources and some money available (Hargreaves & Fink, 2003). The institutionalised character of the latter two studies offers means for the affordances of collaboration and evidences a difference with project-related forms. However, it should be noted that the latter form also offers means of rapid adaptability to diverse social or educational contexts requiring fewer human or material resources. Institutionalised *researchers–teachers partnership* can even have an impact outside the collaborative group as is the case with the work of the IREMs which, for example, has an impact on research communities inside and outside France.

In relation to the former form of collaboration, we are interested in highlighting the work of Coburn et al. (2013), Coburn and Penuel (2016), and Penuel and Hill (2019). We note that the forms of collaboration characterised as researchers-teachers partnership in the studies in Theme B and discussed in the three papers above, reveal some salient commonalities about collaboration. What is presented in these three papers focuses on what the authors refer to as research-practice partnerships. For example, Penuel and Hill note that, "Research-practice partnerships are collaborative research arrangements that seek to transform relationships between researchers, educators and communities" (p. 1).

Despite the differences in the names of the two forms of collaboration, they are consistent in the sense that in both, teachers and researchers work together towards the realisation of a common goal linked to the three levels of teaching work (Esteley, 2014). In the group of works cited at the beginning of this paragraph, it is emphasised that common goals are agreed and worked towards, the work goes beyond a single project and open engagement of the partners with each other is identified. Also, Penuel and Hill discuss issues related to the time and resources required to sustain partnerships. In the researchers–teachers partnership form, the name is chosen because the participants at the ICMI 25 conference themselves identified themselves by that way. Moreover, in some of the cases analysed, this name would emphasise that the research can become a product of the partnership itself. This last observation could be connected to the modifications or changes of projects in the case of the form research-practice partnerships.

Network The network form involves collaboration among groups, communities or institutions and is compatible with a kind of collaboration identified as a *network model* by Hargreaves and O'Connor (2018).[5] These networks may include joint work between communities of researchers from one or several institutions with teachers from several schools and different levels of education. In this case, collaborations can be established not only between communities but also within each community. This aspect of collaboration may imply different content and modes of communication within each community and between the communities involved. In

[5]It should be noted that the authors use the term 'model' in the sense of model or theoretical structure.

some of the reported studies that take this form, networked groups are referred to as communities of practice (Wenger, 1998) or communities of inquiry (Jaworski, 2006, 2008). Other studies, on the other hand, only highlight institutional networks.

In the studies reported, communities are made up of in-service teachers, pre-service teachers, and researchers, and it is possible to identify collaborative interactions inside each community or between communities. The interactions occur in different instances of work, reflection, study, or joint production. Acevedo-Rincón (2020), Koichu et al. (2020), Dörr and Neves (2020) and Canavarro and Serrazina (2020) are examples of collaborative networks. While Acevedo-Rincón shows in detail an example organised in Brazil, Koichu et al. (2020) report a collaboration in Israel.

Both papers report on a complex network of communities. For instance, Koichu et al. report on a collaboration in Israel involving two main communities of inquiry (Jaworski, 2006). One community consists of secondary school teachers and the other of researchers. The communities are organised in the TRIAL (Teacher–Researcher Alliance to Investigate Learning) theoretical–organisational framework. The teachers' community is divided and reorganised into communities or TRIAL teams according to different Israeli regions. The work of each of these teaching communities is done collectively with two researchers. Within each team, "teachers and researchers study pedagogical questions of importance and mutual interest by going through the stages of a research cycle as partners" (p. 369). There were regular work meetings of the teams and a conference that brought together the two main communities.

The work of Dörr and Neves (2020) and Canavarro and Serrazina (2020) are examples of institutional networks. The first is carried out in Brazil through a series of workshops, while the second is developed in Portugal (the latter case will be presented in detail both in Sects. 3.3.3 and in Sect. 3.4).The work reported by Dörr and Neves involves researchers (from a Brazilian university), in-service and pre-service teachers, and students from several public schools that gather and work to "promote school students' development of their mathematical learning processes and contribute to the initial and continuing training of mathematics teachers in the Brazilian Federal District" (p. 294).

A significant number of the interactions and collaborative work activities between researchers, teachers, pre-service teachers and/or students are carried out in a workshop format. The volume of material created in or for these workshops has grown significantly since the start of the collaborations and as the collaborative network has expanded. The activities developed in the framework of the workshops not only offer mathematical experiences, but also bring the possibility of co-planning and co-acting on a shared goal. The authors point out the affordances of this form of collaborative work, and two main limitations. One limitation is linked to the possibility of scientifically validating some of the activities due to the difficulty of formalising the project as an inter-institutional research project. Other limitations concern the dissemination of the activities outside the schools where they took place due to the amount of material produced.

In line with the Dörr and Neves' study, Goodchild et al. (2013) report on collaborative work, carried out in Norway, designing and implementing a series of workshops. This involved extensive fieldwork which was recorded in an extensive database. Lachance and Confrey (2013) report work with teachers from a high school in the United States by implementing workshops. One of the major results reported by these authors is that the workshop provided the catalyst for developing a professional community among the teachers in the school and for their joint work. It seems that workshop-mediated collaborations offer a flexible means of collaborating and have the potential to promote professional community.

The ICMI studies mentioned here show the inter-institutional, organisational and human resource supports that are required for collaboration in the form of a network. These aspects are also highlighted by Hargreaves and O'Connor (2018). Such resources are the supports that give affordances to this form. Perhaps a lack of support for a network could be a limitation to its implementation.

Forms Connected to Specific Purposes We use the term *forms* connected to specific purposes to mean that the collaborative work is organised for achieving a specific goal. Usually, the collaboration is for a specific period and, in some cases, the work may not continue, at least for a similar purpose.

In several of the cases, participants are organised into communities or groups according to the activities that are designed. We highlight groups of an interdisciplinary nature (e.g. Ekici et al., 2020), groups that work in a co-teaching or co-planning design (Eden, 2020; Collura & Di Paola, 2020), groups involving teachers, researchers and curriculum development specialists (Asgari et al., 2020), and groups developed within courses for master's students or with pre-service teachers which focused on issues important to student teachers' future teaching practice (e.g. Pacelli et al., 2020).

This form of collaboration was especially evident in studies that reported on collaborative groups between researchers and teachers in the framework of Ph.D. research (e.g. Kooloos et al., 2020; Masselin et al., 2020) or collaborative works developed by governmental agencies (e.g. Asgari et al., 2020).

For instance, Asgari et al. report that the Iranian Office of Planning and Compilation of School Textbooks (OPCST) produces textbooks and sends them to schools for teachers to use. However, teachers often do not adopt the textbooks produced by the OPCST. In order to find a way to address this issue, Asgari et al. conducted joint work with a heterogeneous group of 21 mathematics teachers (from different regions of Iran) together with experts in mathematics curriculum development. Based on an algebra book produced by the OPCS, the entire collaborative group initiated a cyclical process of analysis–critique production that culminated in the development of an algebra course package for seventh grade. The full cycle and the joint work ended when the collaboratively produced material was sent to the OPCST for further distribution of that material to the Iranian schools.

Focusing on purpose and motivations can, in some cases, promote small and homogeneous collaborative communities. In other cases, there is an emergence of heterogeneous communities that may require time to agree on ways to communicate

ideas or work practices. These and other aspects bring both challenges and richness to collaboration. Details of these forms of collaborative work are provided in Sects. 3.3.3 and 3.3.4.

3.3.2.2 Blended Forms

Only three papers report on blended collaborations. The paper by Hernández et al. (2020) analyses the Seminar on Re-thinking Mathematics (SRM), which brings together teachers and researchers mainly from Latin America (including Spanish or Portuguese speakers) to discuss ways to relate theory and practice. The SRM provides interaction between teachers and researchers via video conferences, online forum discussions and e-mails. Prior to the video conference, the participants read one or more assigned research articles. Once the video conference is over, asynchronous and diachronic interactions are initiated via forum discussions, face-to-face[6] and/or virtual.

Hreinsdóttir (2020) reports on a collaboration developed by the group "The Nordic and Baltic *GeoGebra* Network". This group was founded in 2010 by seven teachers from seven different Nordic or Baltic countries. They privileged online interactions and met face-to-face regularly through conferences. Heck et al. (2020) present a study on the implementation of a PD program, based on the blended 'Mathematics Immersion for Secondary School Teachers' (MIST) model. The program aims for teachers, constituted in communities, to collaborate in activities of mathematical production and pedagogical reflection. They work in sites with four to seven teachers who work face-to-face gathered at the same physical location to promote collaborative learning.

Three or four of these sites work with a facilitator connected through videoconferencing to enhance the learning. One participant at each site is a table leader whose role is to be the eyes and ears for the facilitator who was not in the same room as they were. Table leaders post pictures and periodic verbal updates of participants' work to a shared space for the facilitator. The facilitator's role is to share examples of the participants' work and lead the discussions between the sites. Groups use a collaborative application for asynchronous discussions and to facilitate interactions between table leaders and facilitators.

The three studies exemplified the collaborative work in blended settings. The great advantage of blended collaborations is the scaling up of collaboration that transcends even geographic spaces. A difficulty in the first two cases may stem from the different languages of the participants. In all three cases, time differences between regions can be a problem, as well as the teachers' access to the technology necessary to participate.

[6]Face-to-face meetings take place in each university engaged in the collaboration and involve researchers and/or teachers. For example, different face-to-face meetings take place in Mexico, Colombia, Argentina, etc.

It should be noted that, as discussed in Borba and Llinares (2012) and Borba et al. (2016), the development and study of collaborative activities in a blended format is an issue that is in its infancy. That is reflected in the scarcity of papers presented in Theme B reporting on blended collaborations. Borba and Llinares point out that while the Internet can facilitate the creation of communities, it can also invite members of a community to join and change communities at a rate that could hinder professional development. Recently studies on teacher collaborative work in blended form discussed the strengths, challenges, and further directions (e.g. Chan et al., 2021; Huang et al., 2021).

3.3.3 Collaborative Settings: Focus, Scale and Participants

In the next section, we describe and analyse the main focus and the scale (time and size) of teachers' collaborative work. We also explore possible relationships among them and their influences in the collaborative settings or the related communities.

3.3.3.1 Foci and Goals Beyond Innovation

In some of the studies, as discussed in Robutti et al. (2016), collaborative work focused on the type of innovation including those related to mathematical content, student thinking, task design, curriculum, formative assessment or innovations linked to practices designed to foster pre- or in-service teachers' professional learning. However, other foci also became evident. These were linked to the teaching or learning of certain content and practices to sustain the development of professional learning, but as an integral part of the collaboration.

In such cases, the work focused on developing and sustaining a collaborative community (e.g. Soto et al., 2020) or network (e.g. Hreinsdóttir, 2020), on mathematical discourse (Kooloos et al., 2020) and noticing process (Eden, 2020). The above foci are connected to the goals delimited according to the contexts, origins and needs for collaboration, or the different levels considered, as outlined and discussed in Sect. 3.2 of this chapter.

3.3.3.2 Scale (Time and Size)

Timescale is an important factor in collaborative efforts, and it could be related to different forms and origins of collaboration. Two main categories are identified here, long-term collaborations which last five or more years and those that last less than 5 years. Long-term collaboration can be associated with the institutionalisation and sustainability (Hargreaves & Fink, 2003) of the collaborative work. Collaborations that last less than 5 years may also be institutionalised and provide relevant

information for researchers and agencies interested in the collaborative work. In Sect. 3.3.2, some long-term works are identified in both face-to-face and blended formats.

Researchers–teachers partnerships such as IREM/LéA are examples of long-term collaborations that have an impact from the micro level of the classroom to the macro level of the educational system. (e.g. Modeste & Yvain-Prébiski, 2020). Other examples of long-term collaboration are the case of Asian LS or the GDS (Saturday Group), developed at the State University of Campinas (Brazil). The GDS, is constituted by a network of researchers and teachers (Gonçalves et al., 2014; Rodrigues et al., 2016).

One issue related to long-time collaboration that needs to be addressed is the turnover rate of participants, which could be high among teachers and potentially has a strong impact on the collaboration. One potential impact is connected to changes in the focus and outcomes of such collaborative work because new participants can bring new interests without necessarily changing the main goal of the collaboration set by the community.

Collaborations of less than 5 years can include those developed in the form of project-based researchers–teachers partnership, and those associated with specific purposes connected to a master's or Ph.D. thesis (e.g. Jacques & Clark-Wilson, 2020). We also found collaborations connected to Master's or Ph.D. theses connected to collaborative networks (e.g. Acevedo-Rincón, 2020). Although not exclusive to research associated with doctorates, in such cases teachers are particularly highlighted as stakeholders for research and in this context researchers as stakeholders for teachers' professional learning (Krainer, 2014).

It is worth noting that, in either timescale, the collaborative work was sustainable within the framework of the objectives and contexts in which it was developed (this point is discussed in more detail in Sects. 3.4 and 3.5). For both long- and short-term timescales, there are instances of dissemination of the progress of the work and theoretical contributions either through the groups' own publications (e.g. GDS publications) or through thesis dissertations.

Regarding the *size* of collaborations, according to the three levels proposed by Opfer and Pedder (2011) which interact with teachers' learning and work, we identify three categories: *classroom*, *school* and *educational system.* The *classroom size* corresponds to collaborations involving in-service teachers working together with facilitators in Master's or Ph.D. theses (e.g. Jacques & Clark-Wilson, 2020). The *school size* corresponds to collaborations involving more than one teacher from the same school or teachers from a particular school in the same educational level (e.g. Collura & Di Paola, 2020; Ell, 2020; Kooloos et al., 2020; Richit & Tomkelski, 2020).

At the *educational system size* collaborations involve more than one school, teachers from different educational levels and government or other agencies. At this level, we identify collaborations occurring within one or a few different cities or regions (Climent et al., 2020), at a national level (Heck et al., 2020) and at an international level (Hernández et al., 2020).

When we consider time and size together, we highlight the study of Canavarro and Serrazina (2020). This is a large-scale (time and size) collaborative work with a nationwide initiative developed in Portugal, in all 18 inland districts of the country. The program was developed from 2005–2006 to 2010–2011. It involved more than 14,000 primary school teachers (grades 1–4) and 18 public higher education institutions, each one responsible for primary teacher education in one of the Portuguese districts. The collaboration was developed with the purpose of supporting teachers' work with a new mathematics curriculum (more details on this case are given in Sect. 3.4).

Outside of the studies presented for ICMI 25, the case of the collaborative network known as Escuelas Nuevas[7] (New Schools) stands out as an example of large-scale (time and size) collaboration. This network started in the seventies in Colombia and takes as a reference a transformative pedagogy for small, multi-grade rural schools. Currently it has been extended to multiple Colombian rural and urban schools. Adaptations have been made for the urban contexts and for other countries (Hargreaves & O'Connor, 2018).

3.3.3.3 Participants and Communities

The size of the collaboration has an important link to the participants in the collaboration, both in number and in the communities to which they belong. In this regard, examples of participants for the first four forms are given in Sect. 3.3.2. Other examples can be considered with reference to collaborations that are *connected to specific purposes*. In Ekici et al. (2020), the collaboration was based on implementing a culturally responsive pedagogy (Gay, 2010). They worked with an interdisciplinary community which focused on STEM and mathematical modelling approaches for teaching.

Another example of interdisciplinary collaboration is the study reported by Collura and Di Paola (2020) developed in an Italian scientific upper secondary school. In both cases, the communities that support the collaboration are considered as interdisciplinary communities. They are so-called, not only because the teachers and researcher-facilitators involved come from different areas of knowledge (not only mathematics), but also because they collaborate to plan a lesson or provide answers to local problems of an interdisciplinary nature.

Fiorentini (2013), in considering collaborations between researchers and teachers in the framework of GDS, proposes the idea of borderline communities. For the author, such communities are those that are on "the border between school and university and usually have more freedom of action and ability to define their own agenda of work and study, since they are not institutionally supervised by the school or university" (p. 157). Even group meetings can occur in spaces and times outside the schools or the university. Of course, for the author, the GDS community is a

[7] https://escuelanueva.org/

borderline community. Similarly, when considering two cases of a blended form of collaboration (Hreinsdóttir, 2020; Hernández et al., 2020), the communities related with those studies can also be considered borderline communities.

3.3.4 *Collaborative Settings and Arenas: A Brief Reflection on Conditions That May Inhibit or Foster Collaboration*

As noted in Sect. 3.1, settings refer to what is created by the subjects who develop their activities in interaction with the arena that frames the settings, and the proposed activities. In this sense, activities and experiences are dialectically constituted with the settings (Lave, 1988).

Attendance in collaborative groups implies voluntary participation of the group members. However, in some collaborative settings, and their corresponding social or educational arenas, participation could be required. Although this may be considered a contradiction in terms of a joint work, it is feasible to distinguish settings that can enable the start of a genuine collaborative work while others may inhibit it. However, regarding both required and voluntary participation, the role played by the type and role of the participants in the setting is highlighted as a relevant aspect that can inhibit or favour collaboration.

The required participation of in-service teachers in collaborative initiatives usually originates from government agencies, teachers' unions, non-governmental organisations or a school's requirements. In the studies reported by Trevisan and Elias (2020); Soto et al. (2020) and Canavarro and Serrazina (2020), we identified two possible conditions arising from required participation in collaborative initiatives that could inhibit or promote authentic collaboration. One of these conditions has to do with the way in which the form and focus of the collaboration is agreed upon. Where the form and focus of the collaboration is not sufficiently communicated by the researcher-facilitator, discussed or reworked with the teachers, the collaboration may be compromised (e.g. Trevisan & Elias, 2020—this case will be presented in detail in Sect. 3.4).

However, when the form and focus of the collaboration are thoroughly discussed within the group, collaboration is promoted, as reported by Soto et al. (2020). Another condition that might inhibit or foster collaboration arises from the participants involved and the professional relationship they establish with each other. For instance, Canavarro and Serrazina (2020), due to recognising the importance of this condition as a possible obstacle, highlight certain actions to foster good relations. Among these is the value of promoting trust and complicity between facilitators and teachers, mainly when facilitators enter teachers' classrooms.

Required collaboration in undergraduate or graduate courses designed for teacher training has potential for fostering initiatives for future collaborations when facing the transition from student teacher to teacher (Gueudet et al., 2016). These opportunities for early collaboration offer prospective teachers a learning context for

promoting what Geijsel and Meijers (2005) call professional identity learning, which is considered as a process for "meaning-giving and sense-making essential to bringing fundamental educational change" (p. 420). The process of identity learning could occur only when social construction and individual sense-making become closely related to each other (Anderson et al., 2018).

Pacelli et al. (2020) offer an example of promoting identity learning through an individual−collective−individual cyclical activity. In this case, prospective teachers were asked to solve a mathematical problem individually, and to compare their solution with secondary school students' solutions for a similar problem. Then, the prospective teachers engaged in a collective discussion focused on their individual solutions and their interpretations on secondary school students' solutions. These activities allowed them to work on a typical professional practice, in which collaboration becomes a powerful tool.

In cases in which participation is voluntary and the objectives are shared from the very beginning, it is possible to recognise aspects common to the different collaborative forms that promote collaboration. For example, the value teachers place on participants contributing to interpreting their practices in new ways is highlighted (e.g. Canavarro & Serrazina, 2020; Horoks et al., 2020; Koichu et al., 2020; Richit & Tomkelski; 2020). In relation to the above, the analysis of video recordings of classes is also valued (e.g. Kooloos et al., 2020). Another aspect that can promote collaboration is interaction between secondary and primary school teachers, with their different knowledge, even though, at the beginning, it may seem an obstacle (e.g. Soto et al., 2020). These aspects of teachers' work that promote collaboration align with the work of Robutti et al. (2016).

The above examples highlight the role of participants in promoting or inhibiting collaboration. Although issues related to the participants' role will be expanded in Chap. 4 (Theme C, this volume), what has been discussed seeks to highlight the relationship between the activities developed related to different forms and the arenas that support the application of such forms, and how this relationship can give rise to a truly collaborative work context (we refer to 'context' in the sense of Lave, 1988).

3.3.5 Summary and Comments

Attending to the notion of form and the characterisation of context (Lave, 1988), we have described and analysed illustrative examples offering fine-grained details of the professional learning settings and the social–cultural contexts framing the collaboration. The descriptions and analysis became a way for conveying information on the concepts or actions related to the context of joint work, in connection with the ideas of the people who interact with the activities they engage in (Chaiklin & Lave, 2003).

Considering Theme B studies, as well as other studies, we identified five categories of forms of joint work. We recognise as a limitation of our categorisation the fact

that the categories are not mutually exclusive. Perhaps this fact could be connected to the double facets (differences and similarities) that seem to interweave the diversity of forms. We note that, in a broad way, Denzin and Lincoln (2018) point out that there are certain social phenomena, that at first may come to resist distinct categorisations. Despite the limitation mentioned above, being able to offer categories of form for designing joint work could be a starting point or step towards designing or studying forms of collegiality that comprise joint work and joint responsibility in the different stages of collaboration (Little, 1990). The essential characteristics of each empirical form are summarised below. In such a synthesis, we try to go beyond particular cases to consider the forms in a decontextualised way or as general models.

LS adaptations are a collaborative model characterised by being essentially focused on the micro and/or institutional level; the collaboration involves a community of researchers and teachers who create and apply original resources for the teaching of mathematical content. In all the joint work, the typical LS cycle is followed (Lewis et al., 2009; Fujii, 2014) and research processes are privileged to make evident relevant information from the local context that contributes to the design of the lessons.

The researchers–teachers partnership model is characterised by the development of joint work involving researchers and teachers in professional learning communities. Within each community, objectives are set, processes of inquiry and joint reflection are carried out, and guidelines are established to develop joint work. The community, as a collective agent, carries out activities such as the creation of resources for the teaching of mathematics and joint work to solve issues recognised as problematic in one or several institutions, bringing together teachers from the same or different levels.

The *network* model involves joint work between communities. For example, there may be interactions between a community of researchers and several communities of teachers or pre-service teachers. The communities can be defined according to a school district in the same country or several communities of teachers or researchers in a wide geographical region. The objectives and processes of joint work are varied, as are the communities of practice that compose them. The co-existence of theoretical research activities and scholarly inquiry stands out. Within this model, workshop-mediated collaborations, as well as blended forms, are included, and borderline communities are identified. In the case of blended forms, we highlight the important mediation of technologies for the joint work.

The *connected to specific purposes* model is characterised by the fact that the specificity of the purpose is taken as the focal point of the joint work. In this case, the collaborations may vary in the spectrum of scale (time and size) with a prevalence of short-term and small-size works. The levels of interest range from micro to macro. In these cases, researchers, teachers of the same or different levels or pre-service teachers are involved. Different motives for participation were associated with this model (see Sect. 3.3.4) and learning communities, inquiry communities and interdisciplinary communities were identified as examples.

In these decontextualised characterisations, similarities and differences can be observed that stem from the collaborative nature of the work. The latter comes from the decisions and agreements that are reached among those who promote or participate in the joint work. It is highlighted that the way in which the models are presented is in line with ideas expressed by Matos et al. (2009), who propose that a model of joint work encompasses: "A community of individuals, sharing cultural specificities [. . .], who have particular forms of engagement in the professional development sessions and whose topic of focus or domain pertains to specific aspects of their practice [. . .]" (p. 176).

We note that the models presented above can be refined and modified *a posteriori* in an iterative process as joint work studies progress. However, these models can be useful tools in the framework of research or collaborative work design. In any case, collaborative setting designs are contextualised to the working conditions of each group. In all cases, the settings experienced depend on who is involved, the purposes pursued, the scale of work and the levels (micro, meso, macro) on which the group is focused.

In all models, the settings implemented in the Theme B studies (or other collaborative experiences reported in this section) can account for shared work and responsibilities. Some collaborations included in the researchers–teachers partnership and network models are sustained over time, perhaps due to their institutionalised nature provided by the educational system itself. This aspect is shared by the JLS model. These models have the potential to be scaled up in relation to the number of participants. Models that focus mainly on the micro level, and include a small number of participants, could be used as a means of scaling-up collaborative work. However, in this case, it is important to note that, just as the context adaptations of forms/models are important, adaptations in their scale also deserve special attention.

In this section, the context as a relevant aspect for collaboration was evident in all cases. The collaborative context was manifested as the result of the interrelation between the settings and the arenas (or terrains) upon which the joint work is based (Lave, 1988). Finally, we are interested in highlighting that the experience of each participant in any joint work can signify a valuable point in their professional trajectories (Vezub, 2013). Perhaps this aspect can be made more evident in Sects. 3.4 and 3.5.

3.4 A Comprehensive Examination of Outcomes Related to Teachers' Collaboration

3.4.1 Introduction

In this section, we report and analyse outcomes achieved in collaborative processes in which teachers, mathematics educators and others participated as stakeholders in the whole process. We consider outcomes as the result of an activity or process,

where an activity is defined as a situation in which things are happening, or people are engaged in actions in order to achieve particular, more or less well-defined, aims (Oxford Dictionaries online: www.oxfordlearnersdictionaries.com/definition).

Taking what was presented and discussed in the previous sections as a reference and, in accordance with the introduction to this chapter, it is pointed out that we do not suggest a cause-and-effect relationship, but rather a necessary interplay between the collaborative activities and the results achieved. Considering that most of the studies presented in Theme B have involved research approaches of a qualitative nature, we point out that, in general, qualitative studies highlight processes which lead to outcomes (Denzin & Lincoln, 2018).

It is in that sense that we will examine results, or outcomes, in this section. When we talk about results or outcomes, we are not only thinking about those that were successful in terms of the planned achievements, but we have also tried to recognise those results that were considered not positive, in order to think about what happened in those cases, What can we learn from them? It is important, before getting to the studies themselves, to offer some framing remarks about teacher learning, development, growth and change; we set out, below, how these terms relate to outcomes of collaboration and, then, how they can be studied.

One of the most influential models of professional growth comes from Clarke and Hollingsworth (2002). Following these authors, we take professional growth (which for us is also synonymous with development) as "an inevitable and continuing process of learning" (p. 947). Clarke and Hollingsworth's model offers an image of the change environment as four interconnected, non-linear domains: "the personal domain (teacher knowledge, beliefs and attitudes), the domain of practice (professional experimentation), the domain of consequence (salient outcomes), and the external domain (sources of information, stimulus or support)" (p. 950).

The interactions between these domains occur through the mediating processes of reflection and of enactment. The evident power of this model is indicated by its extensive use. In terms of studying the outcomes of collaboration, we have considered each domain. We follow this model in viewing change as what occurs in one domain, which then triggers change in other domains. Activity is taken to be the catalyst of change.

We now consider how the domains of the change environment can be studied. Changes in the personal domain have been researched via interviews, questionnaires and observations, for instance using attitude surveys or analyses of different forms of teacher knowledge. Coles (2018) notes the perhaps obvious, but still sometimes overlooked, point that questionnaires or surveys can only tell us what teachers *say* about their beliefs and attitudes, which may be different from the beliefs and attitudes enacted in the classroom. Changes in the domain of practice are frequently researched through the use of video and sometimes through self-reports. Changes in outcomes relating to instructional practice, in a review of work on teacher learning by Goldsmith et al. (2014), can be classified into three areas: "mathematical content of lessons, classroom discourse, and students' intellectual autonomy" (p. 13), which can be researched through recordings of lessons and subsequent analysis by researchers.

The Goldsmith et al.'s review also considered the external domain, for instance, curriculum change, and concluded that, as a field, we know little about the impact of curriculum reform on teacher learning. Indeed, that review confirmed a finding from Goldsmith et al. (2009) that we know little about, namely "*how* teachers develop knowledge, beliefs, or instructional practices" (2014, p. 21; *italic in original*). This conclusion relates strongly to our focus here on outcomes of collaboration. In other words, as we review what is said about outcomes, we have been concerned to also note where studies offer us some insight into *how* changes have occurred.

In an overview article on a special double issue on teacher change, Reid and Zack (2010a) identified two themes salient to how teachers learn. Firstly, they noted, "the importance of emotional engagement to the process of change" and secondly, "the vulnerability that occurs as part of change" (p. 372). These themes touch on the personal and external domains mentioned above. Reid and Zack (2010b) also raise the question of whether there is anything specific to *mathematics* teacher change. One potential parallel is between images of teaching mathematics as a process of inquiry, and teachers' investigations into their own practice as a process of inquiry. Reid and Zack (2010b) highlight how changing views of mathematics (as a discipline) can lead to changing approaches to teaching mathematics. In addition, they note the significance of time in the change process, e.g. how destructive it can be in complex change processes to have rigid restrictions on time.

With these preparatory comments in mind, we now turn to giving an account of the results that emerged from studies of collaborative work across 28 papers submitted for Theme B which were analysed by participants of the Study Conference in Lisbon. The themes identified,[8] which grew out of the ICMI conference discussion document, were: (1) *products developed* during the collaborative work as a result of such work, assuming that the subjects involved have opportunities to set their ideas, contributions and knowledge into play or together create new knowledge (Sect. 3.4.2); (2) *outcomes related to learning, professional development, knowledge, beliefs and practices* (Sect. 3.4.3); (3) *difficulties or problematic issues encountered at work* (Sect. 3.4.4); (4) *the sustainability, dissemination and sharing of outcomes* (Sect. 3.4.5). It is evident that these themes cut across Clarke and Hollingsworth's change environment domains.

3.4.2 Products Developed as Result of Collaboration and Their Levels of Application

Since the different outcomes or learning that result from the collaborative work can be related to the activities developed, the objectives set and the participants involved

[8] During the work in Lisbon, Jenny Acevedo-Rincón, Chris Kooloos, Raquel Carneiro Dörr, André Pinzón and Henrique Rizek Elias contributed for selecting some of the themes considered in this section.

(see Sects. 3.2 and 3.3), it is expected that the nature and levels of the outcomes will be related to these factors. Since the collaborative work we have in focus is centred on the practice of mathematics teaching, it is worth remembering that such practices are framed by the interplay between three contexts related to the work of teaching: the classroom, the school and the educational system itself (Opfer & Pedder, 2011).

To start a description and analysis of the levels and nature of the results coming from the processes of collaborative work, we will first consider those results related to developed products that materialise in the form of resources to sustain the collaborative work. In considering what was reported in Sect. 3.3, it is noted that, for several of the collaborative scenarios chosen, specific objectives were set in relation to the development of certain products. In some cases, the expected products are linked to the micro didactical context of the classroom, such as tasks, lessons, textbooks, websites, etc. Despite the didactical nature and the micro level of several of these products, some of them are valuable resources for school institutions and other products could be useful means or inputs for both the educational system and mathematics educators.

Other products are related to curricular aspects, designed mainly to have an impact at the meso (institutional) and macro (educational system) levels. Some examples of these products are curricular designs or materials created specifically to support curricular reforms. Finally, resources designed and produced to support full collaborative work or to carry out research linked to professional development processes are also highlighted. These products could be identified as being transversal since they have implications for all levels of collaborative work.

Due to the complex and intertwined nature of collaborative work, the results in terms of products developed in and for collaboration can be linked in one way or another to different context levels as well as subject levels. However, for analytical purposes, Table 3.1 that follows provides examples of products distributed by the levels to which they can be mainly related.

This table includes two categories of products: material and intellectual. The material products seem to be evident on the micro and meso levels, while the intellectual ones are on the macro level. What is presented here is illustrative and it is assumed that at all levels one can also consider material and intellectual products. Chapter 5 (Theme D, this volume) advances details about some products as resources for, or from, collaboration. Below are examples of collaborative work in which the emergence of some such products is evident.

Among others, the study reported in Canavarro and Serrazina (2020) makes evident the development of products that are related to the micro, meso and macro levels, and with products illustrated in Table 3.1. At the micro didactical level, the teachers chose mathematical content and worked on the didactical knowledge needed for the planning of lessons tackling that content. They also participated in the collective planning of challenging teaching experiments requiring new mathematical knowledge, new kinds of tasks, different teaching strategies and new resources in the classroom, which edges into the meso level. The programme was grounded in teachers' classrooms and placed great emphasis on teacher collaboration in schools, adopting the analysis of students' mathematical productions as a focus to

Table 3.1 Products distributed by levels (Own source)

Main context/levels related to the collaborative work	Main products or resources developed in or for the collaboration
Micro-didactical level Teachers' level or students' level	Lessons plans Mathematical activities Lessons and portfolios Task sequences Tasks of different natures Guidelines for preparing and orchestrating classroom discourse and lesson plans Different resources for teaching Websites STEM and modelling tasks: modelling problems, experimentations, etc. Assessment activities Students' productions
Meso-institutional level and macro level	Curriculum design Curricular materials Teaching material for a new national curriculum
Collaborative work and/or research level	Resources for professional development processes: artefacts (e.g. video records, reactivated forum discussions) A designed collaborative inquiry model involving co-teaching. Interdisciplinary enterprises PD Theoretical and methodological results Portfolios A spiral of inquiry model approach to teaching and learning in mathematics in collaborative settings Models for teacher collaboration in curriculum development

promote reflection on how to improve teaching practice. In that sense, students' productions could be considered as a developed product (by the students) that materialises in the form of resources to sustain collaborative work.

At the end of the year, each teacher was asked to complete a reflective portfolio. Such activity has the aim of "fostering teachers' meta-reflection and learning" (p. 250). The portfolio created by each teacher included a personal reflection on the challenges of teaching within the scope of the new curricular guidelines and on how the Programme of Professional Development in Mathematics (PFCM) had contributed to meet the teacher's needs. "This option acknowledges that teachers learn when they prepare new topics for teaching, but also when they reflect on what happens in the classroom" (p. 249). Some selected portfolios became resources for the professional development (TPD) level of teachers from other schools in the country in the context of PFCM.

Regarding collaborative products connected to the work of teachers or their practice, we take as a reference the study of Dörr and Neves (2020). This paper presents an extensive collaborative work called "Circuito de Vivências em Matemática" (Circuit of Experiences in Mathematics). One result, among others

reported, is a web-site[9] where recreational activities created for the teaching of mathematics were uploaded. Such activities were developed between 2004 and 2018 with the intervention of higher education teachers, basic schoolteachers,[10] researchers and future teachers, and they involved a large number of public or private schools from the same school district.

In the papers mentioned above, as in many other works, several of the products created can be related to results at the micro level and the meso level. In that respect, we mention Ekici et al. (2020). The authors discuss an experience of a collaborative community constituted by 40–50 mathematics and science teachers, education faculty and community partners to work in interdisciplinary collaborative groups which focus their work on the development and implementation of locally relevant integrated STEM projects using mathematical and scientific modelling practices. The group approach begins by integrating STEM problems into schools to foster teacher and student interest, to serve as a catalyst for locally relevant curricula development, and to support "researcher identity in both students and teachers" (p. 309). As discussed in the paper, the products developed during 2 years of collaborative work had impact on the students:

> We observed that students that participated in STEM projects developed [. . .] improved attitudes towards STEM learning, a heightened interest in attending college, and higher degrees of engagement, as they studied problems and concerns situated within their community. (p. 312)

Combining outcomes for teachers and students, as well as the application of a spiral model for inquiry (Timperley et al., 2014), we mention the study by Ell (2020). This study reports on a two-year experience of collaborative work between a researcher and teachers from two Australian primary schools. Those schools were part of a professional learning initiative focused on collaborative inquiry as a "way of tailoring professional learning to local contexts and students' needs" (p. 316). In that sense and considering teaching as an inquiry process, Ell adopted a spiral of inquiry, which began from a students' learning challenge. As reported in the paper, the collection of information about students, both from conversations with students about their learning and from specific diagnostic tools, led to curiosity and enthusiasm on the teachers' behalf. "Results suggest that changes in teacher practice and improvements in student learning can be made by applying a Spiral of Inquiry approach to mathematics teaching and learning" (p. 316).

Just as Ell applies an available model to her work as a valuable tool for her study, among the works presented for Theme B, we highlight papers that propose different and new models as results of their own study. For example, Masselin et al. (2020) present and discuss potentialities and limitations of a collaborative work model, which relates to a doctoral research project from France. The collaboration model displayed on p. 241 is structured in three loops involving a network of three different collaborative groups. Probably, the main limitation of this model is the strong

[9] Website: http://circuitodevivencias.mat.unb.br

[10] In Brazil, the basic school provides services to students between 6 and 14 years of age.

connotation it has in relation to the specificity of the research context, but it also brings contributions:

> it is possible to assert that the model developed provided results for research on the simulation of random experiments. While it has limitations related to the constraints of doctoral research and training over a limited period of time, this collaborative work has helped to explore task simulation use and conceptualize training engineering based on class study (Masselin, 2019). (p. 245)

In Asgari et al. (2020), a cyclical model is presented (p. 236). The model is the result of collaborative face-to-face work focused on the development of curricular material; while Heck et al. (2020, p. 327), set out their "Mathematics Immersion for Secondary Teachers at Scale" model, for a blended form of collaboration.

All the examples presented in this section highlight the products developed as interconnected objects and as contextualised tools that support the professional practices of both teachers and mathematics educators. Many of these objects are aligned to the group's work objectives, while in other cases they become means for achieving them. All of them involve the bringing into play of new or known knowledge and expertise by the group members. When considering the last examples about models, it should be noted that the models created are also related to the context, the problem and the objectives of the work carried out.

This fact highlights one of the main characteristics of a model since, according to Fourez (1995), models are project related. Regarding the applied or created models, they stand out for being considered as valuable tools that can guide the design and analysis of contextualised collaborative settings. We note that the creation of a model may be a valuable form of collaborative activity for those involved.

3.4.3 Outcomes Related to Learning, Professional Development, Knowledge, Beliefs and Practices

In this Sect., to analyse results related to teachers' learning, we take as our starting model the one offered by Clarke and Hollingsworth (2002) and discussed in the introduction. However, some other contributions will also be considered. On the one hand, it is noted that, among the papers presented in Theme B, few offer discussions that conceptualise the process of change itself. For the most part, the papers recognise changes or growth in teachers during the collaborative immersion process and, at the same time, connect those changes or growth to learning.

In several papers, the learning is mainly related to the domains of personal and professional practice. In some cases, such learning is intertwined with the consequence or external domains, perhaps something to be expected since most studies take as their central focus the professional practices of teachers in their school environments. The following is a group of examples in which we seek not only to make evident their outcomes, but also to give some details about the contexts and

settings in which they were developed, in order to make evident the connections between results, contexts and collaborative scenarios.

One example, from the ICMI 25 Study Conference, of a project where there is attention to conceptualising the change process, is the work of Kooloos et al. (2020). This study took place in The Netherlands, involved five mathematics teachers and one researcher, and included guidelines for "discourse-based lessons" which were collaboratively developed. The suggestion from the authors is that, through participating in the study, discourse-based lessons became a "boundary space" (Akkerman & Bakker, 2011) between the teachers' regular teaching practice and academic discourse on mathematics teaching. The study did not aim to collect data on the cross-over or connection between discourse-based lessons and teachers' more regular teaching practice, but Kooloos et al. offered some suggestive comments about likely or possible influences. Change, in this model, is about the development of boundary objects and spaces, initially only minimally interacting with a sphere of practice, and the gradual transformation of the sphere of practice through the influence of the boundary (which itself is also transformed).

The study reported by Asgari et al. (2020) represents an example where the external domain (Clarke & Hollingsworth, 2002) is evident as a stimulus for collaboration. In this case, the authors point out as problematic, in the Iranian context, the gap between the curriculum proposed by experts and the curriculum implemented by teachers. Among the results reported, the authors highlight an increase in teachers' knowledge of the content of the curriculum, a change in their beliefs about their role and opportunities to participate in curriculum development activities, and increased confidence in collaborative curriculum development. In addition, "teachers were found to believe in the effectiveness of teamwork in improving knowledge and sharing their experiences" (p. 230). The above results show that changes in the personal and professional domain of teachers can be related to the external domain.

Horoks et al. (2020) report a long-term collaboration among researchers and teachers. The group worked together to achieve three main objectives. The complete group had the objective of designing resources for the teaching and learning of algebra. Teachers set themselves the goal of enriching their professional practices to offer better support for the learning of all their students while working in algebra. Researchers–facilitators were interested not only in studying the effects of collaborative work, both on the teachers and their students, but also in providing knowledge to the field of mathematical education. Among the several results reported in this work, we highlight those that indicate changes in the teachers and the students. It is worth noticing that the reported changes were not immediately visible but, on the contrary, they became noticeable after the first year of immersion in the collaborative work.

These considerations are in line with those expressed by Reid and Zack (2010b) and with Potari et al. (2010). The first changes of the teachers were linked to the kind of tasks selected by them for their classes, by selecting tasks that favoured the construction of meaning in the algebraic work. Later, changes were noted in their interactions with the students, with the increasing prevalence of interactions that

promoted a certain autonomy among the students while working on sophisticated algebraic tasks. Another important aspect observed concerns professional development at the time of the induction of new members. With respect to the students, changes stand out as they move from an essentially arithmetical way of working to the use of algebraic procedures when solving problems. The results reported make evident an interconnection between three domains: the personal domain, the domain of practice and the domain of consequences.

Richit and Tomkelski (2020) highlight some teachers' growth from participation in jointly carried-out activities. For example, it is noted that, as a result of their involvement in the collaborative development of lesson plans, the teachers identified changes in the group itself that had enabled them to grow closer to each other. They also emphasise the increasing confidence within the group, considering that such achievement "was developed through the respect and care with the needs, interests and anguishes of all participants in the Lesson Study" (p. 419). The authors acknowledge that the participation of the teachers in the LS activities promoted the growth of the teachers in their personal and professional domains.

These results are supported by the teachers' voices:

> As we finish our meeting, I became very happy with what I learned through planning of classes [. . .] I feel the lack of collective planning in the everyday routine. [teacher Erika, p. 415]

> We think alone, we analyse alone, and we find some way out. But what we made here in the group [. . .] is think together and reflect together. There are many people analysing the same question, thinking in the same class. And this made a difference [. . .]. [teacher Ivy, p. 415]

Due to time constraints, at the suggestion of the mathematics educators, part of the teachers' production work was done via *Google Drive*. These activities implied, for the teachers, changes in routines and learning about the use of these resources. As mathematics educators, the authors recognise that, to promote collaborative work in the field of LS, it is necessary to put into practice: "the care, attention and appropriate intervention of those who lead it, so that the group feels welcomed, respected and valued in the collective" (p. 419).

In line with the previously mentioned, we find it interesting to highlight routines and time, as two dimensions of school practices. School routines institute the daily work of teachers and changes in these routines give possibilities for reconfiguration of certain social practices. This is so, because "teachers are not simply learners of techniques; they are also social learners" (Hargreaves, 1996, p. 39). In relation to time, it is a key consideration to understand teachers' problems of change and the structuring of their work. In this regard, as Hargreaves points out, for teachers, time is not only an objective restriction that can be oppressive, but also a horizon of subjectively defined possibilities or limitations.

What is presented in Sects. 3.4.2 and 3.4.3 are some of the main successful results achieved during collaborations. Several of them are considered successful by linking them to expected achievements. Other successful results, as well as issues considered problematic or difficult, emerged during the process itself. In the next section, some of these difficulties are highlighted.

3.4.4 *Unexpected Outcomes: Difficulties or Problematic Issues Encountered at Work*

Approximately half of the papers report some difficulties or issues considered problematic in the development of collaborative work or research itself as emerging results. Difficulties or problematic issues are those elements or situations that, in one way or another, influenced the collaborative work, or the objectives, being only partially achieved. They are hard to categorise, because, as we have mentioned before, they arise from the interaction between the complex process of collaboration and the research that accompanies or dialogues with that process.

The difficulties and problems, as well as the results, are embedded in the contexts and practices developed, such as the collaborative, teaching, learning or research practices. Aspects related to the teachers' lack of time, the need for a long period to observe work progress, the dynamics of school routines and conditions imposed by the institutional or cultural context are recognised, in some way, as limitations to collaborative work or research carried out in these scenarios.

Rafiepour (2020) indicates that the teachers involved in their project admit that the demands on their time and administrative structures limit their possibilities for working with JLS in their schools. Trevisan and Elias (2020) present work developed between 2018 and 2019 in Brazil. This work aims to establish a collaborative group appealing to an adaptation of the LS model and involving primary school teachers and mathematics educators to support the development of their professional learning. The project had support by the Municipal Education Secretariat. In 2018, the work of three primary school teachers with the facilitators–researchers would focus on the development of lesson plans for teaching division and fractions.

However, a series of difficulties made it impossible to achieve the aims. The main difficulties reported are restrictions of context, previous teachers' knowledge or beliefs about the chosen content and some inexperience of the facilitators–researchers in dealing with such issues. The facilitators–researchers, and authors of the report, highlight their recognition and understanding of these issues as a learning experience for them. This allowed them to develop a new collaborative setting in 2019 to work in another school and under different conditions. It could be noted that, in part, the difficulties reported in this case highlight the complexity in achieving productive adaptations of the JLS model.

During discussions at the study conference, the reactor to this paper suggested several aspects that could perhaps have been considered in this study in more detail. Among them were the non-voluntary aspect of starting the work together as an obstacle in this case for collaboration, and the absence of references about the extended use of LS in some countries within a Latin American context.

Heck et al. (2020) focus their analysis on two groups they worked with. From their analysis, they recognise problems related to the partial achievement of certain objectives of their work. Such problems refer to the links that teachers create between producing mathematics and reflecting on their teaching. The authors point out that, even though the teachers showed evidence of their involvement in the

mathematical community, group discussions about the connections between what they experienced in mathematical immersion and teaching were infrequent or lacked depth. The authors also report certain perceived differences in the group's performances according to the facilitators (non-researchers) who intervened in each group. The researchers aim to increase the scale of the study to analyse these problems further.

Horoks et al. (2020) report difficulties associated with school routines and some issues related to collecting data on students' learning. The researchers had planned to evaluate a large number of students. However, due to school organisation issues and the lack of access to digital technologies, this number decreased significantly. This limited their conclusions on the impact of collaborative work on students. Regarding students' learning, it is recognised that some of their developed algebraic skills encounter limitations when operating with complex expressions.

We stress the importance of reporting difficulties, challenges or issues recognised as problematic during collaboration. These difficulties may have been obstacles for those involved, but they are also important inputs for thinking about collaborative practices or for all those who engage in such work. In the next section, we move forward with the analysis of *sustainability*, *dissemination* and *sharing of outcomes.*

3.4.5 The Sustainability, Dissemination and Sharing of Outcomes

In this section, we firstly address the question of the sustainability of outcomes, and then the dissemination and sharing of outcomes. There are two aspects to the sustainability of outcomes of collaboration with teachers. The first is the sustainability of the collaboration itself and the second is the sustainability of any change or learning in relation to teachers' practices, i.e. were changes made in the context of collaboration sustained into long-term changes in the classroom? We will take these aspects in turn.

A significant number of the studies reporting at the ICMI Conference were of collaboration structures that lasted 5 years or more. Most common, of these longer-term projects were structures where the teachers engaged for a year and then a new group of teachers joined (e.g. Koichu et al., 2020). However, some of the research involved even longer-term collaborations. In some cases, this involved a fluid design with some stability of participants and some new members joining (e.g. Hreinsdottir, 2020). In other cases, networks met with the same participants, including one group in Spain who have met for 20 years (Climent et al., 2020). Inevitably, these longer-term collaborations of the same group tend to involve smaller numbers of teachers than those with more fluid memberships or with deliberately changing memberships.

Canavarro and Serrazina (2020) report on a national professional development programme for primary teachers from Portugal, relating to mathematics teaching and discussing and analysing students' mathematical productions. The programme ran

from 2005 to 2011 and, although voluntary, involved over half of the primary teachers in the country, over its lifetime. The programme was dispersed across 18 higher education institutions and involved meetings of teachers in groups of 8–10.

Although not represented in studies for this conference, a programme of professional development on a similar scale has been underway in England for the last 5 years, organised around 42 "Maths Hubs", which are based in schools and funded by the government, and are responsible for offering professional development to their local communities (courses tending to last 1 year). The aim is to create a school-led system for the leadership of professional development (see: https://www.ncetm.org.uk/maths-hubs). A similar aspiration is present in a project in the Virgin Islands (Ekici et al., 2020) that, since 2014, has established yearlong professional learning communities with the aim of developing culturally responsive teaching practices. The work of Dörr and Neves (2020) has sustained itself since 2004, originating from a collaboration between the Brazilian Society of Mathematics Education and associated members in the city of Brasilia. The collaboration appears to be growing in scale.

Three studies in the ICMI 25 programme (Coppé & Roubin, 2020; Masselin et al., 2020; Modeste & Yvain-Prébiski, 2020) related to the French IREMs. IREMs were set up in the 1960s by the French government, to facilitate reform of mathematics teaching via interaction between teachers and academics; involvement of teachers is voluntary and in their free time. IREMs get established in different areas, linked to a university. The work reported at our ICMI 25 Study involved IREMs that had been running for 15 years or more. These groups have an explicit aim of linking theory and practice, and their longevity supports moves towards co-design and co-production of knowledge. They also model the processes of their own collaboration.

The capacity for collaborative groups to emerge into new ways of working appears as significant in several studies, in terms of the sustainability of the co-operation. Coppé and Roubin (2020) discuss three different phases in the evolution of their group, since 2002. An American study (Castro Superfine & Pitvorec, 2020), funded for 5 years, focused on developing practices within teaching algebra and of formative assessment, with 10 teachers. The emergent design of the project allowed a shift in years 4 and 5, towards co-designed collaborations in researcher–teacher dyads.

An unsurprising commonality across many of the collaborations, or collaboration structures, that have lasted 5 years or more, is that they were government initiatives and/or received sustained funding. Another commonality is that several research reports on these long-term collaborations mention the significance of the different levels of interaction facilitated by the work, for instance, between academics and teacher educators (where these are different roles), between academics and teachers, and between teachers and teachers.

Some activities, such as conferences (Hreinsdóttir, 2020) help get new and different actors involved in collaborative processes. Some national contexts, such as government support for professional development, encourage the involvement of

volunteers (Koichu et al., 2020). Sustainability is also developed by taking advantage of working conditions or available resources, but at the same time sustaining a rhythm that adapts conditions and resources to the pace of changes and demands of the context (IREM). Another aspect is that it brings into play flexible and adaptable processes to create, through teamwork, rapid solutions to emerging problems (Horoks et al., 2020).

In terms of our second meaning of sustainability, the extent to which collaborations led to teacher development that sustains in the long-term, there are a diverse set of findings, arising from a range of methodological approaches and a range of conceptualisations of learning and knowledge. For instance, Chen et al. (2020) report on differences in teacher descriptions and interpretations of video-recordings, from having taken part in a Lesson Study group, based on a coding scheme developed by Vrikki et al. (2017). Coppé and Roubin (2020) report on changes in relation to knowledge of specific topics, assessed via pre and post questionnaires. Collura and Di Paola (2020) use interviews to conduct a qualitative analysis of change.

Pacelli et al. (2020) observed changes in teacher beliefs/attitudes, assessed against their novel construct of interpretive knowledge. Eden (2020) researched developments in teacher noticing, both in their observations of video recordings of lessons and also from co-teaching. Jacques and Clark-Wilson (2020) analysed changes in questions or prompts used by teachers in their lessons. Horoks et al. (2020) observed changes in teachers' planning and in their interactions during collaborative meetings. This diversity inevitably makes comparisons across collaborations and collaborative structures next to impossible. Furthermore, the studies reported here analysed change during the process of collaboration, and so we do not have evidence, either way, about the sustainability of changes beyond the collaboration. This is a question relevant for future research.

Two studies (Ekici et al., 2020; Horoks et al., 2020) attempted to track student learning in the classrooms of those teachers involved in collaboration. Horoks et al. aimed to collect data on student knowledge and skills in algebra. Although they reported difficulties with data collection, leading to a small sample, their findings were encouraging. Ekici et al. surveyed student (pre- and post-) attitudes for those students who had taken part in projects linked to their teachers' involvement in professional learning communities and found significant improvements. We suggest further work tracing the impacts of collaboration into the classroom would be highly beneficial to the field.

We now move to consider, more briefly, dissemination and sharing. By far the most common forms of dissemination of outcomes of collaboration were via conferences (for a range of relevant actors) and digital media (websites and resource depositories). However, it seems clear that the main benefit of collaboration is for the collaborators themselves (for instance, in LS collaborative groups taking place in Iran and Brazil). In other words, we surmise that it is the process of reflection, via engaging in collaboration, which is transformative. We suggest, therefore, that a key element of collaboration is the sustainability of the structure within which it takes place (our first element of sustainability, discussed above).

Thus, if the lasting benefits of collaboration are mainly on the actors taking part, then long-term funding and organisation are required such that ultimately all, or a majority of, teachers are able to benefit. The examples from Portugal and England show that, with ambition, it is possible to achieve involvement at a country-wide scale within a relatively short timescale (5 years), through funding of a nationwide network of centres or hubs. There is a need for more research on the long-term impacts of involvement in one-year professional development collaborations.

Returning to our distinction at the beginning of this section, there is clearly a compromise to be made between long-term collaborations involving a few teachers and an iterative design that might involve teachers for only 1 year or less, but then potentially reaches more people. It appears there is little evidence which might, at present, give us confidence as to what length of collaboration, in different contexts, might be needed in order to provoke teacher learning that sustains beyond the time of the collaboration.

3.4.6 Summary and Comments

In general, and independently of the model of collaborative work that is adopted, most studies on papers reported for Theme B take, as a central core, different practices that involve future teachers or in-service teachers. In this sense, some of these results report on classroom practices, but, at the same time, given the complex and collective aspect of collaboration, the results expand to other practices such as research or curriculum development. Of course, the type of results reported acquire particularities according to the purposes of the collaborative project, the scale of these and the contexts that host them.

In Sect. 3.4.2, we considered collectively developed products, as situated results of the collaboration. In Table 3.1, presented in Sect. 3.4.2, the main material or intellectual products are visualised as results of different collaborative work distributed by levels of incidence. It is at the micro level (e.g. lesson plans, task sequences) where there is the largest quantity of products linked to teachers' school practice. The expansion of results outside a micro didactical level shows not only the influence of the aims and focus of the collaborative work, but also the expansion of teachers' work beyond face-to-face activity with their students in the classroom, which implies interactions not only with other teachers, but also with curriculum developers, facilitators or researchers. For example, the second row shows results of the work that impacts on curriculum, while the third row shows results of special interest for researchers.

It is worth noting that, in models of a systemic or structured nature, such as the Japanese LS or the work at French IREMs, the phenomenon of expansion of results to all three levels (micro, meso, macro), in one way or another, forms part of the purposes of such collaborative projects. Similarly, those models of blended collaborative work focus on products linked to websites or other types of material, for example, but it is important to indicate that this is not unique to those models.

What is reported in Sect. 3.4.3 is a range of outcomes related to learning, professional development, knowledge, beliefs and practices. We note that what is reported as an outcome is inevitably coloured by the particular model of collaboration and theoretical perspective of the author. For example, Kooloos et al. (2020) use the notion of boundary objects and, hence, report on the spheres of practice of participants. Horoks et al. (2020) are influenced by the Theory of Didactical Situations, which inevitably influences what is offered to teachers in collaboration and, hence, the outcomes of their work. As well as seeing strong connections between the type of products developed, as a result of the form of collaboration, the feelings of togetherness generated by involvement in a group over time, which has a common focus, appears in several reports. Sections 3.4.2 and 3.4.3 report successful results according to the objectives set for the project and certain variability in results depending on the collaborative work models chosen.

On the other hand, Sect. 3.4.4 presents certain results perceived as unsuccessful by the authors or as emerging results of the work that, in one way or another, hindered the full achievement of the proposed objectives. One such obstacle, which often hinders collaborative work, has to do with certain changes in institutional routines that could not be fully foreseen. Perhaps the most frequent of these are changes in teachers' obligations, activities or times according to emerging institutional requirements during the collaboration. This fact became more evident in those models that involve significant immersion in school institutions, such as the LéA's collaborative working model or those application of the LS model in new contexts.

However, in relation to the application of LS, it should be noted that, when comparing two cases of its application, carried out in the south of Brazil in similar socio-cultural contexts, the results were quite different. In one case, particularly good results were obtained and in another not. When comparing both cases and without wishing to detract from any work, we found two different aspects of the two works that seem important to highlight. One of them is linked to the ways of accessing or approaching school spaces (i.e. whether teachers were volunteers or not) and the other to the expertise of those who intervened as facilitators.

For more details on the first point, see what was discussed in Sect. 3.3.4. Regarding the effect of the role of the facilitator in collaborative work, in another investigation, not within a LS model but in a blended model, certain differences in results depending on the facilitator are also pointed out. This issue of the role of the facilitator in different models deserves more careful and extensive studies, an issue addressed in Chap. 4 (Theme C, this volume).

By way of closing, we wish to highlight and celebrate the large number of long-term collaborations taking place across the world. The model of Japanese LS is one that has been disseminated widely. Nationwide programmes (such as those described above, taking place in Portugal and England) appear to be based on the idea that collaborations are not so much for the purpose of achieving results and outcomes that can then be disseminated outside the collaboration, but rather that the collaborations themselves are the most relevant outcomes. In other words, it is the act of engaging in a collaboration that, time and again, appears to be transformative.

3.5 Mathematics Content and Context for the Collaborative Activity or the Case of a Small-Scale Frame for Collaboration

3.5.1 Introduction

Hiebert and Wearne (1993) proposed that, "what students learn is largely defined by the tasks they are given" (p. 395), and researchers have taken this to apply to pre-service and in-service teachers as well, including in relation to how mathematical tasks are presented, developed, engaged with and resolved (Watson & Mason, 2007). Due to the critical importance of mathematical tasks on student learning and teacher learning, great efforts have been made to explore the nature and roles of mathematical tasks (*Journal of Mathematics Teacher Education*, *10*(4–6) Special Issue edited by Jaworski, 2007), principles and strategies of task design (ICMI Study 22 volume, edited by Watson & Ohtani, 2015), and teachers as partners in task design (*Journal of Mathematics Teacher Education*, *19*(2–3), Special Issue, edited by Jones & Pepin, 2016).

Within the context of teacher PD, Prediger et al. (2019) and Prediger (2020) described a content-specific theory for explaining and enhancing teacher learning in collaborative settings. According to Prediger, two levels of classroom tetrahedron (students, mathematics content, classroom resources and teacher) and PD tetrahedron (teacher, PD content, PD resources and PD facilitator) are interconnected through lifting or nesting. The classroom tetrahedron forms the PD content which must be nested in PD tetrahedron. Mathematics-related content in PD (such as mathematical tasks, student learning artifacts, and teaching artifacts (lesson plan, videotaped lesson) are critically important for what teachers experience, do and learn. By focusing on mathematics content and teacher collaborative learning, we analyse and organise this section into following three aspects: nature and roles of mathematics-related content; design of mathematics-related content; interacting with mathematics-related content and teacher learning.[11]

3.5.2 Nature and Role of Mathematics-Related Content

Building on, and taking a broad view of, the notion of tasks, mathematics-related content was central to the PD activity of many, but not all, of the collaborative groups described in the studies. The mathematics-related content of teachers' collaborative interactions varied in relation to the specificity of the mathematics focus and the nature of teachers' mathematical knowledge to be developed. In relation to

[11] During the work in Lisbon, Raewyn Eden, Freyja Hreinsdóttir, Chris Plyley, Liliana Suárez Téllez and Rachel Zaks contributed for selecting some of the themes considered in this section.

the mathematical content collaborative groups engaged with, a range of learning and teaching objects and processes were of interest including mathematical tasks and problems; mathematical practices such as discourse, modelling and argumentation; the mathematical thinking of learners such as errors, misconceptions and non-standard solutions.

In several studies, the work of the group was not specific to mathematics, and in others the collaborative activity was centered on mathematics generally. For instance, Dörr and Neves (2020) describe a project aimed at promoting interest and engagement in mathematics for public school students in Brazil, through the design and execution of workshops that present mathematics in playful, practical and creative ways. In contrast, other studies focused on specific mathematical domains including algebra and algebraic thinking, probability, measurement, trigonometry, and number and operations. Associated mathematical concepts of interest in the studies included place value, fractions and units of rate and measurement.

The knowledge required for teaching mathematics well is substantial, complex, and evolving (Ball et al., 2008). Building on Ball and colleagues' model of Mathematical Knowledge for Teaching (MKT), Carrillo-Yañez et al. (2018) propose a model of Mathematics Teachers' *Specialised* Knowledge (MTSK), in which teachers' beliefs are central and there is an emphasis on the specialised nature of the knowledge used in teaching mathematics. Drawing on the breadth of mathematical knowledge for teaching, the mathematics focus of some studies included teachers' knowledge of mathematics curricula such as knowledge of standards, content such as knowledge of domains of mathematics including algebra and geometry, and mathematical practices and processes, such as modelling.

Several studies focused on teachers' understanding of the mathematical content including both specialised content knowledge, "the mathematical knowledge and skill unique to teaching" (Ball et al., 2008, p. 400); and horizon content knowledge, "an awareness of how mathematical topics are related over the span of mathematics included in the curriculum" (p. 403). Other studies centred on how knowledge is used in mathematics teaching including teachers' interpretive knowledge involved in making sense of student productions and responding in ways that promote mathematical understanding (Ribeiro et al., 2013).

Making connections between specialised and horizon content knowledge, one study centred on teachers' understanding of learning trajectories for identified aspects of mathematical content; that is "how student understanding develops along the conceptual strands of a discipline" (Castro Superfine & Pitvorec, 2020, p. 255). Arguing that teachers require mathematics knowledge that is both broad and deep, in order to make sense of students' reasoning and anticipate and respond to common obstacles to understanding, Castro Superfine and Pitvorec's study involved teachers and researchers co-constructing a micro learning trajectory for unit rate which was identified as a key aspect of mathematics curriculum content. They found that as they co-constructed, used and reflected on micro-trajectories, teachers were able to deepen their knowledge of mathematics and student learning in the context of their classroom practice, thus expanding the possibilities for responding productively to students' mathematical thinking.

Similarly, Pacelli et al.'s (2020) study is premised on the assumption that teachers require deep and broad knowledge of mathematics, in order to respond to students' mathematical reasoning and productions and support their developing mathematical understandings. Conceptualising such knowledge as interpretive knowledge (IK), the authors argue that collaborative discussion supported prospective teachers to develop novel insights into, and new attitudes towards, students' productions. The study involved the use of interpretive tasks whereby prospective teachers solved mathematics problems and then interpreted a range of students' productions pertaining to the same task. The authors were particularly interested in how prospective teachers could support students to build understanding from their own mathematical productions, including in relation to those students' non-standard, ambiguous and erroneous responses. The authors found that collaborative discussion supported shifts for prospective teachers from evaluative towards more interpretative reasoning about student productions.

Expanding the focus from mathematics alone, interdisciplinary knowledge was highlighted in several studies. For instance, Ekici et al. (2020) reported on a project involving science and mathematics teachers and practitioners collaborating to model locally relevant STEM problems in classrooms as a catalyst to promote interest in these disciplines amongst students and teachers. A continuous cycle of collaborative inquiry involving careful support and planning supported teachers to make culturally responsive adjustments to their practice and fostered students' understanding of the interconnectedness of science and mathematics. Mathematical and scientific modelling of local community issues promoted increased student agency as they started to develop enhanced STEM identities, posed more STEM questions and were more interested in developing mathematical and scientific arguments.

Such interdisciplinary context allowed teachers to expand their disciplinary perspectives and provide more equitable access to mathematics and science experiences for previously underserved students. In her study, Acevedo-Rincón (2020) found that collaborating with peers across disciplinary boundaries expanded the perspectives from which prospective mathematics teachers could understand their practice, their role and the mathematics they were teaching. In both studies, interdisciplinary collaboration afforded teachers of mathematics opportunities to engage with different perspectives and, as such, served as both the context and the catalyst for change. For instance, Ekici et al. (2020) found that the interdisciplinary context provided a pathway for teachers to develop different disciplinary perspectives along the course of their professional lives; a necessity when working in isolated communities. Similarly, Acevedo-Rincón found that engaging within and across interdisciplinary contexts of teaching and learning helped prospective teachers to problematise their professional practices and develop new meanings for the contexts, knowledges and processes of school mathematics, whereby engaging across difference brought new meanings to light.

Developing teachers' pedagogical content knowledge of mathematics and their associated practices was the focus of many of the studies. In his seminal article, Shulman (1986) argued that pedagogical content knowledge (PCK) lies at the intersection of pedagogical and content knowledge—where teachers put their

content knowledge to use in their instructional practice—is key to understanding teachers' knowledge. Studies focused on teachers' pedagogical knowledge of mathematics included for instance teachers' use of "talk moves", formative assessment, mathematics as a focus of teacher inquiry to address student learning needs, and the selection and design of mathematics tasks and task sequences.

A number of studies were centred on teachers' knowledge of students including noticing, interpreting and understanding students' mathematical thinking. For instance, Pacelli et al. (2020) found that engaging prospective teachers in the collaborative analysis of students' productions stimulated deeper reflection and supported norms of mathematical/pedagogical reasoning that were increasingly interpretive.

Teachers' knowledge is relational and mediated by affective factors (Bobis et al., 2012) and teacher collaboration is both supported and constrained by emotions (Brodie, 2020). Brodie suggests that "all learning involves emotion" (p. 40) and emotion can be seen as both a tool for, and an object of, teachers' collaborative activity. The role of emotion in teachers' collaboration is highlighted in several studies, including those focused on teachers' self-knowledge and beliefs. For instance, Trevisan and Elias (2020) in the context of their study aimed at disrupting conservative classroom practices reported one of the teachers in their study as saying, "I felt a little embarrassed. [. . .] Because I do not have mathematical training" and another as reflecting "the exchange of experiences with the other teachers [. . .] helps us to see a way, to feel a little more confident of our practice or even to speak" (p. 439).

In professional development contexts where the focus is on individual teachers, a tension can emerge whereby exposing classroom challenges can direct attention to the quality of the individual teacher's practice and, thus, make the teacher vulnerable to the risk of negative critique. In contrast, in the context of primary and secondary mathematics teachers' collaborating to solve, discuss, implement and reflect on classroom problems, teachers described a shift from negative emotions, such as fear of exposure, to feeling valued and confident, as they jointly reflected on and learned from practice dilemmas (Soto et al., 2020). One of the study's participants, for instance, was described as feeling "valued by the community [. . .] encouraging her more and more to present proposals to their peers to work collaboratively. Her enthusiasm had a multiplier effect" (p. 423). Such shifts illustrate the dual role of trust as both a requirement for, and a product of, teacher collaboration (Eden, 2018), whereby mutual trust is an important element in any such activity.

The notion of teachers' knowledge of themselves, of the epistemologies they bring to their mathematics teaching, might be thought of as meta-knowledge. That is teachers' knowledge of their thinking, beliefs, assumptions and emotions about mathematics teaching and learning can be viewed as an overarching conception of teachers' knowledge.

Together, the studies illustrate the complexity and scope of mathematics as an object of mathematics teachers' professional development through collaborative activity, and suggest that different components of knowledge for mathematics teaching may inform one another in complex ways. In particular, teachers'

collaborative professional development activity focused on specific mathematics-related content might both provide pedagogical tools to expand the possibilities for classroom practice and "influence the collective domain, namely the shared pedagogical content focus on further basic conceptual needs in different mathematical content" as suggested by Prediger (2020, p. 11). The section that follows elaborates on the purpose, selection and design of the specific mathematics-related content that was the focus of the collaborative activity reported.

3.5.3 Selection, Design and Use of Mathematical Content/Tasks

In accordance with the theory elements identified by Prediger (2020), mathematics can be viewed as PD content and/or PD resources at the level of teacher PD, as well as mathematical content at the level of the classroom. Tools that mediate the work and learning of collaborative groups of teachers of mathematics are discussed in Chap. 5 (Theme D, this volume); of interest here are the mathematical content and processes that characterise the collaborative activity of different groups and the ways in which these were selected and designed.

We return first to the importance of mathematical tasks for the professional development of teachers of mathematics, as established at the beginning of this section. A number of studies focused on mathematical tasks and problems, particularly those for use in the classroom, and several studies centred on the selection, design and/or development of tasks for the teaching of mathematics, including learning tasks and diagnostic tools. Of particular interest in several studies focusing on task design and development was the organisation of mathematical content so that it is feasible to be taught. One such study (Asgari et al., 2020), exploring how collaborative curriculum design impacted upon teachers' knowledge of, and attitudes towards, teaching mathematics, found that teachers' knowledge of how algebra is sequenced and organised for teaching, and their trust in the efficacy of the curriculum, were both enhanced.

Soto et al. (2020) describe how the design and use of mathematical tasks for classroom instruction were the focus for the collaborative activity of a community of in-service mathematics teachers from primary and secondary schools, and university-based mathematics educators. The purpose of the collaborative activity was to bridge students' transition from primary- to secondary-level mathematics while deepening teachers' mathematical knowledge for teaching. The design of mathematical tasks was central to an iterative cycle of solving problems together, planning for implementation, using the tasks in classrooms and collectively reflecting on the enactment. The mathematics tasks were thus central to understanding the mathematical transition from primary to secondary school through an iterative process of collective design, enactment and reflection.

Other studies focused on mathematical processes such as modelling, argumentation and simulation (e.g. Ekici et al., 2020; Masselin et al., 2020; Modeste & Yvain-Prébiski, 2020) and practices such as classroom discourse (e.g. Kooloos et al., 2020). The study by Koichu et al. (2020) reports on teachers' co-learning inquiry focused on questions posed by each of four teacher-research communities. One group's focus was on thinking flexibility, and another was on the design and enactment of talking tasks, described as involving opportunities for problem solving, discussion and reflection. Professional development was often framed as addressing weaknesses or gaps in teachers' knowledge, and this was the aim of a number of studies.

For instance, one study examined teachers' professional development related to the role and use of simulation in the teaching of probability and statistics, following its introduction into the French curriculum in 2010 (Masselin et al., 2020). The authors found that a teacher's choice of mathematical objects influenced the mathematical activity and subsequent learning opportunities at the professional development and classroom levels.

Decisions about the focus of the collaborative activity were variously made by teachers or researchers, or by the mutual agreement of both whereby members of the group decided together what to discuss and reflect upon. In a number of studies, the mathematical focus was chosen for its authenticity, and for drawing connections between mathematics and learners' lives. Ekici et al. (2020) report on one such study, in which multidisciplinary groups worked with the community to identify and investigate a locally relevant STEM-focused problem—in this case, water quality—and worked to develop culturally responsive practices aimed at providing equitable access to quality STEM instruction for all learners. In a study by Trevisan and Elias (2020) involving LS, choosing mathematical content in the form of classroom tasks that were close to teachers' current practice appeared to be one of the factors that mitigated resistance and promoted openness to changes in practice.

3.5.4 Interacting with Mathematics-Related Tasks and Teacher Learning

The nature of teachers' interactions with mathematics in the context of their collaborative activity tended to centre on three interrelated dimensions: doing mathematics; talking about mathematics; investigating representations of mathematics teaching and learning. Engaging in *doing* mathematics highlights the active and social nature of mathematics, whereby groups with a shared interest generate problem solutions using mathematics, including experiencing tasks and associated practices as a rehearsal for subsequent classroom use.

For example, in Modeste and Yvain-Prébiski's (2020) study, the researchers proposed to the group an authentic problem involving mathematical modelling such as predicting tree growth or optimising the positioning of a warehouse. The elaboration of the problem was based on characteristics that came from research

results and provided indicators to develop a situation favouring modelling activity in the classrooms. From there, all members of the group collaborated to develop a problem suitable for a modelling activity in the classroom from sixth to twelfth grade. The group designed a first statement and *a priori* analysis; that is, they anticipated student responses to the task prior to using it with their students.

Prior to the group's next meeting, teachers experimented with this problem in their classrooms to contribute to the next iteration of the group's reflections in the development of the problem; a process involving several meetings to reach the final version of the problem. The mathematical content appeared to mobilise the whole process of collaboration that later materialised in the final version of the problem. The problem, then, can be considered a collective product of the group's collaborative activity whereby mathematical objects can be seen as both a vehicle for and the product of teachers' collaboration.

Many studies described collaborative contexts that involved group members' active *discussion about and listening* to mathematics, including raising and answering questions related to mathematical concepts, subjects, processes and procedures in connection with classroom work, sometimes in collaboration with expert partners. In many studies, conversations about mathematics involved groups' collective reflection on classroom events including stories, observations and experiences of individual teachers' classroom practice (e.g. Rafiepour, 2020), the shared practice of co-teaching pairs (e.g. Eden, 2020), and on representations of practice including the analysis of videos of mathematics lessons (e.g. Castro Superfine & Pitvorec, 2020).

The focus of such reflections was the collaborative review of teaching practices related to the specified mathematics content to promote practice improvements: for instance, in Trevisan and Elias' (2020) study focused on the challenges inherent in the reflective and collaborative process of implementing lesson study. In such studies, mathematical discourse can be seen as linking the collaborative activity and the development of mathematical knowledge for teaching. For instance, Eden (2020) found shared planning appeared to set teachers up to be responsive during a co-taught lesson, whereby anticipating how students might respond to a mathematics task supported enhanced noticing. As co-teaching pairs acted to make sense of and reconcile contradictions between what was anticipated and what actually occurred in the lesson, this in turn served to expand the breadth and depth of their reflective thinking after the lesson.

In Chen et al.'s (2020) study, they explored how discourse threads during teachers' collaborative discussion about a video of teaching a geometrical topic may be related to their MKT growth. As such, talking about mathematical content at the classroom level, including related teacher actions and student impacts, serves as both the means of accessing the classroom level as PD content (PD1) and the mechanism for teachers' professional growth (PD2) as described by Prediger (2020). That is, as teachers engage in collaborative discourse about the mathematics of the classroom, they are engaging in sense-making about the mathematics itself, alongside the co-generation of theory about what it means to know, do, teach and/or learn mathematics. Talking about mathematics thus positions mathematical content

simultaneously as a resource or a mechanism for teacher development and the object of that development.

A number of the studies centred on collaborative groups *investigating* authentic problems of practice through collaborative inquiry. An approach common to a number of studies involved collaborative groups developing *research lessons* to address particular aspects of mathematics content including the design, teaching, reflection, revision and re-teaching of mathematics lessons. For instance, Climent et al.'s (2020) study involves two-year projects, each focused on the teaching and learning of an aspect of mathematical process or content, such as problem solving, problem posing or fractions. A group of primary teachers and mathematics education researchers read about and discuss content, and then plan lessons and implement them. The collaboration is centred on authentic problem solving involving reflection on practice as a mechanism for promoting professional development and knowledge generation as connected and reciprocal outcomes.

Again, discourse was a critical element in the professional development process, whereby engagement in reflective discussion was a central mechanism through which teachers could interrogate teaching decisions and their impacts. Through a process of shared critical reflection, participants were able to identify and interrogate the importance of the specialised knowledge of mathematical content they brought to bear in their classroom practice, where previously the teachers had been reluctant to engage with conversations about their own mathematical knowledge. It seems that the collaborative work space created an atmosphere of trust, so that the teachers were willing to communicate ideas about their own practices when teaching fractions.

The collaborative activity of many groups was directed at outcomes related to students' mathematical learning whereby activity at the classroom level is nested within the PD level. For instance, some studies focused on groups collectively interpreting students' productions when solving mathematics problems (e.g. Pacelli et al., 2020). In many cases, mathematics content was central to all aspects of the collaborative process, including teachers developing understanding of mathematical content, anticipating student thinking in relation to that content and designing classroom tasks to engage students with the content.

For instance, Castro Superfine and Pitvorec (2020) report on a study involving teachers and researchers working to co-construct professional learning experiences aimed at developing understandings of learning trajectories in algebra. Reflection on video of classroom discussions that elicited student understandings of algebra engaged teachers in discussion of, and deepened their understandings of, algebra learning trajectories, and expanded the possibilities for their responses to students' thinking in algebra.

Although mathematics learning goals for students were implied rather than specifically addressed in some of the studies, some had an explicit focus on mathematics goals at the classroom level. In a study involving groups of teachers working across disciplines, Collura and Di Paola (2020) found that co-planning and co-teaching mathematics lessons contributed to an expanded view of the nature of mathematics, whereby students saw it "no longer as a discipline in itself, detached from reality and written in a language incomprehensible to many, but as a discipline

that has evolved with others, in history and time, which finds applications in various branches of knowledge" (p. 283).

The connection between mathematics learning at the classroom level and content at the PD level is made explicit in studies involving teachers in inquiry processes, whereby classroom level student learning data informs activity aimed at improving mathematics teaching and learning. In one such study, Ell (2020) found that participating in PD activity aimed at understanding and implementing spirals of inquiry promoted "teacher learning about mathematics concepts, student learning in mathematics, assessment and mathematics teaching" (p. 322), where the chosen inquiry focus was mathematics teaching and learning. In particular, engagement with explicit evidence of students' mathematical thinking was central to the collaborative process, supporting the generation of trust and "sparking curiosity and raising questions" (p. 323) within the inquiry groups.

3.5.5 Summary and Comments: Mathematics Content Is a Mediating Factor Influencing the Nature and Effects of Collaboration

Across the papers reviewed in the preceding sections, mathematical objects (or content) appear both to catalyse and to support professional change by acting as a source of difference. As groups engage in collaborative discussion about mathematical objects, such as classroom tasks, student productions or mathematical processes, those objects simultaneously serve as resources for and objects of the teachers' learning and development. Through teachers' collaborative activity, professional learning and development are catalysed, and new understandings and practices are forged by negotiating across differences. That is, as mathematical content is introduced into, or shaped by, the collaborative activity, new mathematical concepts, tasks and processes serve as contradictions to the status quo and, as Katz and Dack (2014) argue, paying attention to that which interrupts the status quo "holds the potential to yield new professional learning" (p. 36).

Participation in dialogue about mathematical objects simultaneously promotes expanded opportunities for the teachers to act towards the dual objects of their own and their students' learning goals. As teachers consider the mathematics teaching and learning of their students, they also expand their own understandings and practices. Consequently, in the context of collective (or collaborative) interactions, each participant reviews his or her own knowledge and perspectives, and contributes to the co-generation of new shared understandings.

Thus, talking about mathematics positions mathematical content as both a resource or mechanism for teacher development and the object of that development whereby differences act as reflexive objects and catalysts for the on-going transformation of teachers' understandings and work. Making differences in practice visible in the context of teachers collaborative activity appears to open opportunities to learn

about and from those differences (Tobin, 2014). However, more empirical studies on the mechanisms of how mathematics content mediates the teacher learning outcomes are needed.

3.6 Conclusions

In this final section, the progress of the work on Theme B (Sect. 3.6.1) is summarised. Then the answers to the questions that guided the work are presented (Sect. 3.6.2). Finally, further research directions are discussed (Sect. 3.6.3).

3.6.1 A Summary

Throughout this chapter, the main goals, forms and achieved outcomes related to collaborative settings between teachers and others were analysed, synthesised and presented. In all cases, the socio-cultural contexts in which collaboration resided were highlighted. In the analysed studies, there was diversity in goals, design and implementation of settings and outcomes. In this diversity, mathematical knowledge, as part of the collaborative content, emerged as a substantial element for promoting collaborative work.

The studies focused their collaboration on one or more of the following levels: the micro didactic level of the classroom, the meso institutional level or the macro level of the educational system. Accordingly, the outcomes achieved contributed especially to one of these levels. Certain unexpected outcomes emerged while other results may provide input for future studies. Although most of the studies included in-service teachers and researchers, there were also cases in which prospective teachers participated.

3.6.2 Answers to Five Interconnected Questions

From Sects. 3.2, 3.3, 3.4, and 3.5, we present results of a systematic analysis of the studies that were presented and discussed in Theme B to respond to the following five questions:

1. What models of teacher collaboration have been developed? What are the design features, goals and outcomes of the different models?
2. How effective are various models for promoting different outcomes?
3. Which forms of collaboration are appropriate in different contexts?
4. What are the affordances and limitations of each form of teacher collaboration?

5. What are the benefits and the challenges that online teacher collaboration poses to the teachers?

From the analysed sources, we can offer proper answers to the first four questions. Regarding question 5, since only three of the 28 papers, linked to Theme B, presented case studies of collaborations developed in blended format, it is hard to offer an accurate answer to it. But we will provide a brief discussion of relevant issues related to this kind of form. Given the interconnected nature of the questions, we clustered questions 1 and 4, questions 2 and 3, and question 5, and answered each group of questions separately. However, we recognised these questions are strongly interconnected.

3.6.2.1 Forms of Teachers' Collaborative Work: Affordances and Limitations

Drawing on Sects. 3.2 and 3.3, we present our answers to the following questions:

What models of teacher collaboration have been developed?
What are the design features, goals and outcomes of the different models?
What are the affordances and limitations of each form of teacher collaboration?

Although the collaboratives work analysed differed in their forms, the majority of them (25 out of 28) chose to organise their work in a face-to-face format. However, in a few cases, shared work via Google Drive or e-mail were reported (e.g. Richit & Tomkelski, 2020). Of the three cases that adopted blended formats, in two of them, perhaps due to the wide geographical dispersion of the participants, there was a prevalence of general meetings in virtual mode. However, local face-to-face meetings were also held.

Among the *face-to-face* forms, we identify the following forms: *Adaptations of lesson studies*, *researchers–teachers partnerships*, *networks* and *forms connected to specific purposes.* Depending on the participants involved in the joint work, four types of communities have been identified: *Inquiry communities*, *collaborative groups*, *borderline communities* and *interdisciplinary communities*. The first two communities are mainly linked to the first two forms. Borderline communities can be related to a network form. Both the interdisciplinary and inquiry communities are linked to forms connected to specific purposes. We note that the collaborations related to this last form, are mainly those that emerge from an initiative associated with graduate research projects (mainly Ph.D. theses), courses for future teachers or the development of specific materials.

The goals for collaboration are related to the three levels of joint work. At the educational system level, national or multiple-site teacher collaboration programs are typically developed to address national needs and concerns (curriculum reforms and innovative initiatives). At the institutional level, the major concerns for the collaborative work are about developing teachers' MKT or interdisciplinary knowledge, changing teaching practice, building partnerships between teachers and

researchers to promote teachers' growth, and linking research to classroom practice. At the classroom level, the teachers' collaboration mainly focuses on implementing specific teaching strategies and/or research-based, reform-oriented teaching practices.

Regarding scale in relation to duration and size, five studies were large-scale in terms of numbers of participants (teachers, researchers, facilitators, schools or other institutions) and lasted over a long period. Among the five projects, three were of a blended form, while the other two were a face-to-face form. If we consider duration only, many of the works provide long-term collaborative experiences (five or more years). However, most of the long-term collaborations involved issues related to changes in the number (and type) of participants, focus and objectives. Some of the issues are caused by changes in economic, institutional or environmental conditions that afford collaborative work over time, while others are related to unpredictable emerging factors.

All reported forms of collaboration involve in-service teachers or prospective teachers (from one or several schools and/or from one or several teaching levels), researchers/facilitators, facilitators (non-researchers), various institutions and/or agencies. In the case of forms related to interdisciplinary communities, teachers or researchers from areas of knowledge other than mathematics or mathematics education also participate. In one way or another, they become the stakeholders for the collaboration and the corresponding settings. Each form, according to its characteristics, objectives and the activity that is developed will shape its settings. The following provides, for each form a brief summary of its settings, affordances and limitations.

The *LS-adaptative form* could be school-based or district-based, depending on the context and needs. These adaptations include the typical LS cyclical process by considering the local social and school contexts, in order to design and implement the corresponding research lessons. The affordance of this adapted LS depends on certain constraints of the local context (for instance, schools' characteristics, teachers' working environments and educational systems) and/or the expertise of the researchers–facilitators involved.

The *researchers–teachers partnership form* usually includes heterogeneous groups of teachers who may work at different educational levels and at different schools. The joint work is mainly related to processes of collective study of various topics (such as the teaching and learning of certain areas or topics of mathematics, the production and dissemination of resources for the classroom among others). The corresponding settings, although variable, generally involve periodic meetings, including instances of production of educational materials and reflection on what has been worked upon.

In some cases, enactment and class observation are also included. We note that IREM and LéA partnerships have shown a great affordance over time. A great number of the papers presented for Theme B relate to this form. Some of them show limitations with regards to sustainability of the continued participation of teachers. In the case of IREMs, teachers' participation is not sufficiently recognised as part of their regular work. Therefore, for teachers, participation in IREM may

require extra time outside of school. This fact could become an obstacle to sustaining their participation in the joint work.

Considering the *network* form, it is highlighted that, despite differences in the work and settings of networks, they also have certain similarities, in terms of the complex network of participants or communities that support the collaboration. The communities that interact to collaborate involve diverse types of participants (e.g. teachers from more than one school, researchers, facilitators, agencies) and even, in some cases, the number of them. Different groups of stakeholders choose diverse settings (e.g. collaboration in schools or out of schools, learning communities, communities of practice, inquiry communities, joint work within a community and between communities) and foci for their work (applying innovative teaching materials, developing, and enacting large-scale and long-term collaborations).

In all those cases, there was evidence of the affordances of the settings chosen to carry out small- or large-scale, as well as short- or long-term, collaborations. For those large-scale and long-term collaborations, major limitations are related to processes of compilation and analysis of the information (or resources) generated. Working in a long-term network model could require careful scheduling for organising interactions among the members, as well as for enacting the joint work. However, collaborative work through those forms could have favourable implications for the micro, meso and macro levels.

In the case of *forms connected to specific purposes*, the communities are small-size, and the collaboration usually lasts for a short time. The foci are diverse and graduate-project-related. Usually, the form of the setting is connected to a community of inquiry or an interdisciplinary community. In some cases, these forms include co-planning of teaching activities, enacting lessons, co-teaching, collective reflection and interpretative activities. These forms are valuable in terms of the affordances of collaboration, knowledge and/or resource production, and teacher and student learning. When the project is related to future teachers, this model shows the capacity to promote early collaborative work.

When considering *adaptation of LS* models, scalability and sustainability are crucial, while cultural, institutional and leadership factors have an influence on them. For the partnership model, sustainability is crucial, and institutional and leadership factors are important. Some unique effects of using this model could be its capacity for connecting research and practice, as well as developing knowledge which has theoretical and practical implications. Regarding network models, they could be valuable to document what participants learned, as well as the research methodology applied, and to share what was produced.

Each of the forms presented has shown the ability to promote certain affordances or to have some constraints. In any case, each form is promoted and sustained within a particular social, educational, or cultural context (Lave, 1988). In that sense, one similarity across many of the collaborations, or collaboration structures that have lasted 5 years or more, is that they were government initiatives and/or received sustained funding: that is, they are institutionalised models (Hargreaves & Fink, 2003). Examples of this case are the work developed in the IREM or LéA or even the

Asian LS. All of them show important aspects of sustainability and sharing of outcomes to many teachers, researchers or government agencies.

At the same time, we find that several of the long-term collaborations inform processes of changes over time for adapting their work to new emerging conditions. According to Davis and Sengupta (2020), adaptability and the occurrence of emerging events are linked to the complexity of a phenomenon. In this case, we could say that forms for the collaborative work of teachers are connected to a complex phenomenon characterised as being multi-dimensional (multi-level), adaptable, context-sensitive and to have capacity for coping with new, emerging conditions.

Given the particularity and the few cases presented on blended models, next we will focus on their affordances, limitations, benefits and challenges.

3.6.2.2 Blended Form: Affordances, Limitation, Benefits and Challenges

Below, we present partial response to question 5:

What are the benefits and the challenges that online teacher collaboration poses to the teachers?

The three studies that report blended learning are characterised by working in the form of a complex network. They illustrate potential benefits for the collaborative work among teachers such as removing the obstacles of geographical distances and sharing perspectives, resources or documents asynchronously. The potential of this form to sustain and expand collaborative work is also promising. However, the challenges of accessing appropriate technologies, sharing documents synchronously, facilitating productive online discussions and, in some cases, overcoming language barriers (as is the case for the SRPM or the NGGN networks) need to be addressed. These three studies provide valuable insight into development of blended PD, which emphasise connections between online PD and teachers' classroom practices.

However, it seems to be difficult for documenting how the online immersion experience impacts teachers' practices. Previous research suggests that the following factors are crucial for developing a productive online community of practice: trustful relationships among the members; highly qualified mediators for cultivating knowledge generation; ensuring the contents are closely related to teachers' practices (Lantz-Andersson et al., 2018). At the same time, we emphasise that teacher collaboration in online communities of practice has become a popular and necessary approach due to the COVID-19 pandemic (Ferdig et al., 2020). Thus, being able to advance with studies on collaboration in such formats seems to be a need to be covered. For instance, more studies are needed to explore strategies for developing effective, online teacher communities of practice.

3.6.2.3 Models, Outcomes, Content, and Contexts

Next, we provide responses to the following questions:

How effective are various models for promoting different outcomes?
Which forms of collaboration are appropriate in different contexts?

As reported in the previous sections and synthesised in Sects. 3.4 and 3.5, several papers presented for Theme B framed their outcomes on the teaching dimension related to student activities in the classroom, but also on other dimensions, such as those highlighted by Jaworski et al. (2017). Therefore, the outcomes of collaboration are not only dependent on the chosen forms, but also on other factors, such as the purposes and contextual conditions of the work. At the same time, given the complex and collective aspect of collaborative work, as well as the types of communities involved in the collaboration, the outcomes are extended to other practices. These include research practices linked to the production of diverse resources, and educational practices related to teaching disciplines outside of mathematics, as shown in Table 3.1.

For instance, when considering the application of the adaptive LS form at the micro level of the classroom, we argue that they have a great capacity to promote deep teacher learning and produce useful instructional products. However, these results are not independent of contextual aspects linked to the educational institutions or the experience of the facilitators. While considering the possibility of working simultaneously with heterogeneous groups of teachers, the category researchers–teachers partnership has the capability to produce outcomes related to partners' reflections or learning, to the resources for different school levels or for different mathematical content to be taught. In this case, the outcomes can inform aspects connected to the micro level or to the interactions between different levels of the school educational system. Institutionalised researchers–teachers partnership shows affordances for expanding the outcomes to all three levels (micro, meso, macro).

This form also has affordances for supporting the sustainability of collaborations. Meanwhile, the network form has some possibilities for promoting outcomes related to the micro, meso, macro or all three levels. It depends on the project and the type of the community established. In that sense, it has possibilities to yield similar outcomes to the previous model. However, the network model has other capabilities. Among them, we can mention the affordances for yielding outcomes related to the impact of long-term and large-scale collaborative projects on teacher training or on the educational system. It also has the potential to produce outcomes related to the particularities that teacher training requires when working with interdisciplinary projects. Under this last condition, the model also has the potential to promote outcomes related to the impact that the work of teachers and students may have in the social or environmental contexts close to their schools.

Across different models of collaboration, mathematical content catalyses participants' professional learning. Through working on challenging mathematical objects

(both content- and pedagogy-related) and reflecting on teachers' existing knowledge and teaching practice, new understanding of MKT and teaching practice may occur. In the context of collective (or collaborative) interactions, each participant reviews his or her own knowledge and perspectives, and contributes to the co-generation of new shared understandings. Meanwhile, through teachers' collaboration, mathematical objects (e.g. tasks, lesson design) which are more useful or applicable in classroom teaching could be produced. Thus, mathematical objects could be both a vehicle for teacher collaborative learning and the product of teacher collaboration.

For all models, we have highlighted the educational context level with which their outcomes could be connected. In that sense, we offer an answer to the second question selected for this part (which forms of collaboration are appropriate in different contexts?). Regarding this question in a broader notion of context (cultural, social, political), it is important to recognise that collaborative work is a context-sensitive phenomenon. Thus, it is difficult to offer a broader answer, if not impossible. We have presented cases related to the adaptation of LS to some contexts. However, in a very similar social, cultural, and educational context level, some cases were successful while others were not. In any case, as it was discussed, the collaborative work of teachers has a strong relationship with the projects they engage with, and this gives them ample versatility to choose a form, to create a new form or to adapt a known form to a particular context in which the collaboration will develop.

Finally, as pointed out in the bibliography (Sect. 3.1) and evidenced in the developments presented in Sects. 3.2, 3.3, 3.4 and 3.5, we recognise the importance of diversity in collaborative work in terms of objectives, forms and collaboration results. Within the framework of such diversity, we highlight the main contributions of this chapter. Firstly, with the empirical and analytical work carried out, it is possible to establish a categorisation of the forms of collaboration enacted and the corresponding decontextualised models associated with them.

With these categories, it is feasible to unveil the relationships between environments, arenas, subjects, activities and scenarios in collaborative work contexts (Lave, 1988). Secondly, after careful collaborative deconstruction work (Esteley, 2014), detailed and complex pictures of the contexts, forms, outcomes and logics underlying such work are described. With that contribution, on the one hand we consider enriching and extending relevant existing studies (Jaworski et al., 2017; Robutti et al., 2016).

On the other hand, in the process of such deconstruction, it was possible to present voices of participating teachers and researchers that capture the senses attributed to collaboration. After all, we emphasise that, considering the complexity of collaborative work (Davis & Sengupta, 2020), this chapter resorts to its own units of analysis or those of other authors (e.g. Prediger, 2020), as useful tools, to examine such complexity trying not to decomplexify. For instance, as an emergent of the work done in Lisbon, we seek to recover mathematical content in association with didactic systems and its potential to catalyse collaboration among teachers, as described in Sect. 3.5. This chapter also suggests the importance of facilitators and resources of or for collaboration. Both aspects are discussed in depth in the following: Chap. 4, Theme C, and Chap. 5, Theme D.

3.6.3 Further Research Directions

Based on the synthesis of the major findings presented from Sects. 3.2, 3.3, 3.4 and 3.5, this final section provides answers to the questions proposed in Theme B. However, when attempting to answer these questions, some more new issues emerge which need further exploration. This analysis highlights the complexity of the collaborative work of teachers, and the diversity of forms and outcomes achieved (such as multi-dimensionality and multiple levels, adaptability to emergent conditions and context-sensitivity). The realisation of the diversity of the forms of collaborative work represents an interesting contribution to the field and offers insights into future work. However, this diversity also makes comparisons across different collaborative forms and their outcomes a very complicated task, and yields difficulties in creating generalisable knowledge about and patterns in teacher collaboration, if indeed these are possible.

Particularly, if we consider many papers on long-term collaborations, we also note the difficulty of generalising across these contexts and the complexity of comparing approaches and their efficacy. It remains a challenge for the research community to be able to provide generalisable knowledge about and research-based effective models for teacher collaboration across different contexts. It is not yet clear how the results of teacher collaboration are related to school improvements, changes in classroom practices and student learning outcomes. Specifically, when the collaboration work is not job-embedded, there were insufficient outcomes to report impacts on schools where teachers who participated in the collaborative groups work. There is a need to uncover the mechanisms through which teacher collaborations result in their outcomes, and how to measure those outcomes.

Moreover, the unprecedented COVID-19 pandemic has suddenly changed the manner of teacher collaboration tremendously. During 2020, 2021, teacher collaborations through online and/or blended forms of teacher professional development projects became popular, and both advantages and challenges are evidenced (Chan et al., 2021; Huang et al., 2021). Although the pandemic is now over, teacher collaboration will not be returned to previous traditions (Quezada et al., 2020). Developing effective online or blended teacher collaboration models is deemed to be a crucial and challenging task (Desimone, 2020).

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