



The necessity of the sausage-string structure for mode-locking regions of piecewise-linear maps

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ABSTRACT

Piecewise-smooth maps are used as discrete-time models of dynamical systems whose evolution is governed by different equations under different conditions (e.g. switched control systems). By assigning a symbol to each region of phase space where the map is smooth, any period- p solution of the map can be associated to an itinerary of p symbols. As parameters of the map are varied, changes to this itinerary occur at border-collision bifurcations (BCBs) where one point of the periodic solution collides with a region boundary. It is well known that BCBs conform broadly to two cases: *persistence*, where the symbolic itinerary of a periodic solution changes by one symbol, and a *nonsmooth-fold*, where two solutions differing by one symbol collide and annihilate. This paper derives new properties of periodic solutions of piecewise-linear continuous maps on \mathbb{R}^n to show that under mild conditions BCBs of mode-locked solutions on invariant circles must be nonsmooth-folds. This explains why Arnold tongues of piecewise-linear maps exhibit a sausage-string structure whereby changes to symbolic itineraries occur at codimension-two pinch points instead of codimension-one persistence-type BCBs. But the main result is based on the combinatorial properties of the itineraries, so the impossibility of persistence-type BCBs also holds when the periodic solution is unstable or there is no invariant circle.

1. Introduction

Many natural phenomena involve one oscillatory system driving another. Examples include tidal currents driven by the moon's rotation [1], circadian rhythms driven by the spin of the Earth [2], and bridge sway caused by the movement of people [3]. Often the motion of the driven system becomes *mode-locked* to that of the driving system, meaning both systems are periodic with the same period or periods admitting a rational ratio. In two-parameter bifurcation diagrams, mode-locking occurs in *Arnold tongues*. Fig. 1 shows three Arnold tongues for the two-dimensional quadratic map

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} \alpha x_1 + x_2 + 1 \\ \beta x_1 - \frac{1}{2} x_1^2 \end{bmatrix}, \quad (1)$$

where α and β are parameters. In this abstract setting the presence of a driving system is obscured but this is not important for our purposes. Arnold tongues have been described in mathematical models of diverse applications, such as lasers [4], neurons [5], and cardiac dynamics [6].

Below the Neimark–Sacker bifurcation curve in Fig. 1 the map has an attracting invariant circle. Arnold tongues are where this circle contains stable and saddle solutions of some fixed period p and rotation number $\omega = \frac{m}{p}$; elsewhere the dynamics on the circle is quasi-periodic.

Additional bifurcations occur at more negative values of β , such as period-doubling and the destruction of the invariant circle [7–9], but these are not our concern here. The main point is that in Fig. 1 and for models where the equations of motion are smooth, each tongue is usually a connected set. This contrasts Arnold tongues of piecewise-linear maps that usually display a sausage-string structure. Fig. 2 illustrates this for the map

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} \alpha x_1 + x_2 + 1 \\ \beta x_1 - \frac{1}{2} |x_1| \end{bmatrix}, \quad (2)$$

obtained from the previous example by replacing x_1^2 with $|x_1|$. More tongues have been shown in Fig. 2 than Fig. 1 because in the piecewise-linear setting it is considerably easier to compute them to a reasonable degree of precision. Each tongue has zero width at pinch points termed *shrinking points* and is a disjoint union of ‘sausages’. This has been described for diverse applications including power converters [10], trade cycles [11], and mechanical oscillators subject to dry friction [12]. Piecewise-linear maps arise in other ways, e.g. Pumariño et al. [13] use them to qualitatively reproduce the dynamics near homoclinic bifurcations of smooth systems of differential equations. The sausage-string structure occurs in other settings, such as heteroclinic networks [14], delay differential equations [15,16], and homogeneous maps [17,18]. Piecewise-smooth maps with weak nonlinearity in the pieces of the map

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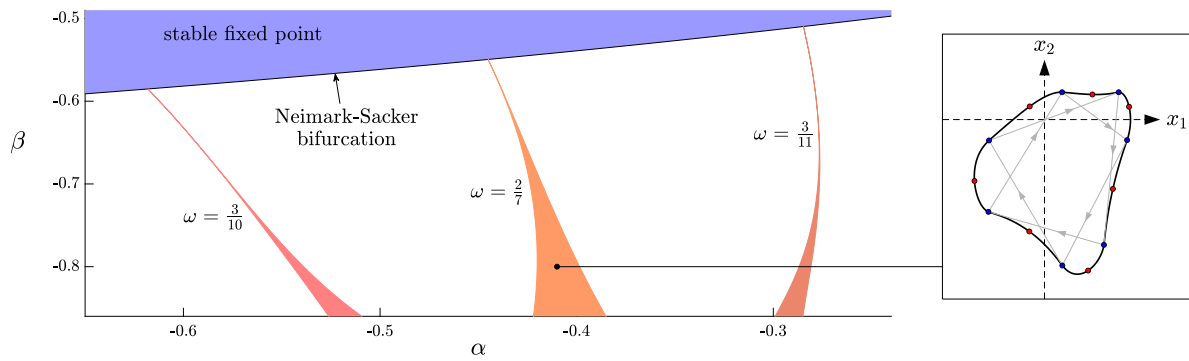


Fig. 1. A two-parameter bifurcation diagram of the smooth map (1). This map has a stable fixed point throughout the blue region which is bounded by a curve of Neimark–Sacker bifurcations. On this curve the stability multipliers associated with the fixed point are $e^{2\pi i\omega}$ where the value of $\omega \in (0,1)$ values continuously along the curve. Arnold tongues emanate from points on the curve at which ω is rational. Three tongues are shown (computed numerically by continuing their boundaries where the associated stable periodic solution undergoes a saddle–node bifurcation). For a point in the middle tongue we provide a phase portrait showing the stable (blue) and saddle (red) periodic solutions on the attracting invariant circle (black). The grey lines show how each point of the stable periodic solution maps to the next point giving a rotation number of $\omega = \frac{2}{7}$.

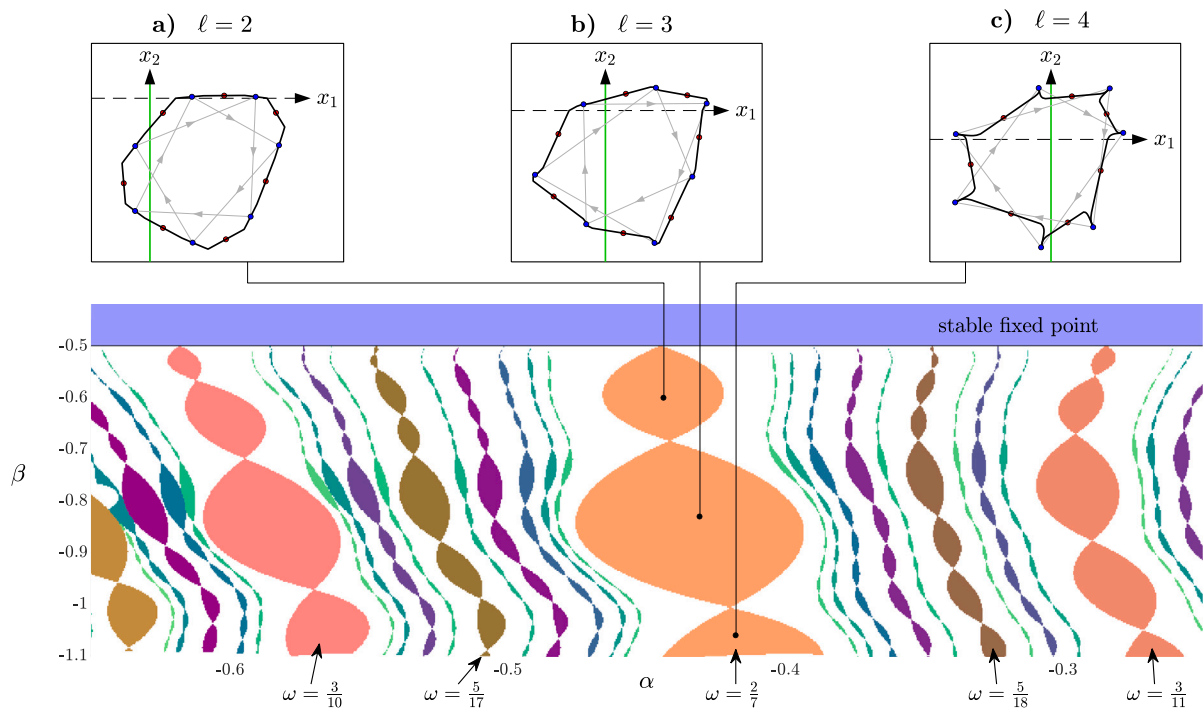


Fig. 2. A two-parameter bifurcation diagram of the piecewise-linear map (2). This map has a stable fixed point throughout the blue region bounded by the line $\beta = -0.5$. This line is a piecewise-linear analogue of a curve of Neimark–Sacker bifurcations [20,21] where the fixed point has stability multipliers $e^{2\pi i\omega}$ with $\omega = \frac{1}{2\pi} \cos^{-1}\left(\frac{a}{2}\right)$. Arnold tongues emanate from points where ω is rational and these have been computed numerically up to period 50 (by solving for the periodic solutions algebraically). Phase portraits for three points in the $\omega = \frac{2}{7}$ tongue are also shown.

exhibit the same structure except the pinch points are fattened due to the presence of saddle–node bifurcations [19].

This paper presents new results for n -dimensional continuous maps of the form

$$x \mapsto \begin{cases} A_L x + b, & x_1 \leq 0, \\ A_R x + b, & x_1 \geq 0, \end{cases} \quad (3)$$

where $x = (x_1, x_2, \dots, x_n)$, A_L and A_R are $n \times n$ matrices differing only in their first columns (for continuity), and $b \in \mathbb{R}^n$. The previous example (2) is a two-dimensional subfamily of (3). As in the smooth setting, Arnold tongues correspond to stable periodic solutions on invariant circles with a given rotation number, but for the piecewise-linear map (3) it is helpful to also consider the number of points ℓ that the solution has in the left half-plane $x_1 < 0$. In general as we move from one sausage to the next the value of ℓ changes by one, as shown in Fig. 2 for the $\omega = \frac{2}{7}$ tongue. These changes occur at codimension-two points,

and there is substantial mathematical theory explaining *how* these occur [22–24], but the theory does not explain *why* the changes must occur in this way. For instance, why cannot increments in ℓ occur in a codimension-one fashion, as suggested in Fig. 3 where one point of the periodic solution simply crosses the switching manifold? Certainly such bifurcations are possible when there is no attracting invariant circle. Fig. 4 shows an example using

$$A_L = \begin{bmatrix} \tau_L & 1 \\ -\delta_L & 0 \end{bmatrix}, \quad A_R = \begin{bmatrix} \tau_R & 1 \\ -\delta_R & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \quad (4)$$

The bifurcation is a border-collision bifurcation (BCB) of persistence type whereby one point of a stable period-9 solution crosses $x_1 = 0$ and the number of points the solution has in $x_1 < 0$ changes from $\ell = 3$ to $\ell = 4$.

Heuristically the impossibility of Fig. 3 can be explained with a simple argument. The invariant circle in the Arnold tongue has stable

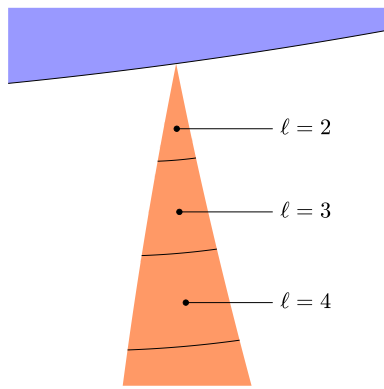


Fig. 3. A theorised Arnold tongue of a piecewise-linear map that we show cannot arise. The tongue is divided by curves of persistence-type border-collision bifurcations where the number ℓ of points of the stable periodic solution located in $x_1 < 0$ changes by one.

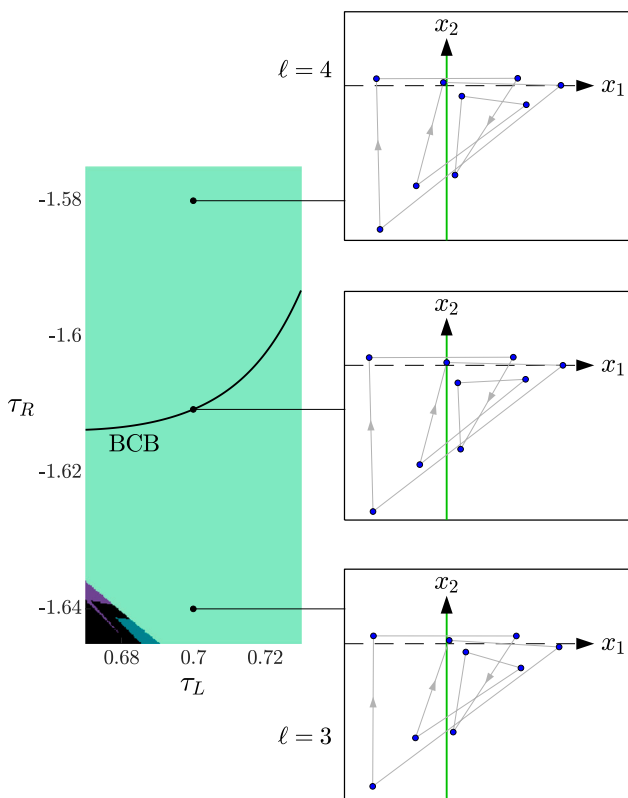


Fig. 4. A two-parameter bifurcation diagram of the two-dimensional border-collision normal form (3) with (4) using $\delta_L = 0.1$ and $\delta_R = 1.2$. This shows a curve of persistence-type BCBs (border-collision bifurcations) where one point of a stable period-9 solution crosses the switching manifold, $x_1 = 0$. (For parameter values in the lower-left the stable solution has higher period or is aperiodic.).

and saddle periodic solutions whose points alternate (stable, saddle, stable, etc.) as we go around the circle. Thus, assuming the circle intersects $x_1 = 0$ at exactly two points, the stable and saddle solutions must have either the same number of points in $x_1 < 0$, or their numbers of points in $x_1 < 0$ differ by one. But they cannot have the same number of points in $x_1 < 0$ because they would have the same symbolic itinerary (i.e. the same sequence of L's and R's) and piecewise-linear maps only have one periodic solution with a given itinerary except at bifurcations. Thus the numbers differ by one and this must be maintained throughout the tongue. So when one point of the stable solution crosses $x_1 = 0$, one

point of the unstable solution must also cross $x_1 = 0$, and these two events cannot happen simultaneously in a codimension-one fashion.

The purpose of this paper is to reach this conclusion in a rigorous manner and in a more general setting. We show that when a piecewise-linear map (3) is invertible and its fixed points do not suffer a degeneracy, the restriction of the map to any attracting invariant circle is a degree-1 circle map (Theorem 7). We characterise the class of symbolic itineraries that are possible for periodic solutions on degree-1 invariant circles intersecting $x_1 = 0$ at exactly two points (Theorem 5); such periodic solutions can be associated to a rotation number and are termed *rotational*. Finally we show that if a rotational periodic solution undergoes a BCB that does not alter its rotation number then this bifurcation must be of nonsmooth-fold type whereby two solutions collide and annihilate (Theorem 8). All three results are new; they are achieved by combining the ideas of Feigin [25] for connecting stability to admissibility, with the basic theory for periodic solutions of piecewise-linear maps, and together with the fundamental principles of circle homeomorphisms.

It is worth stressing that the rotational symbolic itineraries form a two-parameter family. This reflects the two-dimensional sausage-string structure of the Arnold tongues and is an extension of the one-parameter family of minimax itineraries that are well known to characterise periodic solutions of degree-1 circle homeomorphisms in the classical setting [26–28]. In the classical setting the symbols correspond to different branches of the circle map; in our setting the symbols correspond to different sides of $x_1 = 0$.

The remainder of this paper is organised as follows. We start in Section 2 by briefly reviewing circle homeomorphisms. Next in Section 3 we consider continuous maps on \mathbb{R}^n with two smooth pieces and formalise symbolic representations for their periodic solutions following [23]. In this setting we state and prove Theorem 5 that periodic solutions on invariant circles have rotational symbolic itineraries. In Section 4 we return to the piecewise-linear form (3). We show how periodic solutions can be computed explicitly and state and prove Theorem 7. Then in Section 5 we consider BCBs of periodic solutions and state and prove Theorem 8. We then end in Section 6 with a brief discussion.

2. Circle maps

In this section we describe elementary concepts for maps on \mathbb{S}^1 . Further details can be found in the texts [29–32].

We represent the circle \mathbb{S}^1 by the interval $[0, 1)$, where 1 is identified with 0. For any $w \in \mathbb{R}$ we let $\pi(w) = w - [w]$ be the fractional part of w . So the function $\pi : \mathbb{R} \rightarrow \mathbb{S}^1$ is the natural projection of the real line onto the circle.

It is often helpful to extend circle maps onto \mathbb{R} as follows. Note that if $g : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is continuous then $g(1^-) = g(0)$, where $g(1^-)$ is short-hand for $\lim_{t \rightarrow 1^-} g(t)$ (the limit from below).

Definition 1. A lift of a continuous map $g : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is a continuous function $G : \mathbb{R} \rightarrow \mathbb{R}$ with the property that $\pi(G(w)) = g(\pi(w))$ for all $w \in \mathbb{R}$.

Fig. 5 shows a circle map and a corresponding lift. Lifts are unique up to translation by an integer so the value of $G(1) - G(0)$ is independent of the lift. This value is an integer because $g(1^-) = g(0)$ and called the degree of g :

Definition 2. The degree of a continuous map $g : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is the value of $G(1) - G(0)$, where G is a lift of g .

If g is a homeomorphism then its degree must be 1 or -1 . If g is a degree-1 homeomorphism then for any lift G there exists $r \in \mathbb{Z}$ such that

$$G(t) = \begin{cases} g(t) + r, & t \in [0, c), \\ g(t) + r + 1, & t \in [c, 1), \end{cases} \quad (5)$$

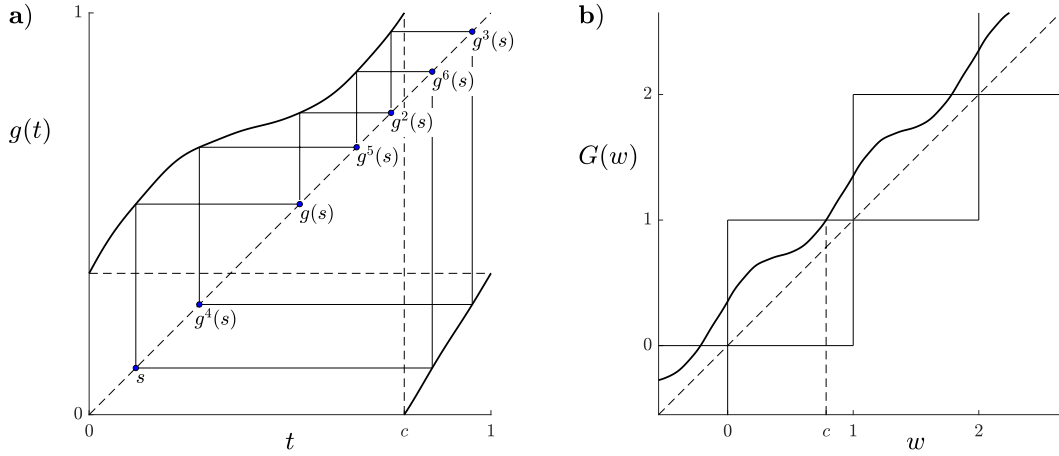


Fig. 5. A continuous degree-1 circle map (panel (a)) and a corresponding lift (panel (b)). The circle map has a stable periodic solution with rotation number $\rho = \frac{2}{7}$.

where $c = g^{-1}(0)$ as in Fig. 5.

Next we define the rotation number. This can be interpreted as the average amount by which iterates of g step around the circle. It is also the fraction of iterates that belong to $[c, 1)$.

Definition 3. The rotation number of a degree-1 homeomorphism $g : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ is

$$\rho = \lim_{i \rightarrow \infty} \pi \left(\frac{G^i(t) - t}{i} \right) \quad (6)$$

for any lift G of g and $t \in [0, 1)$.

As shown originally by Poincaré [33], the rotation number is well-defined because the limit in (6) exists and is independent of t .

We conclude this section by characterising the way by which points in a periodic solution of a degree-1 homeomorphism are ordered. This uses the following definition.

Definition 4. Let m and p be positive coprime integers (i.e. $\gcd(m, p) = 1$). The multiplicative inverse of m modulo p is the unique number $d \in \{1, 2, \dots, p-1\}$ for which $md \bmod p = 1$.

Lemma 1. Let ξ be a period- p solution of a degree-1 homeomorphism $g : \mathbb{S}^1 \rightarrow \mathbb{S}^1$. Let $s \in [0, 1)$ be the smallest point in ξ and let m be number of points in ξ that belong to $[c, 1)$. Then m and p are coprime, the rotation number of g is $\frac{m}{p}$, and

$$s < g^d(s) < g^{2d}(s) < \dots < g^{(p-1)d}(s), \quad (7)$$

where d is the multiplicative inverse of m modulo p .

As an example consider the period $p = 7$ solution in Fig. 5-a. It has $m = 2$ points in $[c, 1)$ and rotation number $\rho = \frac{2}{7}$. Here $d = 4$ and indeed the points are ordered according to (7). A proof of Lemma 1 can be found in [32]; for completeness a proof is provided in Appendix.

3. Symbolic representations of periodic solutions

Consider a map

$$f(x) = \begin{cases} f_L(x), & x_1 \leq 0, \\ f_R(x), & x_1 \geq 0, \end{cases} \quad (8)$$

where f_L and f_R are each smooth and defined throughout \mathbb{R}^n . The hyperplane $x_1 = 0$ is the switching manifold of f . To describe periodic solutions of f we use words comprised of L 's and R 's. For example the upper phase portrait in Fig. 4 shows an admissible stable $LLRRRRLRR$ -cycle. Formally we use the following definition.

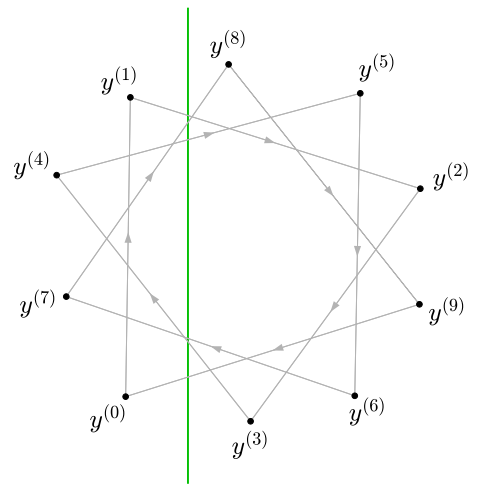


Fig. 6. An $\mathcal{F}[4, 3, 10]$ -cycle ($d = 7$). The green line is the switching manifold $x_1 = 0$.

Definition 5. Let $\Omega = \{L, R\}$ and $p \geq 1$ be an integer. An element of Ω^p is word \mathcal{X} of length p that we write as $\mathcal{X} = \mathcal{X}_0 \mathcal{X}_1 \dots \mathcal{X}_{p-1}$. An \mathcal{X} -cycle of a map (8) is an ordered set $\{y^{(0)}, y^{(1)}, \dots, y^{(p-1)}\}$ of points in \mathbb{R}^n for which

$$\begin{aligned} f_{\mathcal{X}_0}(y^{(0)}) &= y^{(1)}, \\ f_{\mathcal{X}_1}(y^{(1)}) &= y^{(2)}, \\ &\vdots \\ f_{\mathcal{X}_{p-1}}(y^{(p-1)}) &= y^{(0)}. \end{aligned}$$

An \mathcal{X} -cycle is **admissible** if $y_1^{(i)} < 0$ for all i for which $\mathcal{X}_i = L$, and $y_1^{(i)} > 0$ for all i for which $\mathcal{X}_i = R$. An \mathcal{X} -cycle is **virtual** if either $y_1^{(i)} > 0$ for some i for which $\mathcal{X}_i = L$, or $y_1^{(i)} < 0$ for some i for which $\mathcal{X}_i = R$.

Definition 6. The shift map $\sigma : \Omega^p \rightarrow \Omega^p$ is defined by

$$\sigma(\mathcal{X}) = \mathcal{X}_1 \mathcal{X}_2 \dots \mathcal{X}_{p-1} \mathcal{X}_0. \quad (9)$$

A word \mathcal{Y} is said to be a **shift** of another word \mathcal{Z} if there exists an integer i such that $\mathcal{Y} = \sigma^i(\mathcal{Z})$.

If a periodic solution of (8) is an \mathcal{X} -cycle, then it is also a $\sigma^i(\mathcal{X})$ -cycle for any integer i . It follows that any periodic solution of (8) with no points on the switching manifold corresponds to a word that is unique up to a shift.

To motivate the next definition, consider a period- p solution of (8) whose points lie roughly on a circle, Fig. 6. Suppose ℓ of these points lie

to the left of the switching manifold and let $y^{(0)}$ be the lower-most point located left of the switching manifold as indicated in the figure. Let $y^{(i)} = f^i(y^{(0)})$ for each $i = 1, 2, \dots, p-1$, and suppose each point maps m places clockwise. Then each $y^{(i)}$ is located $im \bmod p$ places clockwise from $y^{(0)}$, so $y^{(i)}$ lies to the left of the switching manifold if and only if $im \bmod p \in \{0, 1, \dots, \ell-1\}$.

Definition 7. Let ℓ, m, p be positive integers with $\ell < p$, $m < p$, and m and p coprime. Define the word $\mathcal{F}[\ell, m, p] \in \Omega^p$ by

$$\mathcal{F}_i = \begin{cases} L, & im \bmod p < \ell, \\ R, & \text{otherwise.} \end{cases} \quad (10)$$

A word \mathcal{X} is said to be **rotational** if it is a shift of some $\mathcal{F}[\ell, m, p]$.

The following result provides an equivalent definition of $\mathcal{F}[\ell, m, p]$.

Lemma 2. For any rotational word $\mathcal{F}[\ell, m, p]$ and $j \in \{0, 1, \dots, p-1\}$,

$$\mathcal{F}_{jd \bmod p} = \begin{cases} L, & j < \ell, \\ R, & \text{otherwise,} \end{cases} \quad (11)$$

where d is the multiplicative inverse of m modulo p .

Proof. Notice $j < \ell$ if and only if $jdm \bmod p < \ell$, so by substituting $i = jd \bmod p$ into (10) we obtain (11). \square

In the remainder of the paper we use the following notation. Given $k \in \mathbb{Z}$ and a word $\mathcal{X} \in \Omega^p$, we write \mathcal{X}^k for the word in Ω^p that is identical to \mathcal{X} except in its i th symbol, where $i = k \bmod p$. E.g. if $\mathcal{X} = LLLRR$ then $\mathcal{X}^2 = LLRRR$. The following lemmas will be useful in Section 5.

Lemma 3. Let $\mathcal{X} = \mathcal{F}[\ell, m, p]$ be rotational with $2 \leq \ell \leq p-2$ and d be the multiplicative inverse of m modulo p . Given $j \in \{0, 1, \dots, p-1\}$ the word \mathcal{X}^{jd} is a shift of $\mathcal{F}[\ell-1, m, p]$ or $\mathcal{F}[\ell+1, m, p]$ if and only if $j \in \{0, \ell-1, \ell, p-1\}$.

Proof. Throughout this proof we use (11). Let $\mathcal{Y} = \mathcal{X}^{jd}$. Suppose $j \in \{0, 1, \dots, \ell-1\}$, equivalently $\mathcal{X}_{jd \bmod p} = L$ (the case $\mathcal{X}_{jd \bmod p} = R$ can be treated similarly). If $j = \ell-1$ then $\mathcal{Y} = \mathcal{F}[\ell-1, m, p]$, and if $j = 0$ then $\sigma^d(\mathcal{Y}) = \mathcal{F}[\ell-1, m, p]$. Otherwise $\ell \geq 3$ and \mathcal{Y} has the property that $\mathcal{Y}_{(j-1)d \bmod p} = L$, $\mathcal{Y}_{jd \bmod p} = R$, and $\mathcal{Y}_{(j+1)d \bmod p} = L$, so cannot be a shift $\mathcal{F}[\ell-1, m, p]$. \square

Lemma 4. For any $\mathcal{X} = \mathcal{F}[\ell, m, p]$,

$$\sigma^{\ell d}(\mathcal{X}^{\bar{0}}) = \sigma^{(\ell-1)d}(\mathcal{X}^{\bar{0}}). \quad (12)$$

Proof. By (11) $\mathcal{X}_{jd \bmod p} = L$ if and only if $j \in \{0, 1, \dots, \ell-1\}$. Then $\mathcal{X}_{jd \bmod p}^{\bar{0}} = L$ if and only if $j \in \{1, 2, \dots, \ell-1\}$. So the jd^{th} symbol of the left-hand side of (12) is L if and only if $j + \ell \in \{1, 2, \dots, \ell-1\}$, equivalently $j \in \{p-\ell+1, p-\ell+2, \dots, p-1\}$.

The jd^{th} symbol of $\sigma^{(\ell-1)d}(\mathcal{X})$ is L if and only if $j + \ell - 1 \in \{0, 1, \dots, \ell-1\}$, equivalently $j \in \{p-\ell+1, p-\ell+2, \dots, p-1\} \cup \{0\}$. Flipping the 0th symbol gives the right-hand side of (12), and the above characterisation of the left-hand side of (12). \square

Finally in this section we consider periodic solutions on invariant circles of (8) when this map is a homeomorphism. To be clear, an *invariant circle* of f is a simple closed curve $C \subset \mathbb{R}^n$ for which $f(C) = C$. The restriction of f to C (denoted $f|_C$) is a circle map.

Suppose C intersects the switching manifold at exactly two points and $f|_C$ is degree-1. For any rotational word \mathcal{X} one can certainly engineer a piecewise-smooth map f with such a circle and on which the map has an admissible \mathcal{X} -cycle. The following result provides a converse and shows that rotational words are *exactly* those that correspond

to periodic solutions on invariant circles with the given assumptions. Given what we know about degree-1 circle homeomorphisms, this result should not be surprising because by design $\mathcal{F}[\ell, m, p]$ encodes rotation with rotation number $\frac{m}{p}$.

Theorem 5. Let f be a homeomorphism of the form (8). Suppose f has an invariant circle C intersecting the switching manifold $x_1 = 0$ at exactly two points and $f|_C$ is degree-1. Then for any admissible \mathcal{X} -cycle on C the word \mathcal{X} is rotational.

Fig. 7 provides an example. Panel (a) is a phase portrait of the two-dimensional map (2) with $(\alpha, \beta) = (-0.444, -0.6)$ that has an invariant circle C , and panel (b) shows the restriction of the map to C . Values t in the domain of the circle map correspond to points x with $x_1 < 0$ if $t \in (0, \frac{1}{2})$ (orange), and to points x with $x_1 > 0$ if $y \in (\frac{1}{2}, 1)$ (purple). In the context of the circle map, periodic points are assigned the symbol L if they belong to $(0, \frac{1}{2})$ and assigned the symbol R if they belong to $(\frac{1}{2}, 1)$. This generalises the classical (smooth) situation where symbols are instead used to distinguish the two branches of the circle map, and we recover this scenario in the special case $c = \frac{1}{2}$. In this case periodic solutions correspond to rotational words $\mathcal{F}[\ell, m, p]$ with $\ell = p - m$. The combinatorial properties of such words have been well studied, for instance they are *minimax* [26].

Proof of Theorem 5. Let $\varphi : C \rightarrow [0, 1)$ be a homeomorphism with the $x_1 < 0$ part of C mapping to $(0, \frac{1}{2})$ and the $x_1 > 0$ part of C mapping to $(\frac{1}{2}, 1)$. Then $f|_C$ is conjugate to the circle map $g = \varphi \circ f \circ \varphi^{-1}$. Let $c = g^{-1}(0)$.

The image under φ of the \mathcal{X} -cycle is a period- p solution of g . Since g is degree-1 this solution satisfies (7) where s is the smallest point in the solution and m is the number of points of the solution belonging to $[c, 1)$. Let ℓ be the number of points of the solution in $(0, \frac{1}{2})$. Then $g^{jd}(s) \in (0, \frac{1}{2})$ if and only if $j \in \{0, 1, \dots, \ell-1\}$.

Write the \mathcal{X} -cycle as $\{y^{(0)}, y^{(1)}, \dots, y^{(p-1)}\}$ where $y^{(i)} = \varphi^{-1}(g^i(s))$ for each i . By definition $\mathcal{X}_i = L$ if and only if $y_1^{(i)} < 0$. Here $y_1^{(i)} < 0$ if and only if $g^{\ell d}(s) \in \{0, 1, \dots, \ell-1\}$, so by (11) we have $\mathcal{X} = \mathcal{F}[\ell, m, p]$. \square

4. Periodic solutions of piecewise-linear maps

We continue to consider maps of the form (8) but now suppose f_L is a linear function, i.e.

$$f_L(x) = A_L x + b,$$

where $A_L \in \mathbb{R}^{n \times n}$ and $b \in \mathbb{R}^n$. If f_R is also linear then for f to be continuous map we must have

$$f_R(x) = A_R x + b,$$

where $A_R \in \mathbb{R}^{n \times n}$ differs from A_L in only its first column. This reproduces the piecewise-linear form (3). Notice f is a homeomorphism if and only if $\det(A_L)\det(A_R) > 0$.

Given a word $\mathcal{X} \in \Omega^p$, let

$$f_{\mathcal{X}} = f_{\mathcal{X}_{p-1}} \circ \dots \circ f_{\mathcal{X}_1} \circ f_{\mathcal{X}_0},$$

denote the composition of f_L and f_R in the order specified by \mathcal{X} . Since f_L and f_R are linear functions, so is $f_{\mathcal{X}}$, specifically

$$f_{\mathcal{X}}(x) = M_{\mathcal{X}} x + P_{\mathcal{X}} b,$$

where

$$M_{\mathcal{X}} = A_{\mathcal{X}_{p-1}} \dots A_{\mathcal{X}_1} A_{\mathcal{X}_0}, \quad (13)$$

$$P_{\mathcal{X}} = A_{\mathcal{X}_{p-1}} \dots A_{\mathcal{X}_2} A_{\mathcal{X}_1} + \dots + A_{\mathcal{X}_{p-1}} A_{\mathcal{X}_{p-2}} + A_{\mathcal{X}_{p-1}} + I. \quad (14)$$

If $\det(I - M_{\mathcal{X}}) \neq 0$ then $f_{\mathcal{X}}$ has the unique fixed point

$$x^{\mathcal{X}} = (I - M_{\mathcal{X}})^{-1} P_{\mathcal{X}} b. \quad (15)$$

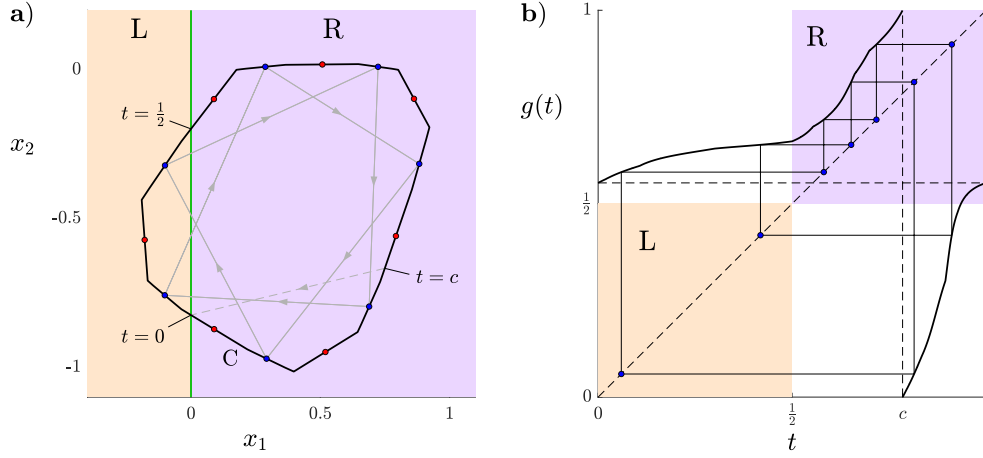


Fig. 7. Panel (a) repeats Fig. 2-a showing a stable $F[2,2,7]$ -cycle and an unstable $F[1,2,7]$ -cycle on an invariant circle C . Panel (b) shows the restriction of the map to C and indicates the corresponding stable period-7 solution. This map was constructed using $t = \varphi(x)$ where $\varphi : C \rightarrow [0, 1)$ is defined by $t = \frac{\theta}{2\pi}$ where θ is the angle between the line segment from x to $(0, -0.5)$ and the corresponding lower half of the switching manifold. Points on C that correspond to $t = 0, \frac{1}{2},$ and c are indicated.

This point is one point of an \mathcal{X} -cycle. The remaining points are generated by iterating $x^{\mathcal{X}}$ under f_L and f_R in the order specified by \mathcal{X} . In fact, since $f_{\mathcal{X}}$ is a linear function, a unique \mathcal{X} -cycle exists if and only if $\det(I - M_{\mathcal{X}}) \neq 0$. But of course the \mathcal{X} -cycle is not necessarily admissible. This is governed by the signs of the first components of its points (Definition 5).

The Jacobian matrix of $f_{\mathcal{X}}$ evaluated at $x^{\mathcal{X}}$ (or evaluated at any point in \mathbb{R}^n) is $M_{\mathcal{X}}$. Thus the stability multipliers associated with an admissible \mathcal{X} -cycle are the eigenvalues of $M_{\mathcal{X}}$. It follows that an admissible \mathcal{X} -cycle of the piecewise-linear map f is asymptotically stable if and only if all eigenvalues of $M_{\mathcal{X}}$ have modulus less than 1.

Next we derive a concise formula for the first component of $x^{\mathcal{X}}$. This formula involves adjugate matrices: if $A \in \mathbb{R}^{n \times n}$ is invertible the adjugate of A , denoted $\text{adj}(A)$, satisfies

$$A^{-1} = \frac{\text{adj}(A)}{\det(A)}. \quad (16)$$

More generally the adjugate is defined as follows [34,35].

Definition 8. For all $i, j \in \{1, 2, \dots, n\}$ let m_{ij} be the determinant of the $(n-1) \times (n-1)$ matrix obtained by removing the i th row and j th column from A . The **adjugate** of A has (i, j) -entry $(-1)^{i+j} m_{ji}$ for all $i, j \in \{1, 2, \dots, n\}$.

Now consider the matrices $I - A_L$ and $I - A_R$. These matrices differ only in their first columns, thus by Definition 8 their adjugates have the same first row. We denote this row ρ^T .

The significance of ρ^T can be seen from the fixed points of f . If $\det(I - A_L) \neq 0$ then f_L has the unique fixed point $x^L = (I - A_L)^{-1}b$. So by (16) the first component of x^L is

$$x_1^L = \frac{\rho^T b}{\det(I - A_L)}. \quad (17)$$

Similarly if $\det(I - A_R) \neq 0$ the first component of the fixed point $x^R = (I - A_R)^{-1}b$ is

$$x_1^R = \frac{\rho^T b}{\det(I - A_R)}. \quad (18)$$

The following result was first obtained in [19] and will be needed in Section 5. A proof is provided below for completeness.

Proposition 6. Let $\mathcal{X} \in \Omega^p$ and suppose $\det(I - M_{\mathcal{X}}) \neq 0$. Then

$$x_1^{\mathcal{X}} = \frac{\det(P_{\mathcal{X}})\rho^T b}{\det(I - M_{\mathcal{X}})}. \quad (19)$$

Proof. By (13) and (14),

$$P_{\mathcal{X}}(I - A_{x_0}) = I - M_{\mathcal{X}} + \sum_{j=1}^{p-1} A_{\mathcal{X}_{p-1}} \cdots A_{\mathcal{X}_{j+1}}(A_{\mathcal{X}_j} - A_{x_0}).$$

For each j the matrices $A_{\mathcal{X}_j}$ and A_{x_0} are either equal or differ only in their first columns. Thus $I - M_{\mathcal{X}}$ and $P_{\mathcal{X}}(I - A_{x_0})$ differ only in their first columns. Thus the first rows of the adjugates of $I - M_{\mathcal{X}}$ and $P_{\mathcal{X}}(I - A_{x_0})$ are the same, i.e.

$$e_1^T \text{adj}(I - M_{\mathcal{X}}) = e_1^T \text{adj}(P_{\mathcal{X}}(I - A_{x_0})),$$

where e_1 is the first standard basis vector of \mathbb{R}^n . By multiplying this equation by $P_{\mathcal{X}}$ on the right we obtain

$$\begin{aligned} e_1^T \text{adj}(I - M_{\mathcal{X}})P_{\mathcal{X}} &= e_1^T \text{adj}(I - A_{x_0}) \text{adj}(P_{\mathcal{X}})P_{\mathcal{X}} \\ &= \rho^T \det(P_{\mathcal{X}}), \end{aligned}$$

and the result follows from (15). \square

The next result shows that if an invariant circle C is stable, meaning *Lyapunov stable* [36], the restriction of f to C is a degree-1 circle map. Fig. 8 provides an example to show that stability is necessary. The proof of Theorem 7 below uses a connection between the admissibility of the fixed points x^L and x^R and their stability multipliers. This type of argument seems to have first been employed by Feigin [25].

Theorem 7. Let f be a map (3) with $\det(A_L)\det(A_R) > 0$, $\det(I - A_L) \neq 0$, $\det(I - A_R) \neq 0$, and $\rho^T b \neq 0$, where ρ^T is the first row of $\text{adj}(I - A_L)$. If f has a stable invariant circle C then $f|_C$ is a degree-1 homeomorphism.

Proof. The map f is a homeomorphism because $\det(A_L)\det(A_R) > 0$. Thus $f|_C$ is a homeomorphism so its degree is 1 or -1 . Let g be a circle map conjugate to $f|_C$ with domain $[0, 1)$. Suppose for a contradiction the degree of g is -1 . Then g has two fixed points and these must correspond to fixed points of f . But $\det(I - A_L)$ and $\det(I - A_R)$ are nonzero so the only possible fixed points of f are x^L and x^R given above. Thus x^L and x^R belong to C and are not virtual. Since $\rho^T b \neq 0$, by (17) and (18) this requires $\det(I - A_L)$ and $\det(I - A_R)$ have opposite signs. Since the determinant of a matrix is the product of its eigenvalues, at least one of $I - A_L$ and $I - A_R$ has a negative eigenvalue, suppose $I - A_L$, without loss of generality. Then A_L has an eigenvalue $\lambda > 1$. So x^L is unstable and f has an orbit that emanates from x^L on one side of x^L (because the eigenvalue is positive). This orbit cannot be contained within C , because g is decreasing so orbits near the corresponding fixed point of g visit both sides of the fixed

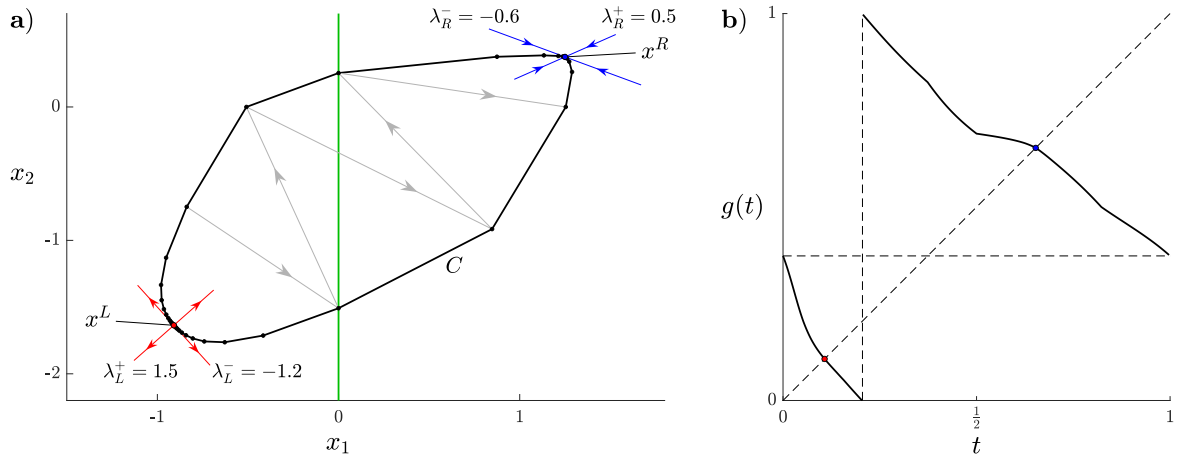


Fig. 8. Panel (a) is a phase portrait of the two-dimensional border-collision normal form (3) with (4) and $(\tau_L, \delta_L, \tau_R, \delta_R) = (0.3, -1.8, -0.1, -0.3)$. The map has an invariant circle C and panel (b) shows that the restriction of the map to C is a circle map g with degree -1 (this map was constructed as in Fig. 7). Panel (a) also shows the fixed points x^L and x^R , their stability multipliers, and the associated eigendirections. The circle C is not stable because $x^L \in C$ is repelling.

point, hence the orbit must emanate from C . But this contradicts the assumption that C is stable, so the degree of g is 1. \square

5. Border-collision bifurcations of rotational periodic solutions

Now we consider one-parameter families of piecewise-linear maps (3). That is, we suppose A_L , A_R , and b vary continuously with a parameter $\eta \in \mathbb{R}$ (while maintaining A_L and A_R differing in only their first columns).

Suppose that at some $\eta = \eta^*$ an admissible \mathcal{X} -cycle undergoes a BCB whereby its k th point collides with the switching manifold. For genericity assume $\det(I - M_{\mathcal{X}}) \neq 0$ at $\eta = \eta^*$. If the bifurcation occurs in a generic fashion then the \mathcal{X} -cycle is admissible on one side of the bifurcation and virtual on the other side of the bifurcation.

Further suppose $\det(I - M_{\mathcal{X}^k}) \neq 0$ at $\eta = \eta^*$. Then, locally, there exists a unique \mathcal{X}^k -cycle that coincides with the \mathcal{X} -cycle at $\eta = \eta^*$. Again, generically, the \mathcal{X}^k -cycle will be admissible on one side of the bifurcation and virtual on the other side of the bifurcation.

From these observations we can identify two distinct cases, as done originally by Brousin et al. [37]. The \mathcal{X} and \mathcal{X}^k -cycles are either admissible on different sides of the bifurcation, or admissible on the same side of the bifurcation. The first case is termed *persistence* because, if we only consider admissible solutions, as we pass through the bifurcation a single periodic solution appears to persist. The second case is termed a *nonsmooth fold* because, with the same mindset, two periodic solutions appear to collide and annihilate, akin to a saddle–node bifurcation, or fold.

Fig. 9 shows an example. This is a one-parameter bifurcation diagram, with parameter τ_R , and corresponds to a vertical slice through Fig. 4. As the value of τ_R is decreased two period-nine solutions become admissible in a nonsmooth fold BCB at $\tau_R \approx -1.5573$. One solution is stable (blue); the other is unstable (red). As τ_R is decreased further the stable solution undergoes a persistence-type BCB at $\tau_R \approx -1.6107$ where one point of the solution crosses the switching manifold from $x_1 < 0$ into $x_1 > 0$; here the stability of the solution is maintained. The solution then undergoes another persistence-type BCB at $\tau_R \approx -1.6489$ where a different point crosses from $x_1 > 0$ to $x_1 < 0$; here the stability of the solution is lost.

Now, motivated by Theorem 5 and our desire to understand periodic solutions in Arnold tongues, we suppose \mathcal{X} and \mathcal{X}^k are rotational and correspond to the same rotation number $\frac{m}{p}$. Lemma 3 tells us what values of k need to be considered. The following result shows that the BCB must be a nonsmooth fold. Notice it makes no assumptions on the stability of the periodic solutions or the existence of an invariant circle.

Theorem 8. *Suppose an η -dependent family of maps (3) has a BCB at $\eta = \eta^*$ that is generic in the sense that $\det(I - M_{\mathcal{X}})$, $\det(I - M_{\mathcal{X}^k})$, and $\rho^T b$ are nonzero at $\eta = \eta^*$, and that the \mathcal{X} and \mathcal{X}^k -cycles are each admissible on exactly one side of the bifurcation. If $\mathcal{X} = \mathcal{F}[\ell, m, p]$ and $k = jd$, where $j \in \{0, \ell - 1, \ell, p - 1\}$, then the BCB at $\eta = \eta^*$ is a nonsmooth fold.*

Proof. By symmetry it suffices to consider $j = 0$. Let $\{y^{(0)}, y^{(1)}, \dots, y^{(p-1)}\}$ denote the \mathcal{X} -cycle and $\{z^{(0)}, z^{(1)}, \dots, z^{(p-1)}\}$ denote the \mathcal{X}^0 -cycle. These vary continuously with η in a neighbourhood of η^* , and at $\eta = \eta^*$ we have $y^{(i)} = z^{(i)}$ for each i . At $\eta = \eta^*$ we also have $y_1^{(i)} = z_1^{(i)} = 0$.

By (19) the first components of $y^{(0)}$ and $z^{(0)}$ are

$$y_1^{(0)} = \frac{\det(P_{\mathcal{X}})\rho^T b}{\det(I - M_{\mathcal{X}})},$$

$$z_1^{(0)} = \frac{\det(P_{\mathcal{X}^0})\rho^T b}{\det(I - M_{\mathcal{X}^0})},$$

where in the second equation we have used the fact that $P_{\mathcal{X}^0} = P_{\mathcal{X}}$ by (14). Thus, by genericity, the value of $\det(P_{\mathcal{X}})$ changes sign at $\eta = \eta^*$. Hence the bifurcation corresponds to persistence if $\det(I - M_{\mathcal{X}})$ and $\det(I - M_{\mathcal{X}^0})$ have the same sign and is a nonsmooth fold if they have different signs.

Eq. (19) also gives

$$y_1^{(\ell-1)d} = \frac{\det(P_{\sigma^{(\ell-1)d}(\mathcal{X})})\rho^T b}{\det(I - M_{\mathcal{X}})}, \quad (20)$$

$$z_1^{(\ell d)} = \frac{\det(P_{\sigma^{\ell d}(\mathcal{X}^0)})\rho^T b}{\det(I - M_{\mathcal{X}^0})}. \quad (21)$$

In the denominator of (20) we have used the fact that the determinant of $I - M_{\sigma^i(\mathcal{X})}$ is independent of i , and similarly for the denominator of (21). To (21) we apply (12) giving

$$z_1^{(\ell d)} = \frac{\det(P_{\sigma^{(\ell-1)d}(\mathcal{X})})\rho^T b}{\det(I - M_{\mathcal{X}^0})}, \quad (22)$$

where we have again used (14) to drop the $\bar{0}$. But $\mathcal{X}^{(\ell-1)d} = L$, so $y_1^{(\ell-1)d} < 0$ by the admissibility assumption, and $\mathcal{X}^0 = R$, so $z_1^{(\ell d)} > 0$ also by admissibility. Thus by (20) and (22) the signs of $\det(I - M_{\mathcal{X}})$ and $\det(I - M_{\mathcal{X}^0})$ are different, hence the BCB is a nonsmooth fold. \square

6. Discussion

The dynamics of two-piece, piecewise-linear maps can be incredibly complex. Chaotic attractors occur robustly [38] and multiple (coexisting) chaotic attractors are even possible [39–41]. But there are limits

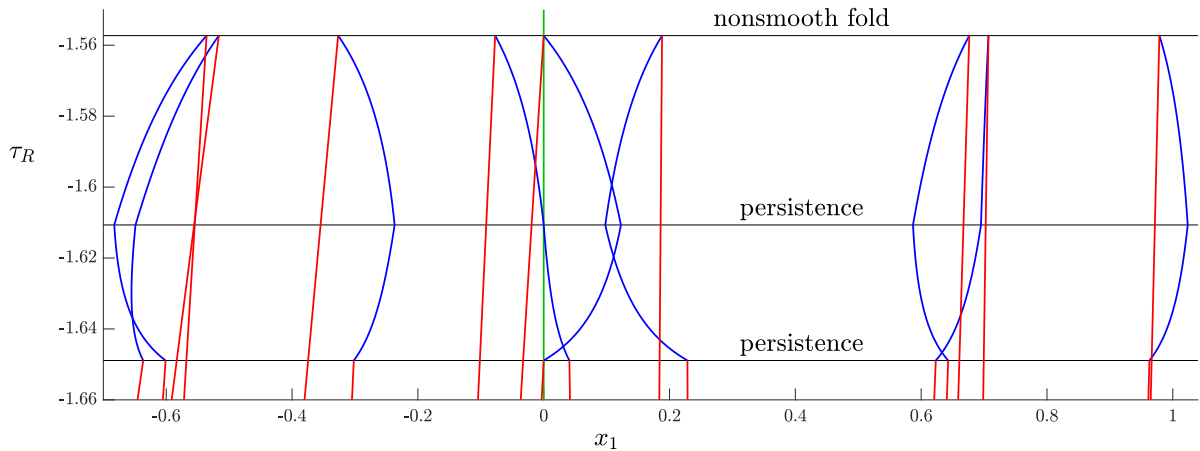


Fig. 9. A bifurcation diagram showing admissible period-nine solutions of the two-dimensional border-collision normal form (3) with (4) using $\tau_L = 0.7$, $\delta_L = 0.1$, and $\delta_R = 1.2$. Blue indicates a stable solution; red indicates an unstable solution. There are three BCBs: two of persistence-type and one nonsmooth fold.

to this complexity, as such maps have an exceedingly simple form, and in this paper we have found new ways to bring these limits to light. We have utilised the fact that the maps cannot have more than one asymptotically stable fixed point to show that stable invariant circles have degree 1. By combining this with the classical theory of circle homeomorphisms we have provided new insights into the sausage-string structure. The results show that two parts of an Arnold tongue corresponding to consecutive values of ℓ cannot be joined by a codimension-one curve of persistence-type BCBs.

It remains to see if similar ideas can advance our understanding of other bifurcation structures that arise in families of piecewise-smooth maps. For one-dimensional piecewise-smooth maps, structures are in the large well understood through renormalisation, asymptotic calculations, and other means [27,42], but in higher dimensions several new structures have recently been described that remain to be fully understood [43,44]. Also, modelling often leads to piecewise-linear maps with more than two pieces that again exhibit novel bifurcation structures, see Campisi et al. [45] for a recent example; a general theory for these remains a tantalising avenue for future research.

CRedit authorship contribution statement

D.J.W. Simpson: Writing – review & editing, Writing – original draft, Visualization, Software, Project administration, Methodology, Investigation, Funding acquisition, Formal analysis, Conceptualization.

Declaration of competing interest

The authors declare the following financial interests/personal relationships which may be considered as potential competing interests: David Simpson reports financial support was provided by Royal Society of New Zealand.

Data availability

No data was used for the research described in the article.

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Appendix. Proof of Lemma 1

Let G be a lift of g and let $r \in \mathbb{Z}$ be as in (5). Since ξ is period- p with m points in $[c, 1)$,

$$G^p(s) = s + rp + m, \tag{A.1}$$

so (6) with $t = s$ gives $\rho = \frac{m}{p}$.

We now define a sequence $S = \{w^{(i)}\}_{i \in \mathbb{Z}}$ of points in \mathbb{R} as follows. First let $w^{(0)}, \dots, w^{(p-1)}$ be the points of ξ in ascending order, i.e.

$$s = w^{(0)} < w^{(1)} < \dots < w^{(p-1)}.$$

Then for each $q \in \{0, 1, \dots, p-1\}$ and $k \in \mathbb{Z}$ let $w^{(q+kp)} = w^{(q)} + k$. Notice S is an increasing sequence, i.e. $w^{(i)} < w^{(j)}$ if and only if $i < j$.

For all $i \in \mathbb{Z}$ there exists $a_i \in \mathbb{Z}$ such that $G(w^{(i)}) = w^{(i+a_i)}$. We will now show that a_i is independent of i . Choose any $i, j \in \mathbb{Z}$ with $i < j$. The number of points in S between $w^{(i)}$ and $w^{(j)}$ is $j - i - 1$. Since G is increasing this is the same as the number of points in S between $G(w^{(i)}) = w^{(i+a_i)}$ and $G(w^{(j)}) = w^{(j+a_j)}$. But the number of points in S between $w^{(i+a_i)}$ and $w^{(j+a_j)}$ is $j + a_j - (i + a_i) - 1$, and matching the two numbers gives $a_i = a_j$.

Thus a_i is independent of i , call it a . Notice $G(s) = G(w^{(0)}) = w^{(a)}$, so $G^i(s) = w^{(ia)}$ for all $i \in \mathbb{Z}$, and in particular $G^p(s) = w^{(pa)} = s + a$. By matching this to (A.1) we obtain $a = rp + m$. Thus

$$g^i(s) = w^{(ia \bmod p)} = w^{(im \bmod p)},$$

for each $i \in \{0, 1, \dots, p-1\}$. From this we observe m and p are coprime because ξ consists of p distinct points. Also

$$g^{jd}(s) = w^{(jdm \bmod p)} = w^{(j)},$$

for each $j \in \{0, 1, \dots, p-1\}$ giving (7) as required. \square

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