# The Development of The Elliptic Functions According To Ramanujan

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# Abstract

Srinivasa Ramanujan (1887-1920) was one of the world's greatest mathematical geniuses. He made substantial contributions to elliptic functions, continued fractions, infinite series, and the theory of numbers. For many years people have studied Ramanujan's work and tried to obtain a better understanding of his work.

The main purpose of my thesis will be to consider some important classical results on elliptic functions and give proofs of these results using the methods which could have been used by Ramanujan. This will give an insight into how Ramanujan may have proved many of his results since his own proofs are often unknown.

This thesis contains five chapters. Chapter 1 is the introduction and this is related to Chapter 2 up to Chapter 4. The goal for Chapter 2 is to write the transformation of $S_{2n+1}(q)$, $\phi_{r,s}(q)$, $U_{2n}(q)$, and $V_{2n}(q)$ in terms of $P(p)$, $Q(p)$, and $R(p)$. Chapter 3 discusses Ramanujan's congruence for partitions and we give a proof for Ramanujan's modulus 5 partition congruence. In Chapter 4, we investigate a method of determining the number of representations of an integer $n$ as the sum of two, four, six, and eight squares and triangular numbers. Then we present two computer programs which are for the sums of squares and triangles. Finally, some interesting relations between the sums of squares and the sums of triangles are shown.

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# Contents

# Introduction

Srinivasa Ramanujan (1887-1920) was an Indian mathematician who had to contend with a lack of education and resources. In 1913, Ramanujan wrote his first letter to G. H. Hardy. Hardy ([19], p. 9) said, "...they defeated me completely; I had never seen anything in the least like them before. A single look at them is enough to show that they could only be written down by a mathematician of the highest class. They must be true because, if they were not true, no one would have had the imagination to invent them. Finally... the writer must be completely honest, because great mathematicians are commoner than thieves or humbugs of such incredible skill."

Ramanujan lived a short life. He died on April 26 1920, and left behind three note-books, a "lost notebook" and other manuscripts, and published papers [21].

The goal of this thesis is to gain a better understanding of Ramanujan's work. All my work, methods, and ideas are based on those in K. Venkatachaliengar's [36] monograph. Venkatachaliengar's work is significant to this thesis.

"We have no idea how he did the marvelous things he did, what led him to them, or anything else," said mathematician Richard Askey ([26], p. 280), a Ramanujan scholar at the University of Wisconsin in Madison. Bruce Berndt ([26], p. 280) said "I still don't understand it all. I may be able to prove it, but I don't know where it comes from and where it fits into the rest of mathematics," after years of working through Ramanujan's notebooks. He also said, "The enigma of Ramanujan's creative process is still covered by a curtain that has barely been drawn." So I think it is worthwhile to study his work and to have a better understanding of Ramanujan's work and of himself.

This thesis consists of five Chapters. Chapter 1 is an introduction to Chapter 2, 3, and 4. We will introduce some of Ramanujan's identities and use some theorems and results due to Venkatachaliengar to prove them. Then we introduce Ramanujan's ${}_1\psi_1$ summation formula, which is related to many different identities. The Jacobi triple product identity and the Jordan-Kronecker function are special cases of Ramanujan's summation formula. Then we introduce the fundamental multiplicative identity, the Weierstrass function and three special cases of $F(a,t)$. Next we give series expansions for $\phi_1(a)$, $\phi_2(a)$, and $\wp(a)$ and introduce the series $P$, $Q$, and $R$.

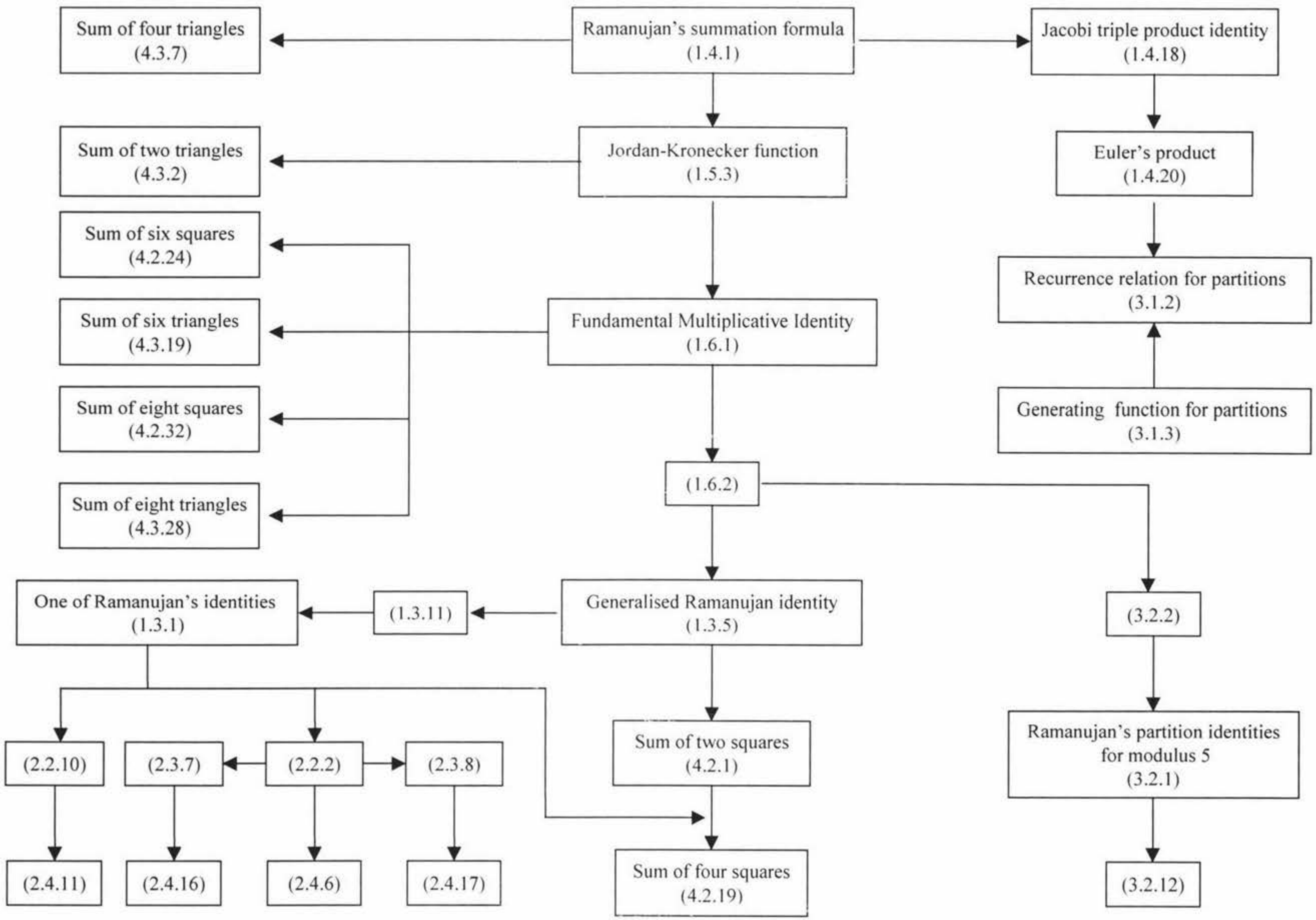
In 1916, Ramanujan published a paper called "On certain arithmetical functions" ([21], pp.136-143) which contain the formulas for $S_n$, $\phi_{r,s}$, and Ramanujan differential equations. According to Venkatachaliengar, the transformation formulas for $S_{2n+1}$, $\phi_{r,s}$, $U_{2n}$, and $V_{2n}$ can be calculated. The aim of Chapter 2 is to write the transformation of $S_{2n+1}(q)$, $\phi_{r,s}(q)$, $U_{2n}(q)$, and $V_{2n}(q)$ in terms of $P(p)$, $Q(p)$, and $R(p)$.

Ramanujan discovered properties of $p(n)$, the number of partitions of $n$, by studying MacMahon's table of values of $p(n)$. Ramanujan observed that the number of partitions of numbers $5m+4$, $7m+5$, and $11m+6$ are divisible by 5, 7, and 11 respectively. In Chapter 3, we will give a proof of Ramanujan's modulus 5 partition congruence and the idea is based on Venkatachaliengar's work.

The problem of the representation of an integer $n$ as the sum of a given number $k$ integral squares and triangles is one of the most interesting in the theory of numbers.

Chapter 4 will present proofs of formulas for the sums of two, four, six, and eight squares and triangles, all based on Ramanujan's summation formula. For instance, Jacobi proved an identity for the sum of six triangles (which is not widely known) but I found a proof using Ramanujan's result. Ramanujan has given a formula for the sum of twenty-four squares. I wrote two computer programs which can find the number of ways of writing an integer as a sums of $k$ squares and triangles, how many groups there are, and their nature. The last section will indicate some interesting relationships between sums of squares and sums of triangles.

A flow chart in the next page represents the main results of this thesis and indicates how we connect some of our results to Ramanujan's works. Each box contains either the equation and the equation number or just the equation number. The equation numbering is standard so that for example (1.4.22) can be found in Chapter 1, section 4, equation 22.


Sum of four triangles
(4.3.7)
Ramanujan's summation formula
(1.4.1)
Jacobi triple product identity
(1.4.18)
Sum of two triangles
(4.3.2)
Jordan-Kronecker function
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Ramanujan's partition identities
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