The Edge Slide Graph of The n-dimensional Cube

A thesis presented in partial fulfilment of the requirements for the degree of

Doctor of Philosophy

in

Mathematics

at Massey University, Manawatū,
New Zealand

Howida Adel AL Fran

2017

Abstract

The goal of this thesis is to understand the spanning trees of the n-dimensional cube Q_n by understanding their *edge slide graph*. An *edge slide* is a move that "slides" an edge of a spanning tree of Q_n across a two-dimensional face, and the edge slide graph is the graph on the spanning trees of Q_n with an edge between two trees if they are connected by an edge slide. Edge slides are a restricted form of an *edge move*, in which the edges involved in the move are constrained by the structure of Q_n, and the edge slide graph is a subgraph of the *tree graph* of Q_n given by edge moves.

The *signature* of a spanning tree of Q_n is the n-tuple $(a_1, \ldots, a_n)$, where a_i is the number of edges in the ith direction. The signature of a tree is invariant under edge slides and is therefore constant on connected components. We say that a signature is *connected* if the trees with that signature lie in a single connected component, and *disconnected* otherwise. The goal of this research is to determine which signatures are connected.

Signatures can be naturally classified as *reducible* or *irreducible*, with the reducible signatures being further divided into *strictly reducible* and *quasi-irreducible* signatures. We determine necessary and sufficient conditions for $(a_1, \ldots, a_n)$ to be a signature of Q_n, and show that strictly reducible signatures are disconnected. We conjecture that strict reducibility is the only obstruction to connectivity, and present substantial partial progress towards an inductive proof of this conjecture. In particular, we reduce the inductive step to the problem of proving under the inductive hypothesis that every irreducible signature has a "*splitting signature*" for which the upright trees with that signature and splitting signature all lie in the same component. We establish this step for certain classes of signatures, but at present are unable to complete it for all.

Hall's Theorem plays an important role throughout the work, both in characterising the signatures, and in proving the existence of certain trees used in the arguments.

Acknowledgement

First of all I'd thank ALLAH almighty, the most merciful and compassionate, for giving me this opportunity, the strength and the patience to complete my dissertation finally, after all the challenges and difficulties. His continuous grace and mercy was with me throughout my life.

Second I would like to thank my supervisor Dr Christopher Tuffley for the patient guidance, encouragement and advice he has provided throughout my time as his student. I have been extremely lucky to have a supervisor who responded to my questions and queries so promptly. He has been very generous with his encouragement and time, especially as I reached submission time. I could not have wished for a better supervisor.

I would also like to thank my co-supervisor Dr David Simpson for his supervision, helpful cooperation, guidance and advice throughout the research. He read through the thesis in very short time and provided valuable feedback.

I would like to thank Dr Angela Feekery for her support and encouragement when it seemed too difficult to complete the thesis. I would have probably given up without her support.

I must express my warmest and deepest appreciation to my husband, Thamer Althagafi, for his patience, assistance, continuous support and providing unending emotional support. Thamer you did everything possible for me to complete this thesis, and I was amazed by the patience you have with me (please keep surprising me).

I would also like to express my appreciation to my wonderful children: Junah, Abdulmalik and Mayar, for being patient with me as I took on yet another challenge which decreases the amount of time I can spend with them. I'm so lucky to have such amazing children, may ALLAH bless you all.

I owe everything to my parents, Salwa Soroji and Adel Al Fran, and my brothers Hisham, Haitham, Hammam and Ibrahim who encouraged and helped me at every stage of my personal and academic life and longed to see this achievement come true. I love you all and cannot live without you.

I would like to thank the Saudi Government, not only for providing the funding which allowed me to undertake this research, but also for giving me the opportunity to attend a conference in Melbourne and meet so many interesting people (I had a wonderful time there).

Contents

Abstract **i**

Acknowledgement **ii**

1 Introduction **1**

1.1 Overview of this thesis . . . 2

2 Preliminaries **5**

2.1 Spanning trees . . . 5

2.2 The n-dimensional cube (hypercube) . . . 8

2.2.1 Counting spanning trees of Q_n . . . 9

2.2.2 The direction and decoupled degree monomials . . . 10

2.2.3 Signatures of spanning trees of Q_n . . . 11

2.2.4 Edge slides . . . 13

2.2.5 Upright spanning trees . . . 15

2.2.5.1 Encoding spanning trees of the graph Q_n . . . 16

2.2.5.2 Notation for upright spanning trees of Q_n . . . 18

3 Signatures of spanning trees of Q_n **20**

3.1 Characterising signatures of spanning trees of Q_n . . . 20

3.1.1 Matchings and Hall's Marriage Theorem . . . 20

3.1.2 Hall's Marriage Theorem and signatures of spanning trees of Q_n . . . 23

3.1.3 Main result . . . 24

3.2 Classification of signatures of spanning trees of Q_n . . . 27

3.2.1 The excess of a signature of Q_n . . . 27

3.3 Summary map . . . 33

4 Local moves on upright spanning trees of Q_n **34**

4.1 Introduction . . . 34

4.2 Local moves . . . 34

4.2.1 The V-move . . . 35

4.2.2 The path move . . . 41

4.3 The local move graph of Q_n . . . 44

4.3.1 The local move graph of the irreducible signature (2, 2, 3) . . . 44

4.4 Summary map . . . 45

5 Splitting spanning trees and signatures of Q_n 48
5.1 Introduction 48
5.2 Splitting the Q_n graph 48
5.3 Upright spanning forests 48
5.3.1 Signatures of rooted spanning forests 49
5.3.2 Rooted spanning forests of $\mathcal{F}_n^{i+}$ and $\mathcal{F}_n^{i-}$ 49
5.3.3 Hall's Theorem and signatures of spanning forests of $\mathcal{F}_n^{n+}$ 50
5.4 The edge slide graph of the irreducible signature (2, 3, 3, 7) 51
5.5 Amenable splitting signatures 56
5.6 Signature moves 57
5.7 Summary map 71

6 Reducible signatures of Q_n 72
6.1 Introduction 72
6.2 Preparatory lemmas 73
6.2.1 Labels of spanning trees of Q_n with a reducible signature 73
6.2.2 Partitioning the n-dimensional cube 75
6.2.3 Edge slides of spanning trees of Q_n with reducible signatures 76
6.2.4 The existence of subtrees with different signatures of spanning trees with reducible signatures 78
6.3 The edge slide graph of a reducible signature of Q_n 80
6.4 Summary map 81

7 Irreducible signatures of Q_n: irreducible splitting signatures 82
7.1 Introduction 82
7.2 Preparatory lemmas 83
7.2.1 Swapping labels in the upper n face 83
7.2.2 Settled trees 91
7.2.2.1 Swapping labels of a settled tree of Q_n in the upper n face . . . 94
7.3 Rearranging the labels in the upper n face 105
7.3.1 The (2, 2, 3) splitting signature of an irreducible signature of Q_4 105
7.3.2 Unidirectional splitting signatures 116
7.3.3 Super rich splitting signatures 117
7.4 Summary map 122

8 Irreducible signatures of Q_n: transforming irreducible splitting signatures 129
8.1 Introduction 129
8.2 Preparatory lemmas 129
8.2.1 The main technical tool of this chapter 129
8.2.2 Ordering irreducible splitting signatures 133
8.2.3 Connecting irreducible splitting signatures 135
8.3 Transforming an irreducible splitting signature of a tree 136
8.4 Summary map 136

9 Irreducible signatures of Q_n: reducible splitting signatures 139
9.1 Introduction 139
9.2 Preparatory lemmas 139

9.2.1 The existence of a vertex with label n that does not lie above the reducing set . . . 140
9.2.2 Increasing the size of the smallest reducing set of reducible splitting signatures . . . 140
9.3 Transforming a tree with a reducible splitting signature into a tree with an irreducible splitting signature . . . 157
9.4 Summary map . . . 159

10 Two infinite families of irreducible signatures of Q_n. 161
10.1 The irreducible signature $\mathcal{I}_n^{(-1)}$ . . . 161
10.1.1 The edge slide graph of signature $\mathcal{I}_n^{(-1)}$ of Q_n . . . 162
10.1.2 Counting the number of upright spanning trees of Q_n with signature $\mathcal{I}_n^{(-1)}$ 164
10.2 The irreducible signature $\mathcal{I}_{(3,n)}^{(-1,-1)}$ . . . 166
10.2.1 The edge slide graph of signature $\mathcal{I}_{(3,n)}^{(-1,-1)}$ of Q_n . . . 167
10.2.2 Counting the number of upright spanning trees of Q_n with signature $\mathcal{I}_{(3,n)}^{(-1,-1)}$ 169

11 Discussion 174

References 177

Index 178

List of Figures

2.1 An example of a spanning tree . . . 6
2.2 An edge move . . . 6
2.3 An example of the tree graph of a connected graph . . . 7
2.4 The construction of 4-cube . . . 9
2.5 Upward and downward edges . . . 11
2.6 A signature of a spanning tree of 2-cube . . . 12
2.7 An edge slide . . . 13
2.8 A downward edge slide . . . 14
2.9 The edge slide graph of 2-cube . . . 15
2.10 An upright spanning tree of 3-cube . . . 16
2.11 Encoding a spanning tree of 3-cube . . . 17
2.12 Notation for an upright spanning tree of 4-cube . . . 19

3.1 A bipartite graph describing the common interests between women and men . . . 21
3.2 A bipartite graph describing the work experience of the candidates . . . 21
3.3 Perfect matching graph for Problem 2 . . . 23
3.4 A bipartite graph of vertices of 3-cube and a signature of 3-cube . . . 24
3.5 An upright spanning tree of Q_3 with signature (1, 3, 3) . . . 25
3.6 A bipartite graph between vertices of n-cube and a signature of n-cube . . . 26

4.1 The process of sliding an edge in direction i . . . 36
4.2 The V-move . . . 37
4.3 The process of the V-move in the case $i \neq j$ . . . 38
4.4 The process of the V-move in the case $i = j$ . . . 39
4.5 The path move . . . 40
4.6 The process of the path move . . . 42
4.7 Diagram for the proof of Lemma 4.2.6 . . . 43
4.8 Upright spanning trees of Q_3 with signature (2, 2, 3) . . . 46
4.9 The local move graph of signature (2, 2, 3) . . . 47

5.1 A rooted upright spanning forest of 3-cube . . . 49
5.2 An upright spanning forest of 3-cube with signature (1, 2, 3) . . . 50
5.3 Sliding a root of an upright spanning forest of 2-cube resulting from splitting an upright spanning tree of 3-cube with the irreducible signature (2, 2, 3) . . . 52
5.4 Diagram 1 for the proof of Proposition 5.4.2 . . . 53
5.5 Diagram 2 for the proof of Proposition 5.4.2 . . . 54
5.6 Diagram 3 for the proof of Proposition 5.4.2 . . . 54

5.7 Diagram 4 for the proof of Proposition 5.4.2 55

6.1 The induced subgraphs of 3-cube 75
6.2 Spanning trees of the induced subgraphs of 3-cube 76
6.3 Edge slides on a spanning tree of 3-cube with a reducible signature 77

7.1 Diagram for the proof of Lemma 7.2.1 85
7.2 Diagram for Case 1 for the proof of Lemma 7.2.2 86
7.3 Diagram for Case 2 for the proof of Lemma 7.2.2 86
7.4 Diagram for the proof of Lemma 7.2.7 88
7.5 Diagram for Corollary 7.2.10 89
7.6 Diagram for the proof of Corollary 7.2.10 92
7.7 A settled tree of 4-cube 93
7.8 Diagram for Case 1 for the proof of Lemma 7.2.14 95
7.9 Diagram for Case 2(a) for the proof of Lemma 7.2.14 96
7.10 Diagram 1 for Case 2(b) for the proof of Lemma 7.2.14 97
7.11 Diagram 2 for Case 2(b) for the proof of Lemma 7.2.14 98
7.12 Diagram for Case (b(II)) for the proof of Lemma 7.2.15 99
7.13 Diagram for the proof of Lemma 7.2.15 100
7.14 Diagram 1 for Example 7.2.16 106
7.15 Diagram 2 for Example 7.2.16 107
7.16 Diagram 3 for Example 7.2.16 108
7.17 Diagram for Case 1(a) for the proof of Lemma 7.2.19 109
7.18 Diagram for Case 1(b(I(i))) for the proof of Lemma 7.2.19 109
7.19 Diagram for Case 1(b(I(ii))) for the proof of Lemma 7.2.19 110
7.20 Diagram 1 for Case 1(b(II)) for the proof of Lemma 7.2.19 111
7.21 Diagram 2 for Case 1(b(II)) for the proof of Lemma 7.2.19 112
7.22 Diagram 3 for Case 1(b(II)) for the proof of Lemma 7.2.19 113
7.23 Diagram 1 for Case 2 for the proof of Lemma 7.2.19 113
7.24 Diagram 2 for Case 2 for the proof of Lemma 7.2.19 114
7.25 Diagram 3 for Case 2 for the proof of Lemma 7.2.19 114
7.26 Diagram 4 for Case 2 for the proof of Lemma 7.2.19 115
7.27 Diagram 5 for Case 2 for the proof of Lemma 7.2.19 115
7.28 Diagram 1 for the proof of Theorem 7.3.1 for signature (2, 3, 4, 6) 123
7.29 Diagram 2 for the proof of Theorem 7.3.1 for signature (2, 3, 5, 5) 124
7.30 Diagram 3 for the proof of Theorem 7.3.1 for signature (2, 4, 4, 5) 125
7.31 Diagram 4 for the proof of Theorem 7.3.1 for signature (3, 3, 3, 6) 126
7.32 Diagram 5 for the proof of Theorem 7.3.1 for signature (3, 3, 4, 5) 127
7.33 Diagram 6 for the proof of Theorem 7.3.1 for signature (3, 4, 4, 4) 128

8.1 Diagram for the proof of Lemma 8.2.1 130
8.2 Diagram 1 for Case (a(I(i))) for the proof of Lemma 8.2.1 137
8.3 Diagram 2 for Case (a(I(i))) for the proof of Lemma 8.2.1 137
8.4 Diagram for Case (b) of the proof of Lemma 8.2.1 138

9.1 Flowchart (part 1) for the proof of Lemma 9.2.2. 141
9.2 Flowchart (part 2) for the proof of Lemma 9.2.2. 142
9.3 Diagram for Case (a) for the proof of Lemma 9.2.2 144

9.4 Diagram 1 for Case (b) for the proof of Lemma 9.2.2 145
9.5 Diagram 2 for Case (b) for the proof of Lemma 9.2.2 146
9.6 Diagram 1 for Case (b(II(i))) for the proof of Lemma 9.2.2 147
9.7 Diagram 2 for Case (b(II(i))) for the proof of Lemma 9.2.2 147
9.8 Diagram 1 for Case (b(II(ii))) for the proof of Lemma 9.2.2 148
9.9 Diagram 2 for Case (b(II(ii))) for the proof of Lemma 9.2.2 148
9.10 Flowchart (part 1) for the proof of Lemma 9.2.3 150
9.11 Flowchart (part 2) for the proof of Lemma 9.2.3 151
9.12 Diagram 1 for Case (a) for the proof of Lemma 9.2.3 152
9.13 Diagram 2 for Case (a) for the proof of Lemma 9.2.3 153
9.14 Diagram 1 for Case (b) for the proof of Lemma 9.2.3 153
9.15 Diagram 2 for Case (b) for the proof of Lemma 9.2.3 154
9.16 Diagram 1 for Case (b(I)) for the proof of Lemma 9.2.3 154
9.17 Diagram 2 for Case (b(I)) for the proof of Lemma 9.2.3 155
9.18 Diagram 1 for Case (b(II)) for the proof of Lemma 9.2.3 155
9.19 Diagram 2 for Case (b(II)) for the proof of Lemma 9.2.3 156
9.20 Diagram 1 for the proof of Theorem 9.1.2 160
9.21 Diagram 2 for the proof of Theorem 9.1.2 160

10.1 Diagram 1 for the proof of Theorem 10.1.1 162
10.2 Diagram 2 for the proof of Theorem 10.1.1 163
10.3 Diagram 3 for the proof of Theorem 10.1.1 163
10.4 Diagram 4 for the proof of Theorem 10.1.1 170
10.5 Diagram 1 for the proof of Theorem 10.2.1 171
10.6 Diagram 2 for the proof of Theorem 10.2.1 171
10.7 Diagram 3 for the proof of Theorem 10.2.1 172