FUNCTIONAL DIFFERENTIAL EQUATIONS ARISING IN THE STUDY OF A CELL GROWTH MODEL

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Allah grants wisdom to whoever He wills. And whoever is granted wisdom is certainly blessed with a great privilege. But none will be mindful "of this“ except people of reason.

— (Surah Al-Baqara, 269)

I dedicate this thesis to my amazing mother who let me live my life my way and my late father who highly valued women higher education.

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Abstract

In this thesis we study a class of functional ordinary and partial differential equations that arise in the study of a size structured cell growth model. We study first and second order pantograph equations, which arise as separable solutions, for various constant and non constant coefficients. We discuss several techniques for solving pantograph equations that use the Laplace and the Mellin transforms, including a novel technique based on Mellin convolutions. These techniques are illustrated by applying them to a simple first order equation. The use of the Mellin transform to solve pantograph equation relies upon solving the transform equation, and this can prove formidable. This motivated us to find another avenue to show the existence of a solution. We consider a simple first order pantograph equation and show the uniqueness of the solution. We extend the study to second order pantograph equations and review a few particular second order pantograph equations with constant and non constant coefficients. These equations are solved using established techniques. Among the second order equation, a cell growth model that involves the Hermite operator is a part of research problems. Two interesting features for the *Hermite Problem* are, the form of the Mellin transform, that is such that the inversion is formidable, and the slow decaying nature of the solution. It is shown that for a range of parameter values α, b and g, there are no pdf solutions to the *Hermite Problem*; however if we drop the integrability and positivity condition, then there are non trivial solutions.

Although the separable solution is the prime candidate for a steady size distribution, showing analytically that it is this distribution requires more advanced techniques. We thus consider the full problem. In particular, we consider the case where cells do not divide when they are under certain size. This problem differs from earlier ones because the eigenvalue for the separable solution is not known explicitly. In order to show the uniqueness of eigenvalue and to show that there is a steady size distribution solution to this problem, we adapt the analysis of Perthame & Ryzhik, who under certain assumptions on the division rate, established the existence and the uniqueness of the solution to a first order ordinary functional differential equation for a non constant division rate. In addition, we show that we have an alternative technique to find the eigenvalue.

A second order partial differential equation arises when there is stochasticity in the growth rate. The earlier studied techniques are of limited use; however, Efendiev *et al.* [18] developed a technique that solves these equations for constant coefficients. They proved that the solution to the problem converges to the separable solution as time goes to ∞. We adapt their analysis and solve a second order functional differential equation with linear growth rate. In addition, we show that the solution to this problem does not have an SSD solution.

Contents

Chapter 1

Introduction

1.1 A brief history

In the early 18th century, Euler published a treatise [20] in which he considered four examples on population dynamics. Euler assumed that the number of births and the number of deaths during the year k are known. Then Euler assumed that the ratio $\frac{B_k}{N_k}$ between births B_k and the population N_k is constant. In addition, he considered the equation

$$\begin{aligned} N_{k+1} &= \lambda N_k \\ &= \lambda^{k+1} N_0, \end{aligned}$$

where $\lambda > 0$ denotes the growth rate which is also called geometric or exponential growth and N_0 is the initial population. Euler considered the population for the interval $k \in [0, 100]$ assuming that after hundred years, $N = 0$. He obtained

$$1 = m \left(1 + \frac{q_1}{\lambda} + \frac{q_1}{\lambda^2} + ... + \frac{q_{100}}{\lambda^{100}} \right),$$

where $q_k = \frac{N_k}{N_0}$ is the survival coefficient and m is the ratio between births and population which is a constant. This equation is known as the *Euler equation.* The first ever population model therefore can be associated with *Euler*.

At the end of 18th century, Malthus [46] publicized Euler's idea. The equation of growth in the *Malthus Model* is

$$\frac{d}{dt} N(t) = \lambda N(t), \quad t \geq 0,$$

where λ is a parameter and $N(t)$ denotes the total population at time t. The *Malthus*

Model strongly influenced the development of the theory of evolution. Darwin and his contemporary naturalist, Wallace, mentioned that the *Malthus Model* inspired their research. In the *Malthus Model* species grow with no limitation on its resources. Integrating the *Malthus Model* gives

$$N(t) = N_0 e^{\lambda t},$$

where N_0 denotes the population size at time $t = 0$. The *Malthus Model* showed that the human population tends to grow exponentially. Such a solution can be valid only for a finite time. As long as the population is small or shows a tendency to decrease, the Malthusian's equation is satisfactory. However for large populations this model is inappropriate. In the middle of the 19th century Verhulst [90] modified the *Malthus Model* and gave the *Logistic Model*

$$\frac{d}{dt}N(t) = \lambda \left(1 - \frac{N(t)}{K}\right) N(t), \tag{1.1.1}$$

where K denotes the carrying capacity of environment and $\lambda > 0$. When $N(t)$ is small compared to K, the *Logistic Model* approximates closely to the *Malthus Model*. A solution to the model (1.1.1) is

$$N(t) = \frac{KN_0 e^{\lambda t}}{K - N_0 + N_0 e^{\lambda t}}.$$

If $N_0 < K$, population increases and $N \to K$ as $t \to \infty$. Similarly for $N_0 > K$, population decreases and $N \to K$ as $t \to \infty$. For $N_0 = K$, population remains constant. Therefore, in contrast to the *Malthus Model*, the *Logistic Model* approaches a non trivial equilibrium as $t \to \infty$.

In the beginning of 19th century, Robertson [70] used the *Logistic Model* to study quantitative relations between the time and amount of growth, and to model the growth of different parts of an organism. In 1911, McKendrick & Pai [50] used the *Logistic Model* for the growth of populations of micro organisms. In 1922, Pearl and Reed [61] applied the *Logistic Model* for the growth of the population of the USA.

The *Malthus Model* and the *Logistic Model* were unstructured models that gave no information regarding the age distribution of the population. First age structured population with continuous variables is associated with Sharpe & Lotka [44], [75]. In 1907, Lotka without knowing about the *Euler Model* obtained

$$1 = \int_0^\infty e^{-\lambda x} N(x) h(x) dx,$$

where $h(x)$ is the fertility rate at age x. This equation is called *Lotka Model*. The *Lotka Model* is a generalization of the *Euler Model*, which considers only the discrete time and age. Ross [71] used the *Lotka Model* in the study of malaria and concluded that if the number of mosquitoes is reduced below the critical threshold then malaria can be eradicated. In 1926 McKendrick studied an epidemic model that took into account the stochastic character of infection and recovery. McKendrick [55] derived the partial differential equation

$$\frac{\partial N}{\partial t} + \frac{\partial N}{\partial x} = -\mu N, \tag{1.1.2}$$

where $N(x,t)$ and $\mu(x,t)$ denote the number density of individuals and death rate at age x and time t respectively. McKendrick showed how this partial differential equation can be converted into an integral equation whereas; Lotka [45] derived a similar result using integral equation.

von Foerster [91] studied the age structured model in terms of cell dynamics using the equation (1.1.2). Mckendrick equation (1.1.2) is sometimes referred as *McKendrick-von Foerster Equation*. McKendrick followed Ross, who in 1917, with Hudson already used equation (1.1.2) to examine the effect of change of the infectivity element in the study of infectious disease (cf. [72], pg. 234, eqn. 87). In the study of malaria when the density of mosquitoes decreased as the distance from a breeding site increased, Ross considered random dispersion in the population model. It was Fisher [22] who first introduced dispersion in the model. He assumed an individual's offspring at position m_0 with favourable genes that do not stay at the same point but disperse randomly in the neighbourhood of m_0. Let $v(x,t)$ be the proportion of the population located at point x at time t that possesses the favourable gene then, using an analogy with physics, he added a diffusion term Υ, to the equation for $v(x,t)$ that led to the partial differential equation

$$\frac{\partial v}{\partial t} = av(1-v) + \Upsilon \frac{\partial^2 v}{\partial x^2},$$

where $0 \leq v \leq 1$, a is a positive parameter and Υ is a constant. The same year 1937, a group of Russian researchers Kolmogorov, Petrovsky and Piskunov [39] modelled the same problem using the partial differential equation

$$\frac{\partial v}{\partial t} = f(v) + \Upsilon \frac{\partial^2 v}{\partial x^2},$$

where $f(v)$ is the rate of increase in v.

In 1967, three groups of scientists, Bell & Anderson [7]; Fredrickson, Ramkrishna & Tsuchiya [24] and Sinko & Streifer [76] independently derived a structured cell size model. Sinko & Streifer [76], [77] developed a deterministic size structured cell model describing the dynamics of a single species population. Their model was applied to populations of the planarian worm *Dugesia tigrina*. Sinko & Streifer assumed that the physiological properties of these organisms can be described by their size alone. They formulated the model in terms of a partial differential equation and solved the problem numerically.

The equations of Lotka and McKendrick model were linear. Gurtin & MacCamy [26], [27] developed the first non linear generalization of the age dependent population model. They allowed division and mortality rate to be non linear functions. May [49] showed that even the simplest non linear population models can behave in a chaotic way. Webb [93] developed a general theory in the formulation of general age structured model that allows general linear or non linear division and mortality rates. These non linearities provided a mechanism by which the population stabilized to a non trivial equilibrium state as $t \to \infty$.

Diekmann & Metz [51] considered the model studied by Sinko & Streifer in a Banach space and proved the existence and uniqueness of the solution. Diekmann [14] studied the problem in terms of a strongly continuous semi group of bounded linear operators and used compactness arguments to establish the existence of a stable size distribution under certain conditions on the growth rate. Hall & Wake [30], [28] used a modified Fokker-Planck equation in the cell growth model, which is based that of Sinko and Streifer.

In this thesis we study an extension of Hall & Wake model.

1.2 Cell growth model with deterministic growth rate

Let $n(x,t)$ be the number density of cells of size x at time t. In this thesis "size" corresponds to biomass or DNA content. It is a quantity that is conserved under division. At steady state, with negligible division or death, the population balance equation given in [66] is

$$\frac{\partial}{\partial t}n(x,t) = -\frac{\partial}{\partial x}(G(x,t)n(x,t)), \tag{1.2.1}$$

where $G(x,t) > 0$ is the rate at which a cell of size x grows. When a cell of size x undergoes the division they are withdrawn from the balance equation (1.2.1) resulting in the growth-fragmentation equation

$$\begin{aligned}\frac{\partial}{\partial t}n(x,t) + \frac{\partial}{\partial x}\left(G(x,t)n(x,t)\right) = &\int_x^\infty B(\xi)W(x,\xi)n(\xi,t)\,d\xi \\ &- \left(\int_0^x \frac{\tau}{x}W(\tau,x)\,d\tau\right)B(x)n(x,t),\end{aligned} \tag{1.2.2}$$

where the fragmentation kernel $W(x,\xi)$ represents the number density of cells of size x produced when one cell of size $\xi > x$ divides and $B(x) \geq 0$ is a division rate. The first term on the right hand side of the equation is the total gain of cells of size x from the division of cells of larger size ξ. The second term represents the loss of cells of size x through division and $\tau W(\tau,x)d\tau$ is the biomass of the cells that arrive in the size interval $(\tau, \tau + d\tau)$ when a cell of size x undergoes division, thus,

$$\int_0^x \tau W(\tau,x)\,d\tau = x. \tag{1.2.3}$$

The factor $\frac{\tau}{x}W(\tau,x)d\tau$ is the fraction of the cell of size x used in the formation of this biomass. Using (1.2.3) in the second term on the right hand side of the equation (1.2.2) gives

$$\int_0^x \frac{\tau}{x}W(\tau,x)d\tau = 1.$$

The second term on the right hand side in the equation (1.2.2) thus reduces to $B(x)n(x,t)$. In general B could be either probabilistic or deterministic. We, however, restrict ourselves to a deterministic division rate. When a cell of size x undergoes division then daughter cells are produced. In this thesis we consider the symmetric cell division case where a cell divides into $\alpha > 1$ daughter cells of equal size. In the cell growth/division model, cell division conserves mass and when a cell divides into α daughter cells, the sum of the mass of the daughter cells is equal to the mass of the original cell. The model assumes that when a division occurs, a cell of size ξ divides into α daughter cells of equal size x. This means that division to the size x occurs only when $\xi = \alpha x$, and this leads to the kernel

$$W(x,\xi) = \alpha\delta\left(\frac{\xi}{\alpha} - x\right), \tag{1.2.4}$$

where δ denotes the Dirac delta function. Substituting (1.2.4) in the first integral on the right hand side in the equation (1.2.2) gives

$$\int_x^\infty B(\xi)W(x,\xi)n(\xi,t)\,d\xi = \alpha^2 B(\alpha\xi)n(\alpha\xi,t).$$

The fragmentation equation (1.2.2) thus reduces to

$$\frac{\partial}{\partial t}n(x,t) + \frac{\partial}{\partial x}\left(G(x,t)n(x,t)\right) + B(x)n(x,t) = \alpha^2 B(\alpha x)n(\alpha x,t). \tag{1.2.5}$$

By adding the mortality rate μ which denotes the death rate of cells, equation (1.2.5) becomes

$$\frac{\partial}{\partial t}n(x,t) + \frac{\partial}{\partial x}\left(G(x,t)n(x,t)\right) + B(x)n(x,t) + \mu(x)n(x,t) = \alpha^2 B(\alpha x)n(\alpha x,t).$$

However in this thesis we consider the cell growth model with $\mu = 0$.

1.3 Cell growth model with stochastic/probabilistic growth rate

Let a cell of size x grows linearly as a function of time t. If we allow the white noise, that is any complicated external or internal influences that changes the system, during the growth then the deterministic cell growth model would be inappropriate. In such a scenario, the behaviour of the chaotic growth of a cell adds stochasticity to the model. In physics, this phenomenon is known as Browinan motion. In 1923 Wiener [94], [95] developed a rigorous theory to study multivariate stochastic processes.

Consider the case where the cell of size x grows as a function of time t and increase in the cell size occurs at random times. At time t_1, a cell of size x_0 undergoes a small increase ϵ_1, where $\epsilon_1 > 0$ is a random variable, and reaches to $x_1 = x_0 + \epsilon_1$. At time t_2 the cell undergoes a increase ϵ_2, where $\epsilon_2 > 0$ is independent of ϵ_1 and reaches the size $x_2 = x_1 + \epsilon_2$. A continuous process of such random growth of cell size thus generates a sequence $\{x(\epsilon)\}$ of independently distributed random variables. Let $S(x,t,\epsilon)$ be the probability distribution function for the sequence $\{x(\epsilon)\}$ and $S(x,t,\epsilon)d\epsilon$ denote the probability of the increase of a cell with size x at random time intervals Δt. Then we have

$$n(x,t) = \int_0^\infty n(x-\epsilon, t-\Delta t)S(x-\epsilon, t-\Delta t, \epsilon)d\epsilon. \tag{1.3.1}$$

Let $t = t + \Delta t$, then in terms of this new scale above equation becomes

$$n(x,t+\Delta t) = \int_0^\infty n(x-\epsilon,t)S(x-\epsilon,t,\epsilon)d\epsilon.$$

Suppose that $S(x-\epsilon,t,\epsilon)$ is infinitely differentiable with respect to x then expanding the integrand on the right hand side as a Taylor series about $n(x,t)S(x,t,\epsilon)$ gives

$$n(x,t+\Delta t) = \int_0^\infty \left[n(x,t)S(x,t,\epsilon) - \epsilon\frac{\partial}{\partial x}(n(x,t)S(x,t,\epsilon)\right.$$
$$\left. + \frac{\epsilon^2}{2!}\frac{\partial^2}{\partial x^2}(n(x,t)S(x,t,\epsilon) + ...\right] d\epsilon$$

Assuming that the probability density function for each S is sharply peaked about its mean for each x and t, we ignore terms that have ϵ of order higher than two; thus

$$n(x,t+\Delta t) = \int_0^\infty \left[n(x,t)S(x,t,\epsilon) - \epsilon\frac{\partial}{\partial x}(n(x,t)S(x,t,\epsilon))\right.$$
$$\left. + \frac{\epsilon^2}{2!}\frac{\partial^2}{\partial x^2}(n(x,t)S(x,t,\epsilon))\right] d\epsilon.$$

Using the Leibniz rule, above equation can be written as

$$n(x,t+\Delta t) = n(x,t)\int_0^\infty S(x,t,\epsilon)d\epsilon - \frac{\partial}{\partial x}\left(n(x,t)\int_0^\infty \epsilon S(x,t,\epsilon)d\epsilon\right)$$
$$+ \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(n(x,t)\int_0^\infty \xi^2 S(x,t,\epsilon)d\epsilon\right). \tag{1.3.2}$$

Let

$$\int_0^\infty S(x,t,\epsilon)d\epsilon = E_0(\epsilon) = 1,$$

$$\int_0^\infty \epsilon S(x,t,\epsilon)d\epsilon = E_1(x,t),$$

and

$$\int_0^\infty \epsilon^2 S(x,t,\epsilon)d\epsilon = E_2(x,t),$$

where E_0, E_1 and E_2 denote the $zero^{th}$, first and second moments respectively. Then, recasting equation (1.3.2) gives

$$\frac{n(x,t+\Delta t) - n(x,t)}{\Delta t} = -\frac{\partial}{\partial x}\left(n(x,t)\frac{E_1(x,t)}{\Delta t}\right) + \frac{1}{2}\frac{\partial^2}{\partial x^2}\left(n(x,t)\frac{E_2(x,t)}{\Delta t}\right).$$

As $\Delta t \to 0$, above equation becomes

$$\frac{\partial}{\partial t}n(x,t) = -\frac{\partial}{\partial x}(G(x,t)n(x,t)) + \frac{\partial^2}{\partial^2 x}(D(x,t)n(x,t)), \tag{1.3.3}$$

where

$$G(x,t) = \lim_{\Delta t \to 0} \frac{E_1(x,t)}{\Delta t},$$

is the mean growth rate and

$$D(x,t) = \lim_{\Delta t \to 0} \frac{E_2(x,t)}{2\Delta t},$$

is the variance of growth rate that we call the dispersion coefficient. For the sake of simplicity we assume $G(x,t) = G(x)$ and $D(x,t) = D(x)$. When a cell undergoes division, equation (1.3.3) becomes

$$-\frac{\partial^2}{\partial^2 x}(D(x)n(x,t)) + \frac{\partial}{\partial t}n(x,t) + \frac{\partial}{\partial x}(G(x)n(x,t)) + B(x)n(x,t) = \alpha^2 B(\alpha x,t)n(\alpha x,t). \tag{1.3.4}$$

Throughout this thesis D, B, G are positive. The Equation (1.3.4) is a Fokker Planck equation. The Fokker Planck equation is an equation of motion for the distribution function of fluctuating macroscopic variables that describes the Brownian motion of particles. This equation usually appears for variables describing the position and velocity for the Brownian motion of a small particle, a current in an electrical circuit, the electrical field in a laser. Above approach to derive the Fokker Planck equation is given in [28] and [9]. Another approach to derive Fokker Planck equation using the Wiener process in a more general context is given in ([9], chap. 5) and [41].

This thesis is concerned with the cell growth/division model (1.3.4) proposed by Hall & Wake. The partial differential equation (Pde) (1.3.4) is supplemented by the decay conditions

$$\lim_{x \to \infty} n(x,t) = 0, \tag{1.3.5}$$

$$\lim_{x \to \infty} \frac{\partial}{\partial x} n(x,t) = 0, \tag{1.3.6}$$

and the boundary conditions

$$\lim_{x \to 0^+} \left(\frac{\partial}{\partial x} (D(x)n(x,t)) - G(x)n(x,t) \right) = 0, \tag{1.3.7}$$

$$\lim_{x \to \infty} \left(\frac{\partial}{\partial x} (D(x)n(x,t)) - G(x)n(x,t) \right) = 0. \tag{1.3.8}$$

We will refer equations (1.3.7) and (1.3.8) as the *No Flux Conditions 1*. The *No Flux Conditions 1* are the special boundary conditions that indicate that the cells cannot enter or leave the system at the boundaries. The cell division problem is of the "initial boundary value" type with the *Initial Condition*

$$n(x, 0) = n_0(x), \tag{1.3.9}$$

for all $x \geq 0$. Here, n_0 is a given initial cell size distribution which may be regarded as a probability density function (pdf) that means $n_0(x) \geq 0$, for all $x \geq 0$ and $\int_0^\infty n_0(x)dx = 1$.

For succinctness, we refer equation (1.3.4) together with the *No Flux Conditions 1* and the *Initial Condition* as *Pde Problem* and the equation (1.2.5) along with *No Flux Conditions 2* that are

$$\lim_{x \to 0^+} G(x)n(x, t) = 0, \tag{1.3.10}$$

$$\lim_{x \to \infty} G(x)n(x, t) = 0, \tag{1.3.11}$$

and the *Initial Condition* as *Pde Problem without Dispersion*.

1.4 Steady Size distributions

There is a special class of solutions to the *Pde Problem* called the steady size distributions (SSD). In case of SSD solutions, the graph of n retains a constant shape. The SSD solutions correspond to the separable solutions as $t \to \infty$. The motivation of the work by Hall & Wake [30] for the study of such solutions came from the experimental observation of this behaviour in the plant cells [32]. Experimental work showed that the shape of n approaches a common shape regardless of the initial data. These solutions are of interest because they represent the long term asymptotic behaviour of the solution to the problem. Roughly speaking, solutions to the problem for any pdf n_0 evolve towards the SSD solution.

Perthame & Ryzhik studied the *Pde Problem without Dispersion* for a constant growth rate and a non constant B. They established the existence of a solution and showed that solution for general choices of n_0 are asymptotic to this solution as $t \to \infty$ in a weighted $L^1[0, \infty)$ norm. Doumic & van Brunt [17] studied the *Pde Problem without Dispersion* for a constant division and a growth rate $G = gx$. They showed that the problem does not have an SSD solution. In chapter 4 we show that in weighted $L^1[0, \infty)$ norm the solution to the *Pde Problem without Dispersion* with $B(x) = bH(x - c)$, where H denotes the Heaviside function and $c > 0$, approaches the separable solution as $t \to \infty$. Using the relative entropy method, Begg *et al.* [6] studied the *Pde Problem* for constant dispersion, constant growth and $B(x) = b\delta(l-x)$, where $l > 0$, $b > 0$. They showed that in weighted $L^1[0,\infty)$ norm, an asymptotic solution to the *Pde Problem* approaches the separable solution as $t \to \infty$. However they did not solve the Pde explicitly. Efendiev *et al.* [18] has recently developed a solution technique that solves the *Pde Problem* analytically for constant coefficients case. They showed that in the $L^1[0,\infty)$ norm there exists a unique solution to the *Pde Problem* that converges to the separable solution as $t \to \infty$. In chapter 6 we study the *Pde Problem* for a constant dispersion, a constant division rate and a linear growth rate.

Above study of Pdes motivates us to look at the separable solutions. We consider the separable solution of the form

$$n(x,t) = N(t)y(x). \tag{1.4.1}$$

Here, y is normalized so that it is a pdf. Substituting separable solutions (1.4.1) in the cell growth model (1.3.4) implies

$$\frac{N'(t)}{N(t)} = \frac{(D(x)y(x))''}{y(x)} - \frac{(G(x)y(x))'}{y(x)} + \alpha^2\frac{(B(\alpha x)y(\alpha x))}{y(x)} - B(x) = \lambda, \tag{1.4.2}$$

where $'$ denotes $\frac{d}{dx}$, $''$ denotes $\frac{d^2}{dx^2}$ and λ is an eigenvalue that appears due to the separation. Note that λ is a constant. Equation (1.4.2) can be written as

$$(D(x)y(x))'' - (G(x)y(x))' + \alpha^2\left(B(\alpha x)y(\alpha x)\right) - (B(x)+\lambda)y(x) = 0, \tag{1.4.3}$$

and

$$\frac{N'(t)}{N(t)} = \lambda. \tag{1.4.4}$$

Solution to the equation (1.4.4) is

$$N(t) = Ke^{\lambda t},$$

where K is a constant.

Equation (1.4.3) is supplemented with the decay conditions

$$\lim_{x\to\infty} y(x) = 0, \tag{1.4.5}$$

$$\lim_{x\to\infty} y'(x) = 0,$$

and the *No Flux Conditions 3* that are

$$\lim_{x\to 0^+} \{(D(x)y(x))^{'} - G(x)y(x)\} = 0, \tag{1.4.6}$$

and

$$\lim_{x\to \infty} \{(D(x)y(x))^{'} - G(x)y(x)\} = 0. \tag{1.4.7}$$

Equation (1.4.3) is a widely known pantograph equation. These equations figure in several applications such as current collection in an electrical locomotive [59], light absorption in the milky way [2], cell growth models [29]. A detailed study of pantograph equations of the form (1.4.3) together with the *No Flux Conditions 3* is given in chapter 3.

Substituting the separable solutions (1.4.1) in the equation (1.2.5) gives the first order pantograph equation

$$(G(x)y(x))^{'} + (B(x) + \lambda)y(x) = \alpha^2 \left(B(\alpha x)y(\alpha x)\right), \tag{1.4.8}$$

which satisfies the *No Flux conditions 4*

$$\lim_{x\to 0^+} G(x)y(x) = 0,$$
$$\lim_{x\to \infty} G(x)y(x) = 0.$$

Since we are interested in those solutions that are integrable therefore $y \in L^1[0, \infty)$. The equation (1.4.8) corresponds to a pdf and consequently the *Pdf Conditions* are

$$y(x) \geq 0, \tag{1.4.9}$$

for all $x \in [0, \infty)$ and

$$\int_0^\infty y(x)dx = 1. \tag{1.4.10}$$

Integrating equation (1.4.8) from 0 to ∞ and using the *No Flux conditions 4* give,

$$\lambda = (\alpha - 1) \int_0^\infty B(x)y(x)dx. \tag{1.4.11}$$

Assuming that y is decaying rapidly enough to have a first moment, and such that xBy is integrable, $xG(x)y(x) \to 0$ as x goes to ∞ or to 0, an alternative expression for λ can be obtained by first multiplying equation (1.4.8) by x and then integrating from 0 to ∞ . This gives

$$\lambda = \frac{\int_0^\infty G(x)y(x)\,dx}{\int_0^\infty xy(x)\,dx}. \tag{1.4.12}$$

Certainly a crux to finding steady size distributions for specific choices of B and G is the determination of λ.

It is not clear for general non negative functions B and G that there is an eigenvalue λ such that equation (1.4.8) yields a pdf. This problem was studied by da Costa *et al.* [10] who established the existence of a positive eigenvalue under the assumption that the non negative functions B and G are such that B/G is integrable on $[0, \infty)$. Perthame & Ryzhik [64] established the existence of a positive eigenfunction for a general class of positive functions B under the assumption that B was uniformly bounded away from 0 and bounded on the interval $[0, \infty)$. In both of these studies, the eigenvalue that produces the positive eigenfunction is unique. Kato & McLeod [35] studied a related problem

$$\begin{aligned} y'(x) &= ay(\beta x) + cy(x), \\ y(0) &= 1, \end{aligned} \tag{1.4.13}$$

where $a \in \mathbb{C}$ and $c \in \mathbb{R}$. It was shown that when $0 < \beta < 1$, for all x, the solution to the problem (1.4.13) is unique. On the other hand for $\beta > 1$, the problem (1.4.13) has infinite number of solutions none of which are analytic at 0. Let $s = Log(x)$, $Log(x)$ is the principal value of the complex logarithm; $h = Log(\beta) > 0$; and $k_m = k_0 + 2m\pi i/Log(\beta), (m = 0, \pm 1, \pm 2, ...)$. In addition, let $g(s)$ be a periodic function of period h and holder continuous with $0 < \theta \leq 1$. Then, it was shown that for $b > 0$ and $\beta > 1$, only one solution to the problem (1.4.13) possess the asymptotic behaviour of the form

$$y(x) = x^{k_0} g(s) + \mathcal{O}(x^{k-\theta}) \tag{1.4.14}$$

as $x \to \infty$. Hall & Wake (*op. cit.*) studied the equation (1.4.8) in which all the rates are constants. In this case λ can be readily obtained from equation (1.4.11). Hall & Wake [30] also examined the case where G is a linear monomial and λ can be obtained from equation (1.4.12). For a constant coefficients case, the solution to equation (1.4.8) can be obtained in a number of ways, which we shall describe in chapter 2.

1.5 Thesis Outline

Chapter 2

In this chapter we discuss several solution techniques for pantograph type equations. These techniques rely upon the Laplace and the Mellin transforms. In addition, we discuss some related elementary theory. We also present a novel technique that uses Mellin convolutions. The techniques discussed in this chapter are illustrated by applying them to a simple first order pantograph equation.

Chapter 3

We begin this chapter with a brief discussion on first order pantograph equations with non constant coefficients and then study a generalization to second order equations with constant and non constant coefficients. All the problems discussed in this chapter are studied using the techniques given in chapter 2. We discuss a few second order pantograph equations specifically the modified Bessel equation and the Airy equations from the literature. We study a particular case of second order pantograph equation with linear growth rate. We show that for all the parameters α, b and g, the problem with linear growth does not have a pdf solution. However, with the help of the Mellin transform, we show that in the absence of the integrability and positivity conditions, there are non trivial solutions that satisfy the *No Flux Conditions 3*.

Chapter 4

In this chapter we discuss a technique developed by Perthame & Ryzhik [64]. Perthame & Ryzhik considered the *Pde Problem without Dispersion* for a general non constant division rate. Under some assumptions they established the existence and the uniqueness of the eigenvalue and the corresponding eigenfunction. They showed that in a weighted $L^1[0, \infty)$ norm, the solution to the *Pde Problem without Dispersion* converges to the separable solution as $t \to \infty$.

Chapter 5

This chapter is a part of our published work [85]. We consider the *Pde Problem without Dispersion* for a specific variable division case in which the cells divide only when they have reached a certain minimum size and after which they divide at a constant rate. Unlike earlier studied problems, the eigenvalue for this problem cannot be determined explicitly. However we define the eigenvalue through the continuity condition. By adapting the analysis in chapter 4, it is shown that the eigenvalue is unique and the solution of the problem under consideration converges to the pdf solution as $t \to \infty$.

Chapter 6

In this chapter we look at the second order case. We adapt a technique developed by Efendiev *et al.* to study the *Pde Problem* for $G(x) = x$. This technique involves the Laplace transform and the determination of a suitable Green's function. For a linear growth, the inversion of the Laplace transformed equation can be formidable, we therefore apply another approach to find the fundamental solution, note that the determination of fundamental solution plays a crucial role in finding the solution to the original problem. In this approach the original equation is converted into the transform equation by applying the Fourier transform to it. The analysis is explained using the *Pde Problem* for constant coefficients. Furthermore, it is shown that in a weighted $L^1[0, \infty)$ norm, the solution to the second order Pde with linear growth rate converges to the separable solution as time goes to ∞.

Appendices

For the sake of simplicity, we have introduced few phrases that we will use throughout in this thesis in lieu of recurring conditions, problems and identities. Lists of those phrases are given in the appendices.

Chapter 2

Elementary theory and techniques

In the previous chapter we discussed the cell growth/division model and showed that separable solutions lead to a pantograph type equation. In this chapter several solution techniques for pantograph type functional differential equations are discussed. Most of the techniques depend on the Laplace and the Mellin transforms. Among these techniques, a Mellin convolution technique is a part of our research. These techniques are applied to a simplified first order pantograph equation with constant coefficients. We will use these techniques later to study second order pantograph equations. We first briefly discuss some basic concepts and notions that are essential to develop and to understand the methodology.

2.1 Pantograph equation with constant coefficients

Let $G(x) = g, B(x) = b$, where b and g are positive constants, then equation (1.4.8) reduces to

$$gy'(x) + (b + \lambda)y(x) = \alpha^2 by(\alpha x). \qquad (2.1.1)$$

Solution to equation (2.1.1) must satisfies the *Boundary Conditions* and the *Pdf Conditions*. Integrating equation (2.1.1) from 0 to ∞ yields

$$\lambda = b(\alpha - 1),$$

and equation (2.1.1) can be written

$$y'(x) + cy(x) = \alpha cy(\alpha x), \qquad (2.1.2)$$

where $c = \frac{b\alpha}{g}$. Equation (2.1.2) subject to the the *Boundary Conditions* and the *Pdf Conditions* will be referred to as *Simplified Problem* for succinctness.

The *Simplified Problem* has been studied in depth and the solution can be represented by the Dirichlet series

$$y(x) = \frac{1}{\prod_{m=1}^{\infty}(1-\alpha^{-m})}\left(e^{-cx} + \sum_{k=1}^{\infty}\frac{(-1)^k e^{-c\alpha^k x}}{\alpha^{k(k-1)/2}\prod_{j=1}^{k}(1-\alpha^{-j})}\right). \qquad (2.1.3)$$

It can be shown that the solution is positive and unique for $x > 0$ (cf. [29]) and unimodal, i.e., the graph has only one local maximum (cf. [10]). Figure 2.1.1 illustrates the shape of the solution for various values of α.

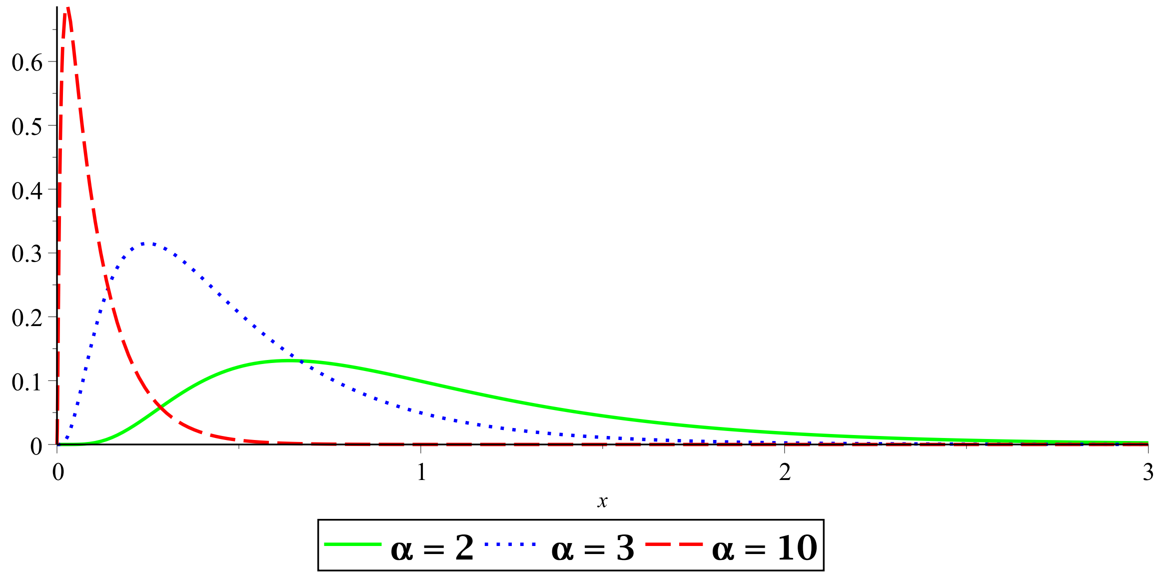

Figure 2.1.1: The shape of $y(x)$ for different values of α with $b = g = 1$

Here we show that the *Simplified Problem* has a unique solution. This proof was discussed in [29]. Suppose that the *Simplified Problem* has two solutions y_1 and y_2 and let $z(x) = y_1(x) - y_2(x)$. Then,

$$z'(x) + cz(x) - c\alpha z(\alpha x) = 0, \qquad (2.1.4)$$

and the *Boundary Conditions* imply

$$z(0) = 0,$$
$$\lim_{x\to\infty} z(x) = 0,$$

and the *Integrability Condition* implies

$$\int_0^\infty z(\xi)d\xi = 0.$$

Let

$$w(x) = \int_x^\infty z(\xi)d\xi. \tag{2.1.6}$$

Integrating equation (2.1.4) from x to ∞ and using equations (2.1.6) yields

$$w'(x) + cw(x) = cw(\alpha x),$$

where

$$w'(x) = -z(x),$$

and

$$\lim_{x\to\infty} w(x) = 0. \tag{2.1.8}$$

Suppose $w(x) \neq 0$ somewhere in $(0, \infty)$ then without loss of generality it can be assumed that

$$w(x^*) > 0,$$

at some $x^* \in (0, \infty)$. Boundary condition (2.1.8) implies that z must have a global maximum, since $z(x) \to 0$ as $x \to \infty$. There must be a greatest value of x at which w achieves this global maximum. Let x_m denotes this value. Then $w'(x_m) = 0$ and

$$w(x_m) = w(\alpha x_m),$$

which implies w achieves its global maximum at $\alpha x_m > x_m$. This contradicts the definition of x_m. Thus w cannot be positive for any $x \in (0, \infty)$. The same argument can be used on $-w$ to show that w cannot be negative in $(0, \infty)$; hence, $w(x) = 0$ for all $x \geq 0$. As a consequence $z = 0$ for all $x \geq 0$. In section 3.2, similar arguments will be used to show the uniqueness of second order pantograph equations with constant coefficients.

In the following sections, we solve the *Simplified Problem* using various techniques based on the Laplace and the Mellin transforms.

2.2 A Laplace transform method

Recall that the Laplace transform of a function $f(t)$ is

$$L(s) = L[f(t)] = \int_0^\infty e^{-ts} f(t)dt,$$

where $t \geq 0$, and the inverse formula for the Laplace transform is

$$L^{-1}(s) = f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} L(s)e^{ts}ds.$$

Let $U(s)$ and $W(s)$ be the Laplace transforms of the functions $f(t)$ and $g(t)$, respectively then their convolution is

$$U(s)W(s) = f(t)\diamond g(t) = L\left(\int_0^t f(t)g(t-\tau)d\tau\right)$$

where $\diamond$ denotes the Laplace convolution. It can readily be verified that in above equation right hand side is equal to the left hand side.

Applying the Laplace transform to the equation (2.1.2) gives

$$sY(s) + cY(s) = cY\left(\frac{s}{\alpha}\right), \tag{2.2.1}$$

where $Y(s) = L[y(x)]$. Note that the *Integrability Condition* implies

$$Y(0) = 1. \tag{2.2.2}$$

Rewriting equation (2.2.1) yields

$$Y(s) = \frac{1}{1+\frac{s}{c}} Y\left(\frac{s}{\alpha}\right).$$

Now,

$$Y\left(\frac{s}{\alpha}\right) = \frac{1}{1+\frac{s}{c\alpha}} Y\left(\frac{s}{\alpha^2}\right);$$

$$Y\left(\frac{s}{\alpha^2}\right) = \frac{1}{1+\frac{s}{c\alpha^2}} Y\left(\frac{s}{\alpha^3}\right);$$

and repeating the argument k times gives

$$Y\left(\frac{s}{\alpha^k}\right) = \frac{1}{1+\frac{s}{c\alpha^k}} Y\left(\frac{s}{\alpha^{k+1}}\right);$$

thus,

$$Y(s) = \frac{1}{1+\frac{s}{c}} \cdot \frac{1}{1+\frac{s}{c\alpha}} \cdots \frac{1}{1+\frac{s}{c\alpha^k}} Y\left(\frac{s}{\alpha^{k+1}}\right).$$

We conclude that

$$\begin{aligned} Y(s) &= Y(0) \prod_{k=0}^{\infty} \frac{1}{1+\frac{\alpha^{-k}s}{c}}, \\ &= \prod_{k=0}^{\infty} \frac{1}{1+\frac{\alpha^{-k}s}{c}}, \end{aligned}$$

where we have used equation (2.2.2). Note that the infinite product converges uniformly in any closed bounded set that does not contain $s = -c\alpha^k$ for $k \in (\mathbb{N} \cup \{0\})$, since for $\alpha > 1$, the series $\sum_{k=0}^{\infty} |s/c\alpha^k|$ converges uniformly in such sets (cf.[81]).

It is easier to determine the inverse Laplace transform of a series than a product, and this motivates us to represent $Y(s)$ as a series. Hall and Wake [29] found a series representation by use of partial fractions and algebraic manipulations followed by a limit process. Here, we use the Mittag-Leffler theorem to find an expression for $Y(s)$ that can be readily inverted.

The product defining Y, as noted, is uniformly convergent in any compact set that does not include $s = -c\alpha^k$, for $k \in (\mathbb{N} \cup \{0\})$. Each factor in the product has a simple pole, and this means that Y must be a meromorphic function with simple poles at $s = -c\alpha^k$, $k \in (\mathbb{N} \cup \{0\})$. We can thus apply the Mittag-Leffler theorem to get

$$\prod_{k=0}^{\infty} \frac{1}{1+\frac{\alpha^{-k}s}{c}} = \sum_{k=0}^{\infty} \frac{b_k}{s+\alpha^k c} + g(s),$$

where $g(s)$ is an entire function and

$$b_k = Res\{Y(s), -c\alpha^k\}.$$

The first few terms are

$$b_0 = \lim_{s\to -c}\left(1+\frac{s}{c}\right)\frac{1}{(1+\frac{s}{c})\prod_{k=1}^{\infty}(1+\frac{s\alpha^{-k}}{c})},$$
$$= \frac{1}{\prod_{k=1}^{\infty}(1-\alpha^{-k})};$$

$$b_1 = \lim_{s\to -\alpha c}\left(1+\frac{s}{c\alpha}\right)\frac{1}{(1+\frac{s}{c})\,(1+\frac{s}{\alpha c})\prod_{k=2}^{\infty}(1+\frac{s\alpha^{-k}}{c})},$$
$$= \frac{1}{(1-\alpha)\prod_{k=2}^{\infty}(1-\alpha^{-k+1})},$$
$$= \frac{1}{(1-\alpha)\prod_{k=1}^{\infty}(1-\alpha^{-k})};$$

and

$$b_2 = \frac{1}{(1-\alpha)(1-\alpha^2)\prod_{k=1}^{\infty}(1-\alpha^{-k})}.$$

The j^{th} term for b_j is

$$b_j = \frac{1}{(1-\alpha)...(1-\alpha^j)\prod_{k=1}^{\infty}(1-\alpha^{-k})}.$$

We thus get

$$Y(s) = \frac{1}{\prod_{m=1}^{\infty}(1-\alpha^{-m})}\left(1+\sum_{k=1}^{\infty}\frac{(-1)^k}{\alpha^{k(k-1)/2}\prod_{j=1}^{k}(1-\alpha^{-j})(s+c\alpha^k)}\right)+\hat{g}(s) \quad (2.2.4)$$

where

$$\frac{(-1)^k}{\alpha^{k(k-1)/2}\prod_{j=1}^{k}(1-\alpha^{-j})} = \frac{1}{(1-\alpha)...(1-\alpha^k)}.$$

The Mittag-Leffler expansion (2.2.4) is uniformly convergent; therefore, we can integrate the series term by term to get the inverse Laplace transform and this yields equation (2.1.3). Note that we can also obtain the series representation through a partition identity (cf. [3], pg. 19)

$$\prod_{n=0}^{\infty}(1-tq^n)^{-1} = 1+\sum_{n=1}^{\infty}\frac{t^n}{\prod_{m=1}^{n}(1-q^m)},$$

where $|t|<1$, $|q|<1$, by choosing $q=\frac{1}{\alpha}$ and $t=-s/c$.

2.3 A convolution technique based on the Laplace transform

Derfel *et al.* [13] developed another method for solving pantograph type equations based on convolutions of the Laplace transform. Here we discuss their technique for the *Simplified Problem* for $c = 1$.

Derfel dealt with the case where the cell size x grows linearly as a function of time t and at random times, a cell of size x divides into α new cells of size x/α. It was assumed that these random times are given by a Poisson distribution and that the cell division from cell size x to x/α occurs at random times

$$0 = t_0 < t_1 < ... < t_n < ....$$

It was also assumed that in every division interval, a cell grows to the size

$$x(t_n) = x + \tau_n, \tag{2.3.1}$$

where $\tau_n = t_{n+1} - t_n$. The sequence $\{\tau_n\}$ thus consists of independent and exponentially distributed random variables, i.e., for $t > 0$, the sequence $\{\tau_n\}$ follows the probability density function

$$P(\tau_n) = e^{-t}.$$

In the following lemma they show that for a cell of size x there exists a limit distribution which is independent of initial cell size.

Lemma 2.3.1. *There exists a limit distribution for the size of a cell which is independent of initial cell size x_0.*

Suppose x_0 is the initial size of a cell. Then relation (2.3.1) implies that the size of a cell at time t_1 is

$$x(t_1) = x_0 + \tau_1.$$

Amid time t_1 and time t_2 a cell of size x_1 further divides into α daughter cells of equal size $\frac{x(t_1)}{\alpha}$. Therefore cell size at time t_2 is

$$x(t_2) = \frac{1}{\alpha}(x_0 + \tau_1) + \tau_2.$$

Continuing this process gives

$$x(t_n) = \frac{x_0}{\alpha^{n-1}} + \tau_n + \frac{\tau_{n-1}}{\alpha} + ... + \frac{\tau_1}{\alpha_{n-1}}, \tag{2.3.2}$$

which is a general form for size of cell at time n. Equation (2.3.2) corresponds to a distribution function of the random variable

$$Z(\eta) = \eta_0 + \frac{\eta_1}{\alpha} + \frac{\eta_2}{\alpha^2} + ... + \frac{\eta_n}{\alpha^n} + ..., \tag{2.3.3}$$

as $n \to \infty$. Here η_k are the exponentially distributed independent random variables. Distribution (2.3.3) is independent of initial cell size x_0.

Let

$$z(x) = \begin{cases} 1 - e^{-x}, & x > 0, \\ 0, & x \leq 0, \end{cases}$$

and

$$y(x) = \begin{cases} e^{-x}, & x > 0 \\ 0, & x \leq 0, \end{cases}$$

denote the cumulative distribution function (cdf) and the pdf for Z respectively. Evidently $y(x) = z'(x)$.

We now look at the strategy given in [13] to show that the *Simplified Problem* for $c = 1$ is satisfied for the cdf z and the pdf y.

Equation (2.3.3) can be recast as

$$\begin{aligned} Z(\eta) &= \eta_0 + \frac{1}{\alpha}\left(\eta_1 + \frac{\eta_2}{\alpha} + \frac{\eta_3}{\alpha^2} + ... \frac{\eta_n}{\alpha^n} + ...\right), \\ &= \eta_0 + \frac{1}{\alpha} Z_1(\eta), \end{aligned} \tag{2.3.4}$$

where function Z_1 has the similar distribution as Z. Let $R(x)$ be a pdf then for any positive number $\beta > 0$ the product $R(x)/\beta$ is given by $R(\beta x)$. Therefore, $z(x)/\alpha$ is $z(\alpha x)$.

In addition, let r_1 and r_2 be two random independent variables with pdfs $R_1(x)$ and

$R_2(x)$ respectively, then pdf of their sum $r_1 + r_2$ is given by the Stieltjes convolution

$$(R_1 \Box R_2)(x) = \int_{-\infty}^{\infty} R_1(x-\tau)dR_2(\tau),$$

where $\Box$ denotes the Stieltjes convolution.

Equation (2.3.4) is the sum of two random variables η_0 and Z_1/α with pdf $z(x)$ and $z(\alpha x)$ respectively. The pdf of their sum is therefore

$$(z(\alpha x)\Box z(x)) = \int_{-\infty}^{\infty} z(\alpha(x-\tau))e^{-\tau}d\tau. \tag{2.3.5}$$

Since $z(x) = 0$ when $x \leq 0$, equation (2.3.5) reduces to

$$\begin{aligned} z(x) &= \int_0^x z(\alpha(x-\tau))e^{-\tau}d\tau, \\ &= (z(\alpha x)\Diamond y(x)), \end{aligned} \tag{2.3.6}$$

where $\Diamond$ denotes the Laplace convolution. Differentiating equation (2.3.6) with respect to x yields

$$\begin{aligned} z'(x) =& z(0)e^{-x} + \alpha\int_0^x z'(\alpha(x-\tau))e^{-\tau}d\tau, \\ =& \alpha\int_0^x z'(\alpha(x-\tau))e^{-\tau}d\tau, \end{aligned} \tag{2.3.7}$$

where,

$$\alpha\int_0^x z'(\alpha(x-\tau))e^{-\tau}d\tau = z(\alpha x) - \int_0^x z(\alpha(x-\tau))e^{-\tau}d\tau. \tag{2.3.8}$$

Using (2.3.6) in (2.3.8) gives

$$z'(x) = z(\alpha x) - z(x), \tag{2.3.9}$$

and differentiating above equation yields

$$\begin{aligned} &y'(x) + y(x) = \alpha y(\alpha x) \\ &y(0) = 0. \end{aligned} \tag{2.3.10}$$

Equations (2.3.9) and (2.3.10) subject to the respective boundary conditions have unique solutions z and y such that $y, z \in L^1[0,\infty)$.

A Solution Method

We now discuss the solution method that Derfel *et al.* have developed by constructing a sequence of convolutions.

Let $\{z_k\}$ be the sequence defined by

$$z_0(x) = 1 - e^{-x},$$

and

$$\begin{aligned} z_{k+1}(x) &= z_k(\alpha x) \Diamond z(x), \\ &= \int_0^x z_k(\alpha(x-\xi))e^{-\xi} d\xi, \end{aligned} \tag{2.3.11}$$

where $k \geq 0$, and $x \geq 0$. Derfel first showed that the sequence $\{z_k\}$ is uniformly convergent on the interval $I = [0, d]$, where $d > 0$ and then proved the following lemma to show the convergence of $\{z_k\} \to z$ as $z \to \infty$.

Lemma 2.3.2. *Each term of the sequence $\{z_k\}$ is a cdf that is differentiable on $[0, \infty)$. The limit of the sequence is also a cdf.*

Each z_k is continuous on $[0, \infty)$. Differentiating equation (2.3.11), using (2.3.7 and the Fundamental Theorem of Calculus therefore imply

$$z'_{k+1}(x) = \alpha \int_0^x z'_k(\alpha(x-\xi))e^{-\xi} d\xi.$$

Let $F(x)$ be a distribution function. Then the Fourier transform $\Phi(t)$ of F

$$\Phi(t) = \int_{-\infty}^{\infty} e^{itx} dF, \tag{2.3.12}$$

is called the characteristic function of F. Here $\Phi(0) = 1$ and $t \in \mathbb{R}$. Since $|e^{itx}| = 1$, $\int_{-\infty}^{\infty} e^{itx} dF$ is absolutely convergent.

Let $\Phi_k(t)$ denote the characteristic functions of $z'_k(x)$ then

$$\Phi_k(t) = \int_{-\infty}^{\infty} e^{itx} z'_k(\xi) d\xi.$$

If $\Phi_1(t)$ and $\Phi_2(t)$ are the characteristic functions of $F_1(x)$ and $F_2(x)$ respectively, then the product $\Phi_1(t)\Phi_2(t)$ is the characteristic function of the convolution $F_1 \Diamond F_2$ (cf. [65]). Therefore, equation (2.3.11) implies

$$\Phi_{k+1}(t) = P_k(t)\psi(t),$$

where $P_k(t)$ and $\psi(t)$ denote the characteristic functions of $z_n(\alpha x)$, and $z(x)$ respectively. We now find $P_k(t)$ and $\psi(t)$.

$$\begin{aligned} P_k(t) &= \int_0^\infty e^{it\xi} dz_k(\alpha x), \\ &= \Phi_k(t/\alpha); \end{aligned}$$

and

$$\begin{aligned} \psi(t) &= \int_0^\infty e^{it\xi} d(1-e^{-\xi}), \\ &= \frac{1}{1-it}. \end{aligned}$$

Evidently,

$$\Phi_{k+1}(t) = \frac{1}{1-it}\Phi_k(t/\alpha),$$

which is a recurrence relation whose one solution is

$$\Phi_{k+1}(t) = \prod_{n=0}^{k+1}\left(1-\frac{it}{\alpha^n}\right)^{-1}.$$

The infinite product

$$\prod_{n=0}^{\infty}\left(1-\frac{it}{\alpha^n}\right)^{-1},$$

converges uniformly on all compact intervals of $\mathbb{R}$; hence, $\Phi_n(t) \to \Phi(t)$, where Φ is continuous on $\mathbb{R}$.

In order to prove the existence of a unique z corresponding to Φ, and to show that $z_n(x) \to z(x)$ as $n \to \infty$, Derfel *et al.* exploited an important result (cf. [65], p. 42) which says that if $\Phi_n(t)$ is the characteristic function of the distribution function $F_n(x)$ and in addition, $\Phi_n(t) \to \Phi(t)$ for all t, then Φ is the characteristic function of a distribution function $F(x)$ and $F_n(x) \to F(x)$ as $n \to \infty$.

2.4 Mellin transform method

Let $f(x)$ be defined on the interval $[0, \infty)$. The Mellin transform of f is defined as

$$M(s) = M[f(x)] = \int_0^\infty x^{s-1} f(x) dx, \tag{2.4.1}$$

where $M(s)$ is a complex valued function. A Mellin transform consists of a function $M(s)$ and a strip of holomorphy $S(a,b) = \{s \in \mathbb{C} : a < \Re(s) < b\}$. A strip of holomorphy is also known as a fundamental strip. The constant a in the fundamental strip $S(a,b)$ is the infimum of all p_1 such that $\int_0^1 x^{p_1-1} f(x)dx$ is integrable and b is the supremum of all p_2 such that $\int_1^\infty x^{p_2-1} f(x)dx$ is integrable. A fundamental strip is the largest open strip in which the integral $\int_0^\infty x^{s-1} f(x)dx$ converges. For any constant $\mu \neq 0$, $M[f(\mu x)] = \mu^{-s} M(s)$, and this makes the Mellin transform particularly useful in the study of pantograph equations. This transform converts a functional differential equation into an algebraic equation.

The inversion formula for the Mellin transform is

$$M^{-1}(f(x)) = f(x) = \frac{1}{2\pi i} \int_{\sigma-i\infty}^{\sigma+i\infty} M(s)x^{-s}ds,$$

where $a < \sigma < b$ and $\Re(s) = \sigma$ is the line of integration in the fundamental strip. One important feature of the Mellin transform with a strip of holomorphicity $S(a,b)$ is the correspondence between the asymptotic expansion of function f near 0 and ∞ and the poles of the transformed function M.

Consider the integral

$$N(L) = \int_{\mathcal{R}} M(s)x^{-s}ds, \tag{2.4.2}$$

where $\mathcal{R}$ is the contour shown in fig (2.4.1). The *Simplified Problem* requires $y \in L^1[0,\infty)$ and hence we focus on the fundamental strip that includes $s = 1$. Let ω be the set of poles of the transformed function M. Suppose $\mathcal{R}$ encloses a finite number of poles $s_m \in \omega$. The contour is chosen in such a way that the sides of $\mathcal{R}$ do not pass through any pole. Let $(\varsigma - iL)$, $(\varsigma + iL)$, $(\sigma - iL)$, and $(\sigma + iL)$ be the corner points of $\mathcal{R}$. The integral in the equation (2.4.2) can be written as

$$\int_{\mathcal{R}} M(s)x^{-s}ds = \int_{\sigma-iL}^{\sigma+iL} M(s)x^{-s}ds + \int_{\sigma+iL}^{\varsigma+iL} M(s)x^{-s}ds + \int_{\varsigma+iL}^{\varsigma-iL} M(s)x^{-s}ds + \int_{\varsigma-iL}^{\sigma-iL} M(s)x^{-s}ds.$$

Let $s = \varsigma + i\tau$, then

$$\begin{aligned}\left| \int_{\varsigma+iL}^{\varsigma-iL} M(s)x^{-s}ds \right| &\leq x^{-\varsigma} \int_{\varsigma+iL}^{\varsigma-iL} |M(\varsigma + i\tau)||x^{-i\tau}|d\tau, \\ &= x^{-\varsigma} \int_{\varsigma+iL}^{\varsigma-iL} |M(\varsigma + i\tau)|d\tau,\end{aligned}$$

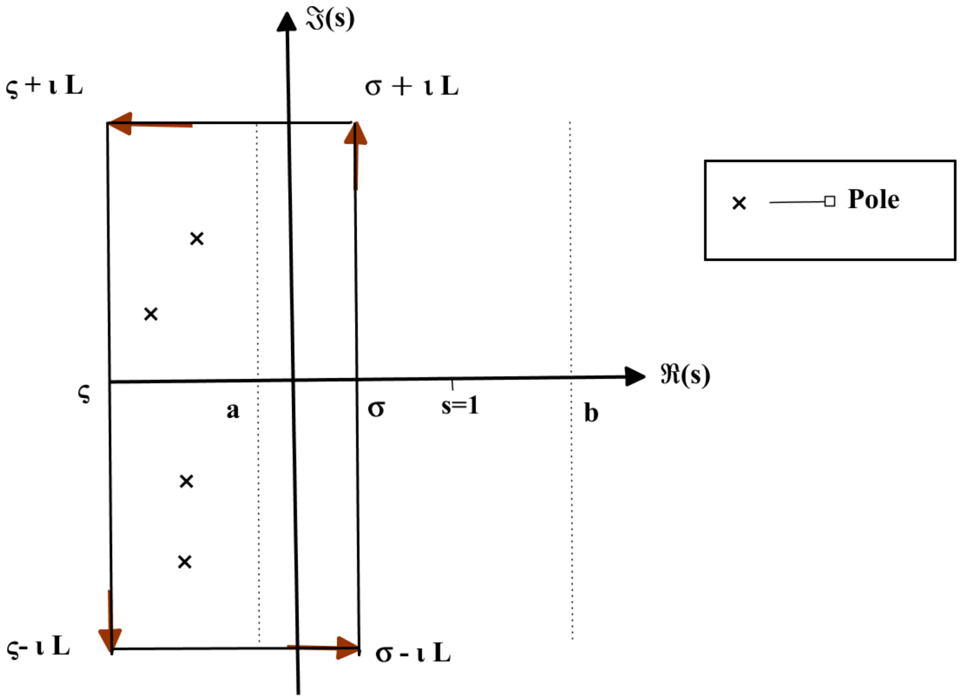

Figure 2.4.1: The contour $\mathcal{R}$ and the fundamental strip $S(a,b) = \{s \in \mathbb{C} : a < \Re(s) < b\}$.

as $L \to \infty$, the Riemann Lebesgue theorem (cf. [82], pg. 11) implies

$$\left| \int_{\varsigma+iL}^{\varsigma-iL} M(s)x^{-s}ds \right| \sim o(x^{-\varsigma}), \tag{2.4.3}$$

where $x \to 0^+$ and $\varsigma < 0$. Suppose $f(x)$ is continuous on the interval $(0, \infty)$ then the Riemann Lebesgue theorem implies that the two integrals along the horizontal lines tend to 0 as $\tau \to \infty$. The integral that lies within the fundamental strip therefore converges to the inverse Mellin transform as $L \to \infty$.

Using the Cauchy residue theorem, equation (2.4.3) therefore implies that

$$f(x) = 2\pi i \sum_{m=1}^{n} Res\left((M(s)x^{-s})_{s=s_m},\ \ s_m \in \omega \right) + \mathcal{O}(x^{-\varsigma}).$$

If there are no poles inside the contour $\mathcal{R}$ or if $\sum_{m=1}^{n} Res[(M(s)x^{-s})_{s=m} = 0$ then $f(x) = \mathcal{O}(x^{-\varsigma})$ which leads to $f(0) = 0$. Following the similar arguments it can be shown that asymptotic behaviour of f as $x \to \infty$ is given by

$$f(x) = -2\pi i \sum_{m=1}^{n} Res\left((M(s)x^{-s})_{s=s_q},\ \ s_q \in \omega \right) + \mathcal{O}(x^{-\eta}),$$

where $\eta > 0$ and s_q are the poles of M in the chosen contour.

van Brunt & Wake [87],[88] applied the Mellin transform to study functional equations that appear in cell growth model. Here, we use the Mellin transform to study the *Simplified Problem*. Applying the Mellin transform to both sides of the equation (2.1.2) yields

$$-(s-1)M(s-1) + cM(s) = c\frac{M(s)}{\alpha^{s-1}}. \tag{2.4.4}$$

Recasting equation (2.4.4) yields

$$M(s+1) = \frac{s}{c\left(1 - \frac{1}{\alpha^s}\right)} M(s). \tag{2.4.5}$$

For $s = 1$, equation (2.4.1) implies that

$$M(1) = 1. \tag{2.4.6}$$

We seek a solution of the form

$$M(s) = F(s)Q(s), \tag{2.4.7}$$

where $F(s)$ is the Mellin transform of a solution to the homogeneous equation

$$y'(x) + cy(x) = 0,$$

i.e., the transform of

$$y_h = e^{-cx},$$

which is

$$F(s) = c^{-s}\Gamma(s).$$

Note that $F(s)$ satisfies

$$-(s-1)F(s-1) + cF(s) = 0. \tag{2.4.8}$$

Substituting solution form (2.4.7) into equation (2.4.4) and using (2.4.8) yields

$$Q(s-1) = \left(1 - \frac{1}{\alpha^{s-1}}\right) Q(s). \tag{2.4.9}$$

A solution to equation (2.4.9) is

$$Q(s) = K \prod_{k=0}^{\infty} \left(1 - \frac{1}{\alpha^{k+s}}\right),$$

where K is a constant, and this gives

$$M(s) = Kc^{-s}\Gamma(s) \prod_{k=0}^{\infty} \left(1 - \frac{1}{\alpha^{k+s}}\right). \tag{2.4.10}$$

The simple poles of $\Gamma(s)$ are cancelled by the first order zeros $Q(s)$. This means that M is an entire function, since the infinite product consists of entire functions and is uniformly convergent in any compact set of $\mathbb{C}$. Condition (2.4.6) implies that

$$K = \frac{1}{\prod_{m=1}^{\infty}(1-\alpha^{-m})}. \tag{2.4.11}$$

As with the Laplace transform, the inverse Mellin transform of an infinite series is easier to derive than that of an infinite product. We thus use the *Euler identity* [3],

$$\prod_{k=0}^{\infty}(1+zq^k) = 1 + \sum_{k=1}^{\infty} \frac{q^{k(k-1)/2} z^k}{\prod_{j=1}^{k}(1-q^j)}, \tag{2.4.12}$$

to convert Q into an infinite series. Let $z = -\alpha^{-s}$ and $q = \alpha^{-1}$, then the *Euler identity* yields

$$Q(s) = 1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{\alpha^{k(k-1)/2} \prod_{j=1}^{k}(1-\alpha^{-j})\alpha^{ks}},$$

and consequently

$$M(s) = \frac{c^{-s}}{\prod_{m=1}^{\infty}(1-\alpha^{-m})}\Gamma(s)\left(1 + \sum_{k=1}^{\infty} \frac{(-1)^k}{\alpha^{k(k-1)/2} \prod_{j=1}^{k}(1-\alpha^{-j})\alpha^{ks}}\right). \tag{2.4.13}$$

Term by term inversion of (2.4.13) is straight forward given that $M[f(\alpha x)] = \frac{1}{\alpha^s}M[f(x)]$. We thus get

$$y(x) = \frac{1}{\prod_{m=1}^{\infty}(1-\alpha^{-m})}\left(e^{-cx} + \sum_{k=1}^{\infty} \frac{(-1)^k}{\alpha^{k(k-1)/2} \prod_{j=1}^{k}(1-\alpha^{-j})} e^{-c\alpha^k x}\right).$$

In the following section, we present a different avenue to show the uniqueness of equation (2.1.2). This approach is based on the proof of uniqueness of solutions to equation (2.4.7).

2.4.1 Uniqueness of the solution

The use of the Mellin transforms to solve problem (2.1.2) relies on solving the transform equation (2.4.4) by assuming the solution form $M(s) = F(s)Q(s)$, where F satisfies (2.4.8). There are problems for which the transform equation itself can prove formidable. One example is a second order pantograph equation that involves Hermite operator that we will discuss in chapter 3. The form of the Mellin transform to this problem is such that we cannot find the inverse Mellin transform. This motivates us to find another avenue to prove the uniqueness of the solution. In this section we introduce a new approach to show the uniqueness of the solution.

Consider the equation (2.1.2). The Mellin transform equation for (2.1.2) is (2.4.4).

Theorem 2.4.1. *There exists a unique solution $M(s)$ to equation (2.4.4) that satisfies the conditions*

A_1. *the inverse y must satisfy $y(x) \to 0$ as $x \to 0^+$ and as $x \to \infty$;*

is unique among functions that are meromorphic in a strip $S(u_0, v)$, where $u_0 < 0$ and $v > 1$;

A_2. $M(1) = 1$; *and*

A_3. M *is invertible.*

Suppose a solution to (2.4.4) is

$$M(s) = M_0(s)R(s), \tag{2.4.14}$$

where

$$M_0(s) = Kc^{-s}\Gamma(s)\prod_{k=0}^{\infty}\left(1-\frac{1}{\alpha^{k+s}}\right). \tag{2.4.15}$$

Then equation (2.4.5) implies that

$$R(s+1) = R(s). \tag{2.4.16}$$

Lemma 2.4.2. *The solution M is entire.*

Proof: Suppose M has only one pole $s_0 = u_0 + iv_0$. Condition (2.4.16) indicates that M must be a meromorphic function. Thus, the Laurent expansion of $M(s)$ for a pole of order m about s_0 is of the form

$$M(s) = \sum_{j=1}^{m}\frac{d_j}{(s-s_0)^j} + \tilde{P}(s), \tag{2.4.17}$$

where $\tilde{P}(s)$ and $\sum_{j=1}^{m}\frac{d_j}{(s-s_0)^j}$ are the analytic and principal parts respectively. Since

$$M[x^a \log^k(x)H(x)] = \left(\frac{(-1)^k k!}{(s+a)^{k+1}}\right)$$

where $H(x) = 1$ for $0 < x < 1$ and $k > 1$, taking the inverse Mellin transform of (2.4.17) gives

$$\begin{aligned} y(x) &= \sum_{j=1}^{m}\frac{d_j \log^j(x) x^{-(u_0+iv_0)\log(x)}}{j!}, \\ &= \sum_{j=1}^{m}\frac{d_j \log^j(x) e^{-(u_0+iv_0)\log(x)}}{j!}, \end{aligned}$$

(cf.[23], pg. 19-34). The asymptotic behaviour of y as $x \to 0^+$ will contain the term

$$\frac{d_m}{m!}\log^m(x)e^{-(u_0+iv_0)\log(x)}.$$

Evidently, if s_0 is the only pole then the boundary condition A_1 is not satisfied. The solution form (2.4.14) indicates that the presence of any pole in M depends only on R and we know from an early discussion in the section 2.4 that $M_0(s)$ is holomorphic for all s. However, the condition (2.4.16) implies that if there is a pole on a real line then the conditions A_1 and A_2 are not satisfied simultaneously. This rules out any poles on the real line. Therefore there must be another pole of order m at $s_1 = u_0 + iv_1$ on the line $\Re(s) = u_0$ and $v_1 \neq v_0$. The Mittag-leffler theorem gives

$$M(s) = \left(\sum_{j=1}^{m} \frac{d_j}{(s - s_0)^j} \right) + \left(\sum_{j=1}^{m} \frac{D_j}{(s - s_1)^j} \right) + \tilde{P}_1(s).$$

Suppose $d_m \neq 0$, $D_m \neq 0$ and $d_m = -D_m$. The asymptotic behaviour of $y(x)$ as $x \to 0^+$ will contain a term of the form

$$\frac{d_m}{m!} \log^m(x) e^{-u_0 \log(x)} \left(e^{-iv_0 \log(x)} - e^{-iv_1 \log(x)} \right).$$

In order to satisfy the boundary condition A_1, we require $v_1 = v_0$. The other poles if any must also influence the asymptotics. We therefore conclude that M has no poles. As a consequence $M \in H(\mathbb{C})$. ■

Corollary 2.4.3. *R is entire.*

Proof: Note that M_0 and M are entire functions. Since

$$R(s) = \frac{M(s)}{M_0(s)},$$

the holomorphicity of $R(s)$ depends on the zeros of M_0; the functions c^{-s}, and $\Gamma(s)$ do not have zeros in $\mathbb{C}$. The only possible zeros of M_0 must come from the infinite product. The zeros for this product are at $s = 0, -1, -2, ...$ and coincide with the poles of $\Gamma(s)$. It is simple to show that these zeros are of first order and occur where $\Gamma(s)$ has a simple pole. This implies that $M_0(-n) \neq 0$, so that M_0 does not vanish for any $s \in \mathbb{C}$. We conclude that $R \in H(\mathbb{C})$. ■

Corollary 2.4.4. $|R(s)| \sim \mathcal{O}\left(e^{\frac{\pi}{2}s}\right)$ *as* $s \to \infty$.

Proof: Let $s = u + iv$. Now

$$|R(s)| = \frac{|M(s)|}{|M_0(s)|},$$

where

$$|M_0(s)| = \left| K c^{-(u+iv)} \Gamma(u + iv) \prod_{k=0}^{\infty} \left(1 - \frac{1}{\alpha^{(u+iv)+k}} \right) \right|.$$

For $u > 0$,

$$|\alpha^{u+k+iv}| = |\alpha^{u+k}||e^{iv\log(\alpha)}|$$
$$= |\alpha^{u+k}| > 1.$$

Therefore,

$$0 < \prod_{k=0}^{\infty}\left(1 - \frac{1}{|\alpha^{u+k+iv}|}\right) \leq \left|\prod_{k=0}^{\infty}\left(1 - \frac{1}{\alpha^{u+k+iv}}\right)\right|,$$

that is $\left|\prod_{k=0}^{\infty}\left(1 - \frac{1}{\alpha^{u+k+iv}}\right)\right|$ is bounded away from 0 for all v and $\frac{1}{\prod_{k=0}^{\infty}\left(1-\frac{1}{|\alpha^{u+k+iv}|}\right)}$ is $\mathcal{O}(1)$ as $v \to \infty$. Evidently, $|c^{-(u+i\nu)}| = c^{-u}$, as $v \to \infty$. In addition, as $|v| \to \infty$, $|\Gamma(s)| \sim \mathcal{O}(e^{\frac{-v\pi}{2}})$, (cf.[1], pg. 257), and $|M(u+iv)| \sim o(1)$ by using Riemann-Lebesgue lemma [82]. Hence,

$$|R(s)| \sim \mathcal{O}(e^{\frac{v\pi}{2}}), \tag{2.4.18}$$

as $v \to \infty$ for $\Re(s) > 0$. Condition (2.4.16) immediately shows that (2.4.18) is also true for $\Re(s) \leq 0$. We therefore conclude that as $v \to \infty$, $|R(s)| \sim \mathcal{O}(e^{\frac{v\pi}{2}})$, for all $\Re(s)$. ■

To show that R must be a constant we use Carlson's theorem (cf.[81], pg. 186):

Theorem 2.4.5 (Carlson's theorem). *Let*

$$F \in H(\mathbb{C}),$$

and

$$F(z) = 0, \quad \textit{for} \quad z = 0, 1, 2, ...$$

If for any $0 \leq m_0 < \pi$, $F(z) \sim \mathcal{O}(e^{m_0|z|})$ *as* $z \to \infty$ *then* $F(z)$ *is a constant.*

Corollary 2.4.6. $R(s)$ *is constant.*

Proof: Let

$$H(s) = R(s) - R(0),$$

then condition (2.4.16) implies

$$H(s+1) = H(s),$$

which implies that $H(s) = 0$ at $s = 0, 1, 2, 3, ...$ We immediately see that $H(s) \in H(\mathbb{C})$ and $H(s) \sim \mathcal{O}(e^{\frac{\pi}{2}|s|})$. Carlson's theorem thus implies that $R(s)$ is a constant. ■

Hence; the uniqueness of the solution to the *Simplified Problem* is established. This technique can be applied to the second order pantograph equations provided conditions A_1, A_2, and A_3 are satisfied. However, this technique cannot be utilized for the *Hermite Problem* that will be discussed in section 3.3.3, because the Mellin transform for the *Hermite Problem* does not satisfy the condition A_2.

2.5 A technique based on the Mellin transform

The Mellin transform $M(s)$ derived in section 2.4 is of the form $M(s) = F(s)Q(s)$, where Q is represented as an infinite product. The inversion of M relies crucially on the conversion of Q to an infinite series via the *Euler identity*. The *Euler identity*, however, is specialized. For example, the equation

$$y''(x) + xy'(x) - y(\alpha x) = 0,$$

has a Mellin transform solution

$$M(s) = K\Gamma(s/2)\prod_{k=0}^{\infty}\left(1 - \frac{1}{\{s+2(k+1)\}\alpha^{s+2k+1}}\right), \tag{2.5.1}$$

for which the *Euler identity* is of limited use. There may be other partition function identities for particular cases, but in the absence of such identities we are confronted with somehow converting the infinite product to an infinite series. This can prove a formidable task, and it motivates us to seek another method whereby the conversion is avoided. Assuming that we have the Mellin transform in the form $M(s) = F(s)Q(s)$, where Q is an infinite product, it is natural to examine the inversion of $M_n(s) = F(s)Q_n(s)$, where $Q_n(s)$ is a partial product. Since we are dealing with products, this brings the Mellin convolution to the fore. If $M_1(s)$ and $M_2(s)$ are the Mellin transform of functions $f(x)$ and $g(x)$ then the Mellin convolution formula is

$$M^{-1}[M_1(s)M_2(s)] = f(x) * g(x) = \int_0^{\infty} f\left(\frac{x}{\tau}\right) g(\tau)\frac{d\tau}{\tau},$$

where * denotes the Mellin convolution and M^{-1} denotes the inverse Mellin transform. If $f(x)$ and $g(x)$ are two pdfs for random variables X and Y, then the Mellin convolution is the pdf of the product of those random variables [19]. The technique we describe has yet to be applied to transforms such as (2.5.1), but we illustrate it with the simple pantograph equation (2.1.2). The Mellin transform for equation (2.1.2) is given by equation (2.4.10). The partial product for Q is

$$Q_n(s) = K \prod_{k=0}^{n} \left(1 - \frac{1}{\alpha^{k+s}}\right).$$

The idea is to form a sequence of Mellin transforms

$$M_n(s) = F(s)Q_n(s), \tag{2.5.2}$$

and use the convolution formula to construct another sequence of inverse transforms

$$y_n(x) = M^{-1}[F(s)Q_n(s)].$$

We then show that

$$\lim_{n\to\infty} y_n(x) = y(x),$$

where y is the solution given by the equation (2.1.3) and K is given by the equation (2.4.11). Let

$$P_k(s) = 1 - \frac{1}{\alpha^{k+s}},$$

for $k = 0, 1, 2, ...$ The Mellin inverse transform of P_k is given by

$$p_k(x) = \delta(x-1) - \frac{1}{\alpha^k}\delta(\alpha x - 1),$$

where δ denotes the Dirac delta function. Equation (2.5.2) implies

$$\begin{aligned} M_0(s) &= KF(s)P_0(s), \\ M_1(s) &= M_0(s)P_1(s), \\ &\vdots \\ M_{n+1}(s) &= M_n(s)P_{n+1}(s), \end{aligned}$$

and in terms of convolutions we have

$$\begin{aligned} y_0(x) &= KM^{-1}[F(s)P_0(s)] = Kf(x) * p_0(x), \\ y_1(x) &= M^{-1}[M_0(s)P_1(s)] = y_0(x) * p_1(x), \\ &\vdots \\ y_{n+1}(x) &= M^{-1}[M_n(s)P_{n+1}(s)] = y_n(x) * p_{n+1}(x), \end{aligned}$$

where $f(x)$ denotes the inverse Mellin transform of $F(s)$. In this manner, we can construct a sequence $\{y_n\}$, that we will show converges to the solution (2.1.3). For our example, $f(x) = e^{-cx}$; hence,

$$\begin{aligned} y_0(x) &= Ke^{-cx} * \{\delta(x-1) - \delta(\alpha x - 1)\}, \\ &= K\int_0^\infty \frac{e^{-cx/\tau}}{\tau}\{\delta(\tau-1) - \delta(\alpha\tau - 1)\}d\tau, \\ &= K(a_{0,0}e^{-cx} - a_{1,0}e^{-c\alpha x}), \end{aligned}$$

where $a_{0,0} = 1$, $a_{1,0} = 1$. Continuing this process, we find

$$\begin{aligned} y_1(x) &= K(a_{0,1}e^{-cx} - a_{1,1}e^{-c\alpha x} + a_{2,1}e^{-c\alpha^2 x}), \\ y_2(x) &= K(a_{0,2}e^{-cx} - a_{1,2}e^{-c\alpha x} + a_{2,2}e^{-c\alpha^2 x} - a_{3,2}e^{-c\alpha^3 x}), \\ &\vdots \\ y_n(x) &= K(a_{0,n}e^{-cx} - a_{1,n}e^{-c\alpha x} + a_{2,n}e^{-c\alpha^2 x} - a_{3,n}e^{-c\alpha^3 x} + ... + (-1)^n a_{k+1,n}e^{-c\alpha^{k+1} x}), \end{aligned}$$

where, for

$$S_j^1 = \sum_{m_1=0}^{j} \frac{1}{\alpha^{m_1}} = \frac{1-(\frac{1}{\alpha})^{j+1}}{1-\frac{1}{\alpha}},$$

we have

$$\begin{aligned} a_{0,n} &= 1, \\ a_{1,n} &= S_n^1, \\ &= \left(1 + \frac{1}{\alpha} + \frac{1}{\alpha^2} + ... + \frac{1}{\alpha^n}\right), \\ &= \frac{1-(\frac{1}{\alpha})^{n+1}}{1-\frac{1}{\alpha}}, \end{aligned}$$

and

$$a_1 = \lim_{n\to\infty} a_{1,n} = \frac{1}{1-\alpha^{-1}}.$$

Continuing the same procedure for a_2 gives

$$a_{2,n} = \frac{1}{\alpha}\sum_{m_2=0}^{n-1} \frac{1}{\alpha^{2m_2}} S_{n-1-m_2}^1,$$

where

$$S^1_{n-1-m_2} = \sum_{m_1=0}^{n-1-m_2} \frac{1}{\alpha^{m_1}},$$
$$= \frac{1-(\frac{1}{\alpha})^{n-m_2}}{1-\frac{1}{\alpha}};$$

hence,

$$a_{2,n} = \frac{1}{\alpha}\sum_{m_2=0}^{n-1}\frac{1}{\alpha^{2m_2}}\left(\frac{1-(\frac{1}{\alpha})^{n-m_2}}{1-\frac{1}{\alpha}}\right),$$
$$= \frac{1}{\alpha(1-\alpha^{-1})}\left(\sum_{m_2=0}^{n-1}\frac{1}{\alpha^{2m_2}} - \sum_{m_2=0}^{n-1}\frac{1}{\alpha^{n+m_2}}\right),$$
$$= \frac{1}{\alpha(1-\alpha^{-1})}\left(\frac{1-(\frac{1}{\alpha^2})^n}{1-\frac{1}{\alpha^2}} - \frac{1}{\alpha^n}.\frac{1-(\frac{1}{\alpha})^n}{1-\frac{1}{\alpha}}\right),$$

and thus

$$a_2 = \lim_{n\to\infty} a_{2,n} = \frac{1}{\alpha(1-\alpha^{-1})(1-\alpha^{-2})}.$$

Proceeding in the same manner, we have

$$a_{3,n} = \frac{1}{\alpha^3}\sum_{m_3=0}^{n-2}\frac{1}{\alpha^{3m_3}}S^2_{n-2-m_3},$$

where

$$S^2_{n-2-m_3} = \frac{1}{(1-\alpha^{-1})}\left(\frac{1-(\frac{1}{\alpha^2})^{n-1-m_3}}{1-\frac{1}{\alpha^2}} - \frac{1}{\alpha^n}.\frac{1-(\frac{1}{\alpha})^{n-1-m_3}}{1-\frac{1}{\alpha}}\right).$$

Consequently,

$$a_{3,n} = \frac{1}{\alpha^3}\frac{1}{(1-\alpha^{-1})(1-\alpha^{-2})}\left(\frac{1-(\frac{1}{\alpha^3})^{n-1}}{1-\frac{1}{\alpha^3}} - \frac{1}{\alpha^{n-1}}.\frac{1-(\frac{1}{\alpha^2})^{n-1}}{1-\frac{1}{\alpha^2}}\right)$$
$$-\frac{1}{\alpha^3\alpha^{n-1}(1-\alpha^{-1})(1-\alpha^{-2})}\left(\frac{1-(\frac{1}{\alpha^3})^{n-1}}{1-\frac{1}{\alpha^3}} - \frac{1}{\alpha^{n-1}}.\frac{1-(\frac{1}{\alpha^2})^{n-1}}{1-\frac{1}{\alpha^2}}\right),$$

and

$$a_3 = \lim_{n\to\infty} a_{3,n} = \frac{1}{\alpha^3(1-\alpha^{-1})(1-\alpha^{-2})(1-\alpha^{-3})}.$$

Continuing this process we get

$$a_{k,n} = \frac{1}{\alpha^{k(k-1)/2}}\sum_{m_k=0}^{n-(k-1)}\frac{1}{\alpha^{km_k}}S^{k-1}_{n-(k-1)-m_k},$$

where S^k_j denotes the product of k geometric series where every series has j terms in it. This leads to

$$a_k = \lim_{n\to\infty} a_{k,n} = \frac{1}{\alpha^{k(k-1)/2}\prod_{j=1}^{k}(1-\alpha^{-j})}.$$

We thus get the solution

$$y(x) = \lim_{n\to\infty} y_n(x) = K\left(e^{-cx} + \sum_{k=1}^{\infty}(-1)^k a_k e^{-c\alpha^k x}\right).$$

The *Integrability Condition* gives

$$K = \frac{1}{\prod_{r=1}^{\infty}(1-\alpha^{-r})}.$$

2.6 A numerical Solution

A functional differential equations such as

$$(G(x)y(x))' + (B(x)+\lambda) = \alpha^2 B(\alpha x)y(\alpha x)$$

cannot, in general, be solved analytically. For example, the above techniques are of limited help for equations such as

$$y' + (bx + \Lambda)y(x) = b\alpha^3 xy(\alpha x). \tag{2.6.1}$$

For such problems a numerical approach is useful. Here, we discuss a numerical approach to solve a functional differential equation by applying it to equation (2.1.2).

Consider a truncated domain $[0, L]$, where $L > 0$. For a truncated domain equations (2.1.2) yields

$$\begin{cases} y_L'(x) + cy_L(x) = c\alpha y_L(\alpha x), \quad 0 < x < L/\alpha, \\ \\ y_L'(x) + cy_L(x) = 0, \quad L/\alpha \le x < L \\ \\ y_L(0) = y_L(L) = 0, \quad y_L(x) > 0 \quad \text{for} \quad x > 0, \end{cases} \tag{2.6.2}$$

and

$$\int_0^L y_L(x)dx = 1. \tag{2.6.3}$$

The functional term $y_L(\alpha x)$ in the equation (2.6.2) vanishes when $x > L/\alpha$. We now use finite difference scheme to solve the *Simplified Problem*. A backward difference approximation for the derivative of a function y_L is defined as

$$y_i' = \frac{y_i - y_{i-1}}{h},$$

where $y_i = y(x_i)$. Dividing interval $(0, L)$ into n subintervals, choosing the interval size equal to $\frac{L}{n}$ and applying the finite difference method to equation (2.6.2) gives a discretized equation

$$\begin{cases} \left(\frac{n}{L} + c\right) y_i - \frac{n}{L} y_{i-1} = \alpha c y_{i\alpha}, \quad 1 < i < L/h, \\ \\ \left(\frac{n}{L} + c\right) y_i - \frac{n}{L} y_{i-1} = 0, \quad L/h \le i < L, \\ \\ y_0 = y_L = 0, \end{cases} \quad . \tag{2.6.4}$$

where $y_{i\alpha} = y(\alpha x_i)$. To evaluate $y(\alpha x_i)$ at any grid point x_i, we set $y(\alpha x_i) = 0$ for all $\alpha x_i \ge L$. For an integer value of α, discretized equation (2.6.4) generates a square matrix and for a non integer α, a linear interpolation technique can be employed to generate a square matrix. The later case is detailed in chapter 4. Solving equation (2.6.4) together with the condition (2.6.3) and the trapezoidal rule gives a solution that is shown in fig 2.6.1.

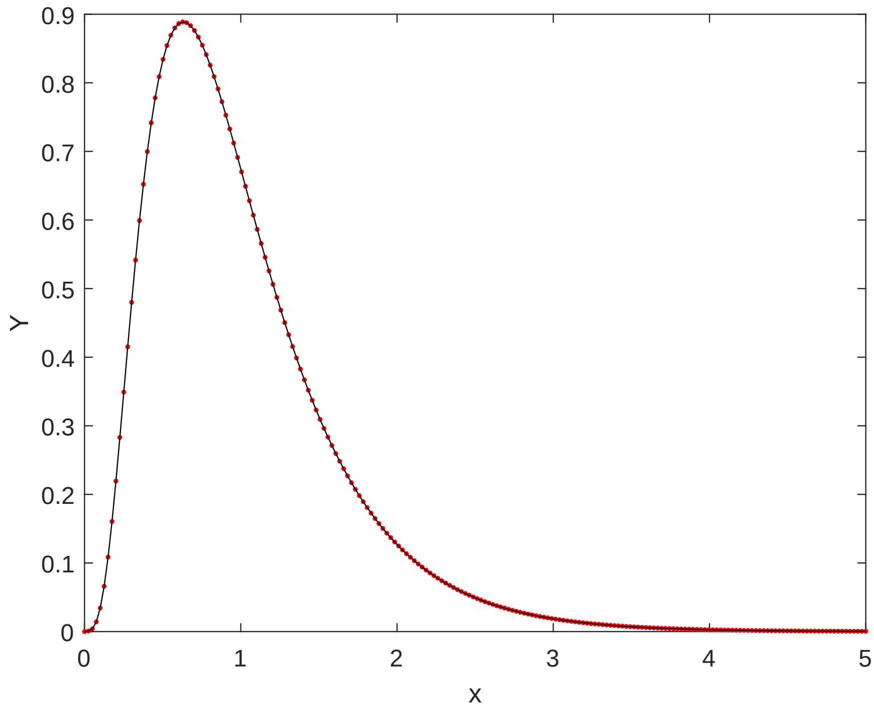

Figure 2.6.1: Graph of numerical solution of equation $y'(x) + cy(x) = c\alpha y(\alpha x)$ for $\alpha = 2$, $c = 2$, $L = 5$, $n = 200$

Numerical approach is useful when analytical techniques are of limited help. Equation (2.6.1) has not been solved analytically but it is straightforward to implement the numerical scheme.

Integrating equation (2.6.1) from 0 to ∞ gives

$$\Lambda = b(\alpha - 1) \int_0^\infty xy(x)dx. \tag{2.6.5}$$

Let $\int_0^\infty xy(x)dx = E_1$ then equation (2.6.5) yields,

$$\Lambda = b(\alpha - 1)E_1. \tag{2.6.6}$$

Now multiplying equation (2.6.1) by x and then integrating from 0 to ∞ gives

$$\Lambda E_1 = 1. \tag{2.6.7}$$

Equations (2.6.6) in (2.6.7) shows that,

$$\Lambda = \sqrt{b(\alpha - 1)}. \tag{2.6.8}$$

Fig 2.6.2 that is drawn for different values of Λ suggests that a positive solution to equation (2.6.1) exists for $\Lambda = \sqrt{b(\alpha - 1)}$. There is no direct application of this tech-

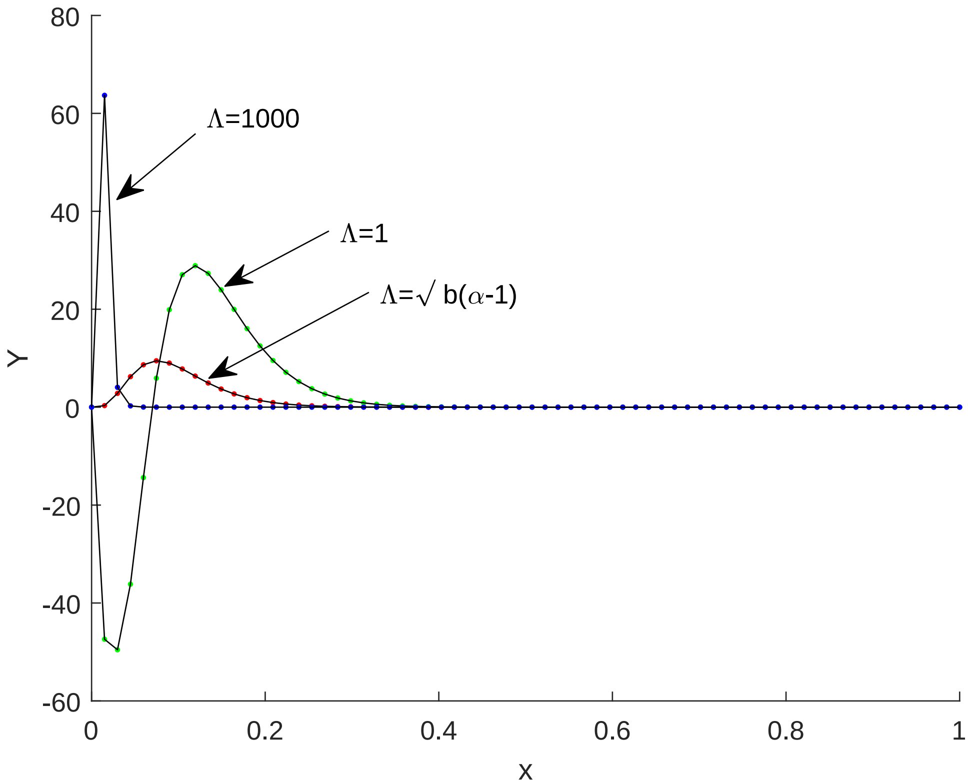

Figure 2.6.2: Solution graph of equation (2.6.1) for different values of Λ with $\alpha = 2$, $b = 100$, $L = 1$ and $n = 68$.

nique to any problem later in the thesis. However in chapters 4 and 5, when we apply the Perron Frobenius theorem to show the uniqueness of the the eigenvalue for a first order functional differential equation with a non constant division rate, we discretize the equation under consideration using the finite difference scheme along with some matrix properties.

Chapter 3

Pantograph equations

In this chapter, we begin with a brief discussion of the first order pantograph equations with non constant coefficients and then consider a generalization to second order pantograph equations with constant and non constant coefficients. Pantograph type equations arise in the cell growth model when the Pde that governs the cell growth model has a separable solution. A detailed discussion of pantograph type equations is given in chapter 1. Various techniques that can be used to solve pantograph type equations are given in chap 2. Here, we discuss a few particular second order pantograph equations with non constant coefficients such as modified Bessel equation and Airy equations. In addition, we study a pantograph equation that involves a Hermite type operator and show that there exists a class of non trivial solutions to this problem.

3.1 First order pantograph equations with non constant coefficients

The nature of solutions to pantograph equations depends on a different choices of growth and division rates. Perthame & Ryzhik [64] showed that under certain conditions on the division rate, there exists a unique solution to pantograph equations with a constant growth rate. Review of Perthame & Ryzhik's work will be discussed in chapter 4. van Brunt & Hulstman [87] studied the equation

$$y'(x) + bx^n y(x) = \lambda \alpha^n x^n y(\alpha x), \tag{3.1.1}$$

for $n \geq 0$, subject to the *Boundary Conditions*

$$y(0) = 0, \quad \lim_{x \to \infty} y(x) = 0,$$

and the *Pdf Conditions.* Taking the Mellin transform to the equation (3.1.1) gives

$$-(s-1)M(s-1)+bM(s+n)=\frac{\lambda}{\alpha^s}M(s+n).$$

They obtained

$$M(s)=\frac{K}{n+1}\left(\frac{b}{n+1}\right)^{\frac{-s}{n+1}}\Gamma\left(\frac{s}{n+1}\right)\prod_{k=0}^{\infty}\left(1-\frac{\lambda}{b\alpha^{k(n+1)+1+s}}\right), \tag{3.1.2}$$

where K is a constant. Equation (3.1.2) is defined for all $\Re(s)>0$. The *Integrability Condition* implies that

$$M(1)=1, \tag{3.1.3}$$

and this condition can be satisfied for a suitable choice of K provided

$$\prod_{k=0}^{\infty}\left(1-\frac{\lambda}{b\alpha^{k(n+1)+2}}\right)\neq 0, \tag{3.1.4}$$

or in other words $\lambda \neq b\alpha^{k(n+1)+1}$ for any non negative integer k. However by integrating equation (3.1.1) from 0 to ∞ they obtained

$$\left(b-\frac{\lambda}{b\alpha}\right)M(n+1)=0,$$

which implies that for $s=0$ the infinite product (3.1.4) can be zero if $\lambda=b\alpha$ or $M(n+1)=0$. van Brunt & Hulstman thus found that the spectrum of eigenvalues $\{\lambda_m\}$ are

$$\lambda_m=b\alpha^{m(n+1)+1},$$

where m is a non negative integer. The choice of eigenvalues $\lambda\in\{\lambda_m\}$ ensured that $M(s)$ is defined for $s\leq 0$. van Brunt & Hulstman then inverted the M and obtained

$$y_m(x)=K_m\left(e^{\frac{-bx^{n+1}}{n+1}}+\sum_{k=1}^{\infty}p_k(\lambda_m)e^{\frac{-b\alpha^{k(n+1)}x^{n+1}}{n+1}}\right).$$

They first showed that the first eigenvector y_0 is positive and is a unique solution to a boundary value problem (3.1.1) under the condition

$$\int_0^{\infty}x^n y(x)dx<\infty,$$

and then established the uniqueness for the higher eigenvectors in using similar arguments that they used to show the uniqueness for the first eigenvector.

Hall & Wake [30] studied the equation

$$g(xy(x))' + (bx^n + \lambda)y(x) = b\alpha^2(\alpha x)^n y(\alpha x), \tag{3.1.5}$$

where $n \geq 0$, and g and b are positive constants, subject to the *Pdf Conditions.* This equation arises from the equation (1.4.8) for a linear growth and a non constant division rate. Using (1.4.12) they obtained $\lambda = g$. Hall & Wake used the transformation $y(x) = w(x)$, where

$$w(x) = \frac{x^n}{n}, \tag{3.1.6}$$

to convert the equation (3.1.5) into the form solved in their earlier paper [29], which was detailed in section 2.2. Here, we apply the Mellin transform to obtain the solution of equation (3.1.5).

For $\lambda = g$, equation (3.1.5) becomes

$$gxy'(x) + bx^n y(x) + 2gy(x) = b\alpha^2(\alpha x)^n y(\alpha x). \tag{3.1.7}$$

Taking the Mellin transform of both sides of the equation (3.1.7) gives

$$-g(s-2)M(s) + bM(s+n) = b\frac{1}{\alpha^{s-2}}M(s+n). \tag{3.1.8}$$

We seek a solution of the form $M(s) = F(s)Q(s)$. Evidently,

$$F(s) = \frac{1}{n}\left(\frac{b}{gn}\right)^{-\frac{s-2}{n}} \Gamma\left(\frac{s-2}{n}\right),$$

where $F(s)$ is the Mellin transform of the solution to the homogeneous equation

$$gxy'(x) + (bx^n + 2g)y(x) = 0,$$

that is

$$y_h = x^{-2}e^{-\frac{b}{gn}x^n};$$

and satisfies

$$-g(s-2)F(s) + bF(s+n) = 0. \tag{3.1.9}$$

Substituting solution form (2.4.7) into equation (3.1.8) and using equation (3.1.9) gives

$$Q(s) = K \prod_{k=0}^{\infty} \left(1 - \frac{1}{\alpha^{s+kn-2}}\right),$$

where K is a constant, and this gives

$$M(s) = \frac{K}{n}\left(\frac{b}{gn}\right)^{-\frac{s-2}{n}} \Gamma\left(\frac{s-2}{n}\right) \prod_{k=0}^{\infty}\left(1 - \frac{1}{\alpha^{nk+s-2}}\right).$$

Although F has a simple pole at $s = 1$, a simple pole at $s = 1$ appears for $n = 1$ only, Q has a zero of order 1 at $s = 1$; the situation is the same at $s = 2$. We thus see that M is a function holomorphic in the half plane $\Re(s) > 0$. The *Euler identity* can be applied to Q and this leads to

$$y(x) = \frac{K}{n} x^{-2} \left(e^{-\frac{b}{gn}x^n} + \sum_{k=1}^{\infty} p_k e^{\frac{-b}{gn}\alpha^{nk}x^n} \right),$$

where

$$p_k = \frac{(-1)^k}{\alpha^{nk(k-1)/2} \prod_{j=1}^{k}(1 - \alpha^{-nj})},$$

and K can be obtained using the *Integrability Condition.*

Generally, a first order pantograph equations with non constant G and B cannot be solved analytically. Nonetheless, it is possible to extract some analytical information. Here, we look briefly at the influence G and B have on the decay of the solutions as $x \to \infty$.

Consider the equation

$$(G(x)y(x))' + (B(x) + \lambda)y(x) = \alpha^2 B(\alpha x)y(\alpha x), \tag{3.1.10}$$

where $G \geq 0$, $B \geq 0$. Suppose $y(x)$ is a solution to (3.1.10) that satisfies the *Boundary Conditions*, and the *Pdf Conditions.* Further suppose that

$$By \in L^1[0, \infty). \tag{3.1.11}$$

Integrating equation (3.1.10) from 0 to ∞ and then using the *No Flux Conditions 4*

$$G(x)y(x) \to 0 \quad \text{as} \quad x \to 0^+ \quad \text{and} \quad x \to \infty,$$

gives

$$\lambda = (\alpha - 1)\int_0^\infty B(\xi)y(\xi)d\xi.$$

Let

$$y(x) = \frac{w(x)}{G(x)}, \tag{3.1.12}$$

then recasting equation (3.1.10) yields

$$w^{'}(x) + (h(x) + p(x))\, w(x) = \alpha^2 h(\alpha x) w(\alpha x), \tag{3.1.13}$$

where

$$h(x) = \frac{B(x)}{G(x)},$$

and

$$p(x) = \frac{\lambda}{G(x)}.$$

In the following theorem we prove that if

$$h(x) \in L^1[0, \infty), \tag{3.1.14}$$

and

$$p(x) \in L^1[0, \infty), \tag{3.1.15}$$

then any solution to equation (3.1.10) cannot meet the *No Flux Conditions 4*.

Theorem 3.1.1. *If $h, p \in L^1(x_1, \infty)$ for some $x_1 \geq 0$ then $w \not\to 0$ as $x \to \infty$; consequently, a pdf solution y to (3.1.10) that satisfies (3.1.11) cannot satisfy the No Flux Conditions 4.*

Proof: Let $H'(x) = h(x)$ and $P'(x) = p(x)$. Then equation (3.1.13) implies

$$\left(w(x)e^{H(x)+P(x)}\right)^{'} = \alpha^2 e^{H(x)+P(x)} h(\alpha x) w(\alpha x). \tag{3.1.16}$$

Since y is non trivial and non negative, there exists an $x_0 > 0$ such that

$$w(x_0) > 0.$$

Integrating equation (3.1.16) from x_0 to x yields

$$w(x)e^{H(x)+P(x)} = c + \alpha^2 \int_{x_0}^{x} e^{H(\xi)+P(\xi)} h(\alpha\xi)w(\alpha\xi)d\xi, \tag{3.1.17}$$

where $c = w(x_0)e^{H(x_0)+P(x_0)}$. Since

$$h, p \in L^1[x_0, \infty),$$

there is a number l such that

$$\lim_{x\to\infty} (H(x) + P(x)) = l;$$

moreover, since h and p are non negative we have $H(x)+P(x) \leq l$ and thus $e^{-(H(x)+P(x))} > e^{-l}$. Equation (3.1.17) therefore implies

$$w(x) \geq ce^{-l} > 0. \tag{3.1.18}$$

Thus,

$$y(x) \geq \frac{ce^{-l}}{G(x)},$$

for all $x \geq x_0$ and consequently

$$G(x)y(x) \geq ce^{-l} > 0,$$

for all $x > x_0$ and this contradicts the *No Flux Conditions 4*. Hence we conclude that if $h, p \in L^1[x_0, \infty)$ then the *No Flux Conditions 4* is not satisfied. ■

It is illuminating to consider the special case when $G(x) = x^r$ and $h \in L^1[x_0, \infty)$. If $r > 1$, then the conditions of the previous theorem are satisfied and we know that the *No Flux Conditions 4* as $x \to \infty$ cannot be satisfied. Now, $G(x)$ is the growth rate of a cell of size x so that

$$x'(t) = G(x) = x^r;$$

consequently,

$$\frac{x^{1-r}(t)}{1-r} = t + x_0,$$

where x_0 is a constant. The above expression shows that

$$x(t) = (r-1)^{\frac{1}{1-r}}(x_0 + t)^{\frac{1}{1-r}},$$

and since $r > 1$, the cell size "blows up" in finite time.

Suppose now that $r = 1$, the $e^{P(x)} = x^{\lambda}$, and equation (3.1.17) shows that

$$y(x) \geq \frac{c^*}{x^{1+\lambda}}, \tag{3.1.19}$$

for $x \geq x_0$, where $c^* > 0$ is a constant. Equation (3.1.19) indicates if a solution to (3.1.10) decays to zero then it must be an algebraic decay.

If $r < 1$, then $e^{P(x)} = e^{\lambda\frac{x^{1-r}}{1-r}}$ so that we get, for $x \geq x_0$ and some constant $c^* > 0$,

$$y(x) \geq \frac{c^*}{x} e^{-\lambda\frac{x^{1-r}}{1-r}}.$$

In this case it is possible that y decays exponentially as $x \to \infty$.

We note that the condition $p, h \in L^1[0,\infty)$ is not necessary for solutions of (3.1.10) to fail or to satisfy either the *No Flux Conditions 4* or the pdf condition on y. A tractable example is when $G(x) = x$ and $B(x) = b$, where $b > 0$ is a constant. In this case neither p nor h are in $L^1[x_0,\infty)$ for any choice of $x_0 > 0$. If y is a pdf then we know that $\lambda = b\alpha$ and hence equation (3.1.10) is

$$(xy(x))' + b\alpha y(x) = b\alpha^2 y(\alpha x). \tag{3.1.20}$$

Suppose that there exists a pdf solution y such that the *No Flux Conditions 4* are satisfied. Then, for $xy(x) = w(x)$, equation (3.1.20) becomes

$$xw'(x) + b\alpha w(x) = b\alpha w(\alpha x). \tag{3.1.21}$$

Since y is a pdf, w is a non trivial solution and the *No Flux Conditions 4* indicates that $w(x) \to 0$ as $x \to 0^+$ and as $x \to \infty$. We conclude that w must have a positive global maximum. Since $w(x) \to 0$ as $x \to \infty$, there exists a largest value $x_m < \infty$ at which w achieves this maximum. Now, $w'(x_m) = 0$, and equation (3.1.21) gives

$$w(x_m) = w(\alpha x_m),$$

but $\alpha > 1$ so that w must achieve its maximum at $\alpha x_m > x_m$ contradicting the definition of x_m. We thus conclude that there is no solution to equation (3.1.20) that is a pdf and satisfies the *No Flux Conditions 4*. Note that an obvious non trivial solution to (3.1.20) is $y(x) = \frac{y_0}{x}$, where y_0 is a constant.

We can deal with the case $B(x) = b$ and $G(x) = x^r$ where $r < 1$. In this case equation (3.1.10) becomes

$$(x^r y(x))' + (b+\lambda)y(x) = b\alpha^2 y(\alpha x). \tag{3.1.22}$$

In the above equation, $h = \frac{b}{x^r} \in L^1[0,\infty)$ and $p = \frac{\lambda}{x^r} \in L^1[0,\infty)$. Let

$$y(x) = \frac{w(x)}{x^r},$$

then equation (3.1.22) becomes

$$x^r w'(x) + (b+\lambda)w(x) = b\alpha^{2-r} w(\alpha x). \tag{3.1.23}$$

Let

$$\psi(z(x)) = w(x),$$

where

$$z(x) = \frac{x^{1-r}}{1-r}, \tag{3.1.24}$$

so that $z'(x) = x^{-r}$. Then

$$\begin{aligned} x^r w'(x) &= x^r \Psi'(z) z'(x), \\ &= \Psi'(z), \end{aligned}$$

and

$$\begin{aligned} z(\alpha x) &= \alpha^{1-r} \frac{x^{1-r}}{1-r}, \\ &= \beta z(x), \end{aligned}$$

where $\beta = \alpha^{1-r} > 1$. Equation (3.1.23) can thus be written

$$\Psi'(z) + (b+\lambda)\Psi(z) = b\alpha\beta\Psi(\beta z). \tag{3.1.25}$$

The *Boundary conditions* give

$$\Psi(z) \to 0 \quad \text{as} \quad z \to 0^+, \quad z \to \infty. \tag{3.1.26}$$

Equation (3.1.25) is a first order pantograph equation that together with the boundary condition (3.1.26) has a solution of the form

$$\Psi(z) = \sum_{k=0}^{\infty} c_k e^{-b\alpha\beta^k z},$$

which in terms of y is

$$y(x) = \frac{\sum_{k=0}^{\infty} c_k e^{-b\alpha\beta^k \frac{x^{1-r}}{1-r}}}{x^r}.$$

We conclude problem (3.1.23) together with the boundary conditions, decays exponentially for $r < 1$ as $x \to \infty$.

3.2 Second order pantograph equations with constant coefficients

In this section we solve a second order pantograph equation with constant coefficients using the techniques developed in chapter (2). Recall the second order pantograph equation (1.4.3)

$$(D(x)y(x))'' - (G(x)y(x))' - (B(x) + \lambda)y(x) + \alpha^2 B(\alpha x)y(\alpha x) = 0.$$

This equation is supplemented with the *No Flux Conditions 3* and the *Pdf Conditions*.

Kim [38] studied the related equation

$$y''(x) + ay'(x) + by(x) + cy(\alpha x) = 0, \tag{3.2.1}$$

with the *Integrability Condition*, boundary conditions (**??**), and $y'(0)+ay(0) = b+c/\alpha$, where a, b, c are positive. The observation that, the first order version of equation (3.2.1) has a Dirichlet series solution, motivated Kim to construct a solution to equation (3.2.1) of the form

$$y(x) = \sum_{k=0}^{\infty} a_k e^{-\alpha^k r x}, \tag{3.2.2}$$

where a_k and r are constants. Substituting (3.2.2) and its derivatives y' and y'' in (3.2.1) gives

$$r^2 \sum_{k=0}^{\infty} a_k \alpha^{2k} e^{-\alpha^k r x} - ar \sum_{k=0}^{\infty} a_k \alpha^k e^{-\alpha^k r x} + br^2 \sum_{k=0}^{\infty} a_k e^{-\alpha^k r x} + c \sum_{k=0}^{\infty} a_{k-1} e^{-2\alpha^k r x} = 0.$$

Equating the coefficients of $e^{-\alpha r x}$ and $e^{-\alpha^k r x}$ yields the indicial equations

$$r^2 - ar + b = 0,$$

and

$$\left(\alpha^{2k} r^2 - a\alpha^k r + b\right) a_k = -c\alpha a_{k-1}, \quad k \geq 1,$$

respectively. Kim showed that for $|c| < a|b|$, there exists a unique and positive solution to the problem (3.2.1) that has exactly one maximum. van Brunt *et al.* [89] studied the equation

$$y''(x) - ay'(x) - by(x) + \lambda y(\alpha x) = 0, \tag{3.2.3}$$

with the *Boundary Conditions* and the *Integrability Condition*, for $a > 0$, $b > 0$. We refer equation (3.2.3) along with the *Boundary Conditions* and the *Integrability Condition* as *Constant Coefficient Problem 1.* van Brunt *et al.* regarded the *Constant Coefficient Problem 1* as a type of singular Sturm-Liouville problem and in this sense, λ played the role of an eigenvalue parameter. They found the Dirichlet series solution

$$y(x) = C_0 \left(e^{-r_1 x} + \sum_{k=1}^{\infty} \frac{(-\lambda)^k e^{-r_1 \alpha^k x}}{\prod_{j=1}^{k} (\alpha^{2j} r_1^2 + a\alpha^j r_1 - b)} \right),$$

where

$$C_0 = r_1 \left(1 + \sum_{k=1}^{\infty} \frac{(-\frac{\lambda}{\alpha})^k}{\prod_{j=1}^{k} (\alpha^{2j} r_1^2 + a\alpha^j r_1 - b)} \right)^{-1},$$

and

$$r_1 = \frac{-a + \sqrt{a^2 + 4b}}{2}.$$

Let

$$F(\lambda) = \left(1 + \sum_{k=1}^{\infty} \frac{(-\lambda)^k}{\prod_{j=1}^{k}(\alpha^{2j} r_1^2 + a\alpha^j r_1 - b)}\right).$$

Evidently, $y(0) = C_0 F(\lambda)$. van Brunt *et al.* found that eigenvalues are the zeros of the entire function F, and a solution to the *Constant Coefficient Problem 1* exists for certain values of λ. Using the Laplace transform they gleaned the information regarding the uniqueness of corresponding eigenfunctions. It was shown that there exists a unique solution to the *Constant Coefficient Problem 1* for $\lambda \in \Omega$, where Ω is the set of zeros of F such that λ is a real eigenvalue and $b\alpha^n \leq \lambda \leq b\alpha^{n+1}$, for some $n = 1, 2, ...$

Wake *et al.* [92] studied the *Constant Coefficient Problem 1* for $\lambda = b\alpha$, $b > 0$, $a > 0$, subject to the boundary conditions $y'(0) - y(0) = 0$, and (**??**). Hall & Wake [28], [29] solved a first order analogue to the *Constant Coefficient Problem 1* for $\lambda = b\alpha$. In [92], a Dirichlet series solution was constructed and it was shown that the solution is positive and unique.

We now consider a case of the *Constant Coefficient Problem 1*.

Let $b \to b\alpha$ and $\lambda \to b\alpha^2$, then the *Constant Coefficient Problem 1* becomes

$$y''(x) - gy'(x) - (b + \lambda)y(x) + b\alpha^2 y(\alpha x) = 0, \tag{3.2.4}$$

where g and b are positive constants. Integrating equation (3.2.4) from 0 to ∞ and using the *No Flux Conditions 3* and the *Integrability Condition* gives

$$\lambda = b(\alpha - 1).$$

Equation (3.2.4) thus reduces to

$$y''(x) - gy'(x) - b\alpha y(x) + b\alpha^2 y(\alpha x) = 0. \tag{3.2.5}$$

For simplicity, we refer equation (3.2.5) along with the *No Flux Conditions 3* and the *Integrability Condition* as *Constant Coefficient Problem 2*.

For the sake of completeness we give the uniqueness proof for the *Constant Coefficient Problem 2*. This approach will be used in section 3.3.3 to prove the uniqueness of the *Hermite Problem* subject to the boundary condition (**??**). This uniqueness proof is given in [88]. Following the same line of proof given in [88] we use a cumulative distribution function to show that if the *Constant Coefficient Problem 2* has a pdf solution then it must be positive for $x > 0$ and that the solution must be unique. Let

$$w(x) = \int_x^\infty y(\xi)d\xi.$$

The *Integrability Condition* shows that

$$w(0) = 1, \tag{3.2.6}$$

and the definition of w implies

$$\lim_{x\to\infty} w(x) = 0. \tag{3.2.7}$$

Integrating the equation (3.2.5) from x to ∞ gives

$$w''(x) - gw'(x) - b\alpha w(x) + b\alpha w(\alpha x) = 0. \tag{3.2.8}$$

We refer equation (3.2.8) along with conditions (3.2.6) and (3.2.7) as *w Problem.* We first establish that any solutions to the *w Problem* cannot have a point at which the derivative vanishes. More precisely, we have the following lemma.

Lemma 3.2.1. *Any solution to the w Problem cannot have a point $x_0 > 0$, at which $w'(x_0) = 0$, unless $w(x) = 0$ for all $x \geq 0$.*

Proof: We first show that w cannot have local extrema.

Suppose $w(x)$ has a positive local maximum at $x_0 > 0$ then $w'(x_0) = 0$, $w''(x_0) \leq 0$, $w(x_0) > 0$, and equation (3.2.8) implies

$$w(x_0) \leq w(\alpha x_0).$$

Condition (3.2.7) implies that there exists another local positive maximum $x_1 \geq \alpha x_0 > x_0$ and $w(x_0) \leq w(\alpha x_0)$. Repeating this argument we can thus construct $\{x_k\}$ and $\{w(x_k)\}$ such that

$$x_k \geq \alpha^k x_0, \tag{3.2.9}$$

and

$$w(x_k) \geq w(x_0) > 0. \tag{3.2.10}$$

Evidently (3.2.9) implies $x_k \to \infty$ as $k \to \infty$ and (3.2.10) shows that condition (3.2.7) cannot be satisfied. We thus conclude that we cannot have a positive local maximum.

A similar argument can be used to show that w cannot have a non negative local minimum at any $x > 0$. The absence of these extrema preclude the existence of local negative maxima and positive local minima.

Suppose that $w'(x_0) = 0$ at some $x_0 > 0$. The above arguments show that w does not have a local extremum at x_0 and thus $w''(x_0) = 0$. Equation (3.2.5) implies $w(x_0) = w(\alpha x_0)$ and since w can have no local extrema we have

$$w(x) = w(x_0), \tag{3.2.11}$$

for all $x \in [x_0, \alpha x_0]$. The continuity of w, w' and w'' indicates that $w'(\alpha x_0) = w''(\alpha x_0) = 0$, so that we can repeat the argument to show that equation (3.2.11) is satisfied for all $x \in [x_0, \alpha^2 x_0]$. It is clear that we can repeat this argument *ad infinitum* to show that the equation (3.2.11) must be satisfied for all $x \geq x_0$. Condition (3.2.7) implies that $w(x_0) = 0$ and thus $w(x) = 0$ for all $x \geq x_0$. It may be that w is non zero for $x < x_0$. Since the equation (3.2.11) holds for all $x \geq x_0$, we see that the equation (3.2.8) reduces to the ordinary differential equation

$$w''(x) - gw'(x) - b\alpha w(x) = 0, \tag{3.2.12}$$

for all $x \in [x_0/\alpha, x_0]$. The ordinary differential equation (3.2.12) has the initial conditions $w(x_0) = w'(x_0) = 0$. We can apply Picard's existence and uniqueness theorem to show that the only solution to the equation (3.2.12) that satisfies these initial conditions is the trivial solution $w(x) = 0$. We thus have that equation (3.2.11) is satisfied for all $x \geq x_0/\alpha$. The above argument can be repeated to show that (3.2.11) is satisfied for all $x \geq x_0/\alpha^2$, and it can be repeated *ad infinitum* to show that (3.2.11) is satisfied for all $x > 0$. The continuity of w at $x = 0$ shows that $w(x) = 0$ for all $x \geq 0$.

■

Theorem 3.2.2 (Positivity). *Any solution to the Constant Coefficient Problem 2 must be positive.*

Proof: The *Integrability Condition* shows that w must satisfy equation (3.2.6) and hence cannot be the trivial solution. Lemma (3.2.1) shows that $w'(x) \neq 0$ for all $x > 0$, and this means that $y(x) \neq 0$ for all $x > 0$. The *Integrability Condition* also implies that y must be positive for some $x > 0$. Since y cannot change sign, we conclude that $y(x) > 0$ for all $x > 0$. ■

Theorem 3.2.3 (Uniqueness). *If a solution exists to the Constant Coefficient Problem 2, then it is unique.*

Proof: Let $Z(x) = y_1(x) - y_2(x)$ where $y_1(x)$ and $y_2(x)$ are two solutions of the *Constant Coefficient Problem 2.* Then $Z(x)$ is a solution of equation (3.2.5); therefore, equation (3.2.5) implies

$$Z''(x) - gZ'(x) + b\alpha Z(x) - b\alpha^2 Z(\alpha x) = 0, \tag{3.2.13}$$

and the *Pdf Conditions* imply

$$Z(0) = 0, \tag{3.2.14}$$

$$\lim_{x\to\infty} Z(x) = 0, \tag{3.2.15}$$

and

$$\int_0^\infty Z(\xi)d\xi = 0.$$

Let

$$v(x) = \int_x^\infty Z(\xi)d\xi.$$

Integrating equation (3.2.13) from x to ∞ and using equations (3.2.15) yields

$$v''(x) - gv(x) + b\alpha v(x) - b\alpha v(\alpha x) = 0, \tag{3.2.16}$$

where

$$v'(x) = -Z(x), \tag{3.2.17}$$

$$v''(x) = -Z'(x),$$

and

$$\lim_{x\to\infty} v(x) = 0. \tag{3.2.18}$$

Equation (3.2.16) is the same as the equation (3.2.8), and the condition (3.2.18) is the same as condition (3.2.7). Lemma (3.2.1) can thus be applied to show that either v is the trivial solution or there is no $x_0 > 0$ at which $v'(x_0) = 0$. Conditions $v(0) = 0$ and (3.2.18), however precludes the last case since a non trivial v must have at least one non zero extremum. We thus see that $v(x) = 0$ for all $x \geq 0$, and condition (3.2.17) implies that Z is a constant function for $x \geq 0$. Condition (3.2.14) shows that $Z(x) = 0$ for all $x \geq 0$ and therefore $y_1(x) = y_2(x)$ for all $x \geq 0$. ∎

We anticipate that the *Constant Coefficient Problem 2* has a Dirichlet series solution. Let

$$y(x) = \sum_{k=0}^{\infty} c_k e^{-r\alpha^k x},$$

where $r > 0$ is a constant to be determined. Then

$$y'(x) = -\sum_{k=0}^{\infty} r\alpha^k c_k e^{-r\alpha^k x},$$

$$y''(x) = \sum_{k=0}^{\infty} r^2\alpha^{2k} c_k e^{-r\alpha^k x},$$

and

$$y(\alpha x) = \sum_{k=1}^{\infty} c_{k-1} e^{-r\alpha^k x}.$$

Substituting these expressions into (3.2.5) gives

$$\sum_{k=1}^{\infty}[r^2\alpha^{2k}c_k - gr\alpha^k c_k - b\alpha c_k + b\alpha^2 c_{k-1}]e^{-r\alpha kx} + (r^2 - gr - b\alpha)c_0 = 0.$$

Assuming $c_0 \neq 0$, and balancing coefficients of $e^{-r\alpha kx}$ gives

$$(r^2 - gr - b\alpha) = 0, \tag{3.2.19}$$

and

$$\left(r^2\alpha^{2k}c_k - gr\alpha^k c_k - b\alpha c_k + b\alpha^2 c_{k-1}\right) = 0. \tag{3.2.20}$$

Equation (3.2.19) gives two solutions for r, only the choice $r = (g + \sqrt{g^2 + 4\alpha b})/2$ leads to a convergent series. Rearranging equation (3.2.20) and using (3.2.19) gives

$$c_k = \frac{-(b\alpha^2/r)}{(\alpha^k - 1)(r\alpha^k + r - g)} c_{k-1},$$

so that

$$c_k = \frac{-(b\alpha^2)/r)^k}{\prod_{m=1}^{k}((\alpha^m - 1)(r\alpha^m + r - g))} c_0.$$

We thus have

$$y(x) = c_0[e^{-rx} + \sum_{k=1}^{\infty} \frac{-(b\alpha^2)/r)^k}{\prod_{m=1}^{k}((\alpha^m - 1)(r\alpha^m + r - g))} e^{-r\alpha^k x}], \tag{3.2.21}$$

where c_0 can be determined using the *Integrability Condition*.

Note that the Dirichlet series solution (3.2.21) can be constructed from the Laplace transform (cf. [89]).

3.3 Second order pantograph equations with non constant coefficients

In this section we first review some second order pantograph equations with non constant coefficients with known solutions. We then study a second order pantograph equation that involves a Hermite type operator.

3.3.1 Modified Bessel equation

Let $D(x) = \Upsilon x$, $G(x) = g$ and $B(x) = b$ where b, g and Υ are positive then equation (1.4.3) reduces to

$$\Upsilon x y''(x) - (g - 2\Upsilon)y'(x) - (b + \lambda)y(x) + b\alpha^2 y(\alpha x) = 0. \tag{3.3.1}$$

Integrating equation (3.3.1) from 0 to ∞ and using the *No Flux Conditions 3* and the *Integrability Condition* gives

$$\lambda = b(\alpha - 1),$$

so that equation (3.3.1) becomes

$$xy''(x) - (g/\Upsilon - 2)y'(x) - (b\alpha/\Upsilon)y(x) + \left(b\alpha^2/\Upsilon\right) y(\alpha x) = 0. \tag{3.3.2}$$

Let $a = (g/\Upsilon - 2)$, $b_1 = (b\alpha/\Upsilon)$ then equation (3.3.2) reduces to

$$xy''(x) - ay'(x) - b_1 y(x) + b_1 \alpha y(\alpha x) = 0. \tag{3.3.3}$$

We refer equation (3.3.3) along with the *No Flux Conditions 3* and the *Integrability Condition* as *Modified Bessel Problem*.

van Brunt & Wake [88] solved the *Modified Bessel Problem* using a Mellin transform method. The homogeneous equation of the *Modified Bessel Problem* is

$$xy''(x) - ay'(x) - b_1 y(x) = 0,$$

and the general solution y_h of above equation is of the form

$$y_h = B_1 x^{\nu/2} I_\nu\left(2\sqrt{b_1 x}\right) + B_2 x^{\nu/2} K_\nu\left(2\sqrt{b_1 x}\right),$$

where B_1, B_2 are constants, $\nu = a + 1 = g/\Upsilon - 1$, and I_ν, K_ν denote the modified Bessel functions of order ν. The function I_ν is not bounded as $x \to \infty$. van Brunt & Wake therefore applied the Mellin transform to the class of homogeneous solutions of the form

$$y_h = B_2 x^{\nu/2} K_\nu(2\sqrt{b_1 x}).$$

They obtained the Mellin transform

$$M(s) = \frac{Cb_1^{\frac{(v+2)}{2}}}{\Gamma(v+1)} \frac{\Gamma(s)\Gamma(s+v)}{b_1^{\frac{(2s+v)}{2}}} \prod_{k=0}^{\infty}\left(1 - \frac{1}{\alpha^{s+k}}\right),$$

where C is a constant.

The simple poles of the gamma function $\Gamma(s)$ for $s = -m$, where $m = 0, -1, -2, ...$ are killed by the first order zeroes of the infinite product. The singularities thus come from $\Gamma(s+\nu)$. It was shown that if $\nu > 0$, the $M(s)$ is holomorphic in the half plane $\Re(s) \geq 0$ and if $\nu < 0$ then the fundamental strip is $0 < \Re(s) < \infty$ because M has a simple pole at $s = 0$. For the case $\nu = 0$,

$$M(s) = Cb_1^{1-s}\Gamma^2(s) \prod_{k=0}^{\infty}\left(1 - \frac{1}{\alpha^{s+k}}\right),$$

and the residue at $s = 0$ is

$$\begin{aligned} Res(M(s)x^{-s},\ s = 0) &= \tilde{C} \lim_{s \to 0} s\Gamma(s)\Gamma(s+1) \frac{\left(1 - \frac{1}{\alpha^s}\right)}{s} \\ &= \tilde{C} \log \alpha, \end{aligned}$$

where $\tilde{C} = Cb_1 \prod_{k=1}^{\infty}\left(1 - \frac{1}{\alpha^k}\right)$. van Brunt & Wake inverted M using the *Euler identity* and obtained

$$y(x) = \frac{2Cb^{(\nu+2)/2}}{\Gamma(\nu+1)} \sum_{k=0}^{\infty} (\alpha^k x)^{\nu/2} p_k K_\nu\left(2\sqrt{b_1 \alpha^k x}\right), \tag{3.3.4}$$

where

$$p_k = \frac{(-1)^k}{\alpha^{k(k-1)/2}\prod_{j=1}^{k}(1-\alpha^{-j})}.$$

It was shown that

(a) if $\nu > 0$, the solution (3.3.4) is uni-modal with $y(0) = 0$;

(b) if $\nu < 0$, then $y(x) \to \infty$ as $x \to 0^+$ and y decreases in the interval $(0, \infty)$; and

(c) if $\nu = 0$ then $y(x) \to b\log\alpha$, as $x \to 0^+$ and y decreases in the interval $(0, \infty)$.

Since for any value of ν, K_ν decays exponentially as $x \to \infty$ (cf. [1], pg. 375) therefore the solution y decays exponentially. van Brunt & Wake showed that the solution y is the unique pdf solution with $\nu = (g/\Upsilon - 1)$.

3.3.2 Airy type equation

Another example is given by an Airy type operator. Note that the Airy type equation does not directly correspond to any choice of D and G in equation (1.4.3). Nonetheless, it provides another illustration. The form of the equation is

$$y''(x) - xy(x) + \lambda xy(\alpha x) = 0. \tag{3.3.5}$$

We refer equation (3.3.5) along with the *Boundary Conditions* as *Airy Problem.*

Kim [38] solved the *Airy Problem* for a general λ where a series of Airy functions was used to solve the problem. Kim showed that if $|\lambda| \leq \alpha^2$ and $\int_0^\infty xy(x)dx$ exists then for all $x \geq 0$, a positive and unique solution exists to the problem (3.3.5) that decreases in the interval $[0, \infty)$. The solution that Kim obtained is

$$y(x) = K\left(Ai(x) + \sum_{k=1}^{\infty}\frac{(-\lambda)^k}{\prod_{m=1}^{n}(\alpha^{3m}-1)}Ai(\alpha^k x)\right),$$

where K can be determined by using the *Integrability Condition.* It was observed that $y(0) = 0$ when $\lambda = \alpha^{3n}$, $n = 1, 2, ...$

Here, we use the Mellin transform to solve the *Airy Problem* together with the *Integrability Condition.* The homogeneous equation is

$$y_h''(x) - xy_h(x) = 0, \tag{3.3.6}$$

which has the general solution

$$y_h(x) = C_1 Ai(x) + C_2 Bi(x),$$

where Ai and Bi are the Airy functions and C_1 and C_2 are constants. Note that Bi is not bounded as $x \to \infty$ (cf. [1], pg. 449) therefore Bi does not have a Mellin transform for $s = 1$. Let

$$y_u(x) = C_1 Ai(x).$$

Taking the Mellin transform of (3.3.5) gives

$$(s-2)(s-1)M(s-2) = \left(1 - \frac{\lambda}{\alpha^{s-1}}\right) M(s+1). \tag{3.3.7}$$

Taking the Mellin transform both sides of equation (3.3.6) yields

$$(s-2)(s-1)\frac{F(s-2)}{F(s+1)} = 1, \tag{3.3.8}$$

where F is the Mellin transform of y_u. Substituting the solution form (2.4.7) into equation (3.3.7) and using equation (3.3.8), gives

$$Q(s) = K_1 \prod_{k=0}^{\infty} \left(1 - \frac{\lambda}{\alpha^{s+1+3k}}\right),$$

where K_1 is a constant. The Mellin transform of y_u is

$$F(s) = C_1 \frac{1}{2\sqrt{\pi}} 3^{\frac{2s}{3} - \frac{7}{6}} \Gamma\left(\frac{s}{3}\right) \Gamma\left(\frac{1}{3} + \frac{s}{3}\right).$$

Substituting above expressions for $F(s)$ and $Q(s)$ in (2.4.7) yields

$$M(s) = K \frac{1}{2\sqrt{\pi}} 3^{\frac{2s}{3} - \frac{7}{6}} \Gamma\left(\frac{s}{3}\right) \Gamma\left(\frac{1}{3} + \frac{s}{3}\right) \prod_{k=0}^{\infty} \left(1 - \frac{\lambda}{\alpha^{s+1+3k}}\right), \tag{3.3.9}$$

where $K = C_1 K_1$. Using the *Integrability Condition*, equation (3.3.9) gives

$$K = \frac{2\sqrt{3\pi}}{\Gamma(\frac{1}{3})\Gamma(\frac{2}{3}) \prod_{k=0}^{\infty} \left(1 - \frac{\lambda}{\alpha^{2+3k}}\right)}.$$

The simple poles generated by $F(s)$ at $s = -3m$, where m is a non negative integer, are balanced by the first order zeros of $Q(s)$, if for $s = -3m$,

$$\prod_{k=0}^{\infty} \left(1 - \frac{\lambda}{b\alpha^{-3m+1+3k}}\right) = 0.$$

This gives spectrum of eigenvalues $\lambda_m = \alpha^{1+3m}$, $m = 0, 1, 2, ...$ for which $M(s)$ is a holomorphic function in the right half plane. The infinite product, defining $Q(s)$ can be expressed in a series form by use of the *Euler identity*. Let $q = \alpha^{-3}$ and $z = \alpha^{-s}$, then the *Euler identity* implies

$$Q_m(s) = 1 + \sum_{k=1}^{\infty} p_k \frac{\lambda_m/\alpha}{\alpha^{ks}},$$

where

$$p_k = \frac{(-1)^k}{\alpha^{\frac{3k(k-1)}{2}} \prod_{j=1}^{k}(1-\alpha^{-3j})}.$$

Recasting the equation (3.3.9) gives

$$M_m(s) = K_m \frac{1}{2\sqrt{\pi}} 3^{\frac{2s}{3}-\frac{7}{6}} \Gamma\left(\frac{s}{3}\right) \Gamma\left(\frac{1}{3}+\frac{s}{3}\right) \left(1 + \frac{\lambda_m}{\alpha} \sum_{k=1}^{\infty} \frac{p_k}{\alpha^{ks}}\right). \tag{3.3.10}$$

Term by term inversion of (3.3.10) implies

$$y_m(x) = K_m \left(Ai(x) + \frac{\lambda_m}{\alpha} \sum_{k=1}^{\infty} p_k Ai(\alpha^k x)\right).$$

3.3.3 Second order pantograph equation involving Hermite operator

Earlier in this chapter we discussed those second order pantograph problems that can be solved by the Mellin transform. Here, we study a problem with a linear growth rate and constant dispersion that cannot be solved explicitly like previous problems. Nonetheless, it is possible to glean some information regarding the nature of the solution via the Mellin transform. This problem is a part of our research.

Let $D(x) = \Upsilon$, $G(x) = gx$ and $B(x) = b$, where b, g and Υ are positive constants. Then equation (1.4.3) is

$$\Upsilon y''(x) - (gxy(x))' - (b+\lambda)y(x) + b\alpha^2 y(\alpha x) = 0. \tag{3.3.11}$$

Integrating equation (3.3.11) from 0 to ∞ and using the *No Flux Conditions 3*

$$\lim_{x\to 0^+} \left(\Upsilon y'(x) - gxy(x)\right) = 0, \tag{3.3.12}$$

$$\lim_{x\to \infty} \left(\Upsilon y'(x) - gxy(x)\right) = 0, \tag{3.3.13}$$

and the *Integrability Condition* yields

$$\lambda = b(\alpha - 1),$$

so that equation (3.3.11) becomes

$$y''(x) - \left(\frac{g}{\Upsilon}xy(x)\right)' - \frac{b\alpha}{\Upsilon}y(x) + \frac{b\alpha^2}{\Upsilon}y(\alpha x) = 0. \tag{3.3.14}$$

We refer equation (3.3.14) along with the no flux conditions (3.3.12), (3.3.13) and the *Integrability Condition* as *Hermite Problem*.

Some qualitative results

We show that if the no flux conditions (3.3.12) and (3.3.13) are satisfied then

$$y(0) = L; \tag{3.3.15}$$

where $L > 0$,

$$y'(0) = 0; \tag{3.3.16}$$

and

$$\lim_{x \to \infty} y'(x) = 0. \tag{3.3.17}$$

In addition, we investigate the term $g > b\alpha(\alpha - 1)$ so that there exists a pdf solution to the *Hermite Problem*.

Let

$$\phi(x) = -\Upsilon y'(x) + gxy(x). \tag{3.3.18}$$

Then equation (3.3.14) becomes

$$\phi'(x) + b\alpha y(x) = b\alpha^2 y(\alpha x),$$

that is

$$\phi(x) = b\alpha \int_x^{\alpha x} y(\xi) d\xi. \tag{3.3.19}$$

The *Pdf Conditions* implies

$$0 \le \phi(x) \le b\alpha. \tag{3.3.20}$$

Further, equation (3.3.18) can be written as

$$\left(y(x)e^{-gx^2/2\Upsilon}\right)' = -e^{-gx^2/2\Upsilon}\phi(x).$$

Integrating above equation from x to ∞ gives

$$y(x) = e^{gx^2/2\Upsilon}\int_x^\infty e^{-g\xi^2/2\Upsilon}\phi(\xi)d\xi; \tag{3.3.21}$$

for $x = 0$, above equation implies

$$y(0) = \int_0^\infty e^{-g\xi^2/2\Upsilon}\phi(\xi)d\xi.$$

If $y(0) = 0$, then $\phi(x) = 0$ for all $x \ge 0$ which contradicts the *Pdf Condition* (1.4.10). As a consequence y satisfies the condition (3.3.15) that together with the *No Flux Condition* (3.3.12) leads to (3.3.16). In order to find the behaviour of y near ∞, multiplying equation (3.3.21) by x gives

$$xy(x) = xe^{gx^2/2\Upsilon}\int_x^\infty e^{-g\xi^2/2\Upsilon}\phi(\xi)d\xi$$

$$= \frac{\int_x^\infty e^{-g\xi^2/2\Upsilon}\phi(\xi)d\xi}{\frac{e^{-gx^2/2\Upsilon}}{x}}.$$

Clearly, $xy(x) \to 0$ as $x \to \infty$. The *No Flux Condition* (3.3.13) therefore yields (3.3.17).

Equation (3.3.14) at $x = 0$ becomes

$$y''(0) = (g - b\alpha(\alpha - 1))\frac{y(0)}{\Upsilon}.$$

Conspicuously, $g > b\alpha(\alpha - 1)$ induces $y''(0) > 0$. For the existence of a pdf solution, the *Hermite Problem* with a positive curvature at origin and (3.3.16) must have at least one maximum. We show that y cannot have any maximum if $g > b\alpha(\alpha - 1)$.

Let $x = x_m$ be the position of the final global maximum of y. Then equation (3.3.14) implies

$$-y''(x_m) + (g + b\alpha)y(x_m) = b\alpha^2(y(\alpha x_m),$$

and since $y''(x_m)$ is negative and $y(\alpha x_m) < y(x_m)$,

$$(g + b\alpha)y(x_m) < b\alpha^2(y(x_m),$$

which implies $g < b\alpha(\alpha - 1)$. We conclude that for $g > b\alpha(\alpha - 1)$, a pdf solution to the *Hermite Problem* does not exists. We now look at the second case when $g < b\alpha(\alpha - 1)$. Equations (3.3.18) and (3.3.19) give

$$-\Upsilon y'(x) + gxy(x) = b\alpha \int_x^{\alpha x} y(\xi)d\xi.$$

Suppose

$$y'(x) < 0, \tag{3.3.22}$$

for all $x \in (x_0, \infty)$, where $x_0 > 0$. Then,

$$gxy(x) \leq b\alpha \int_x^{\alpha x} y(\xi)d\xi,$$

that is

$$\begin{aligned} y(x) &\leq \frac{b\alpha}{gx} \int_x^{\alpha x} y(\xi)d\xi, \\ &\leq \frac{b\alpha}{gx}. \end{aligned}$$

Iterating above step gives,

$$\begin{aligned} y(x) &\leq \frac{b\alpha}{gx} \int_x^{\alpha x} \frac{b\alpha}{g\xi} d\xi, \\ &\leq \frac{b\alpha}{gx}, \\ &= \frac{(b\alpha)^2}{g^2 x} \log(\alpha) \\ &\vdots \\ &\leq \frac{b\alpha}{gx} \left(\frac{b\alpha \log(\alpha)}{g} \right)^k. \end{aligned}$$

Suppose $\frac{b\alpha}{g} \log(\alpha) < 1$, then

$$\frac{1}{\alpha - 1} < \frac{b\alpha}{g} < \frac{1}{\log(\alpha)}, \tag{3.3.23}$$

and $y(x) = 0$ for all (x_0, ∞). As a consequence, equation (3.3.14) reduces to the ordinary differential equation

$$y''(x) - \left(\frac{g}{\Upsilon} x y(x)\right)' - \frac{b\alpha}{\Upsilon} y(x) = 0, \tag{3.3.24}$$

for all $x \in [x_0/\alpha, x_0]$. The ordinary differential equation (3.3.24) has the initial conditions $y(x_0) = y'(x_0) = 0$. We can apply the Picard's existence and uniqueness theorem to show that the only solution to the equation (3.3.24) that satisfies these initial conditions is the trivial solution $y(x) = 0$. We thus have that equation (3.3.24) is satisfied for all $x \geq x_0/\alpha$. The above argument can be repeated to show that (3.3.24) is satisfied for all $x \geq x_0/\alpha^2$, and it can be repeated *ad infinitum* to show that (3.3.24) is satisfied for all $x > 0$. The continuity of y at $x = 0$ shows that $y(x) = 0$ for all $x \geq 0$ which contradicts the condition (3.3.15). Therefore, the *Hermite Problem* along with the conditions (3.3.22) and (3.3.23) has no solution. However, it is not obvious that for the case when the condition (3.3.22) is coupled with $\frac{b\alpha}{g} = \frac{1}{\alpha-1}$, the *Hermite Problem* has any solution. In addition, if there is a solution, it is not clear that the resulting solution is a pdf. In fact there are solutions to the problem if we relax the integrability and positivity conditions. We construct a Mellin transform solution and use this to show the existence of non trivial solutions.

A Mellin transform solution

The homogeneous equation associated with equation (3.3.14) is

$$y_h''(x) - \left(\frac{g}{\Upsilon} x y_h(x)\right)' - \frac{b\alpha}{\Upsilon} y_h(x) = 0, \tag{3.3.25}$$

which has a general solution of the form

$$y_h = x\left[A_{11}F_1\left(1 + \frac{b\alpha}{2g}, \frac{3}{2}, \frac{gx^2}{2\Upsilon}\right) + A_2 U\left(1 + \frac{b\alpha}{2g}, \frac{3}{2}, \frac{gx^2}{2\Upsilon}\right)\right],$$

where U and ${}_1F_1$ denote Kummer functions and A_1 and A_2 are constants. The Kummer functions ${}_1F_1$ and U are confluent hyper-geometric functions, the properties of which can be found in [78] and [1]. Note that

$${}_1F_1(a, b, z) \sim \frac{\Gamma(b)}{\Gamma(a)} e^z z^{a-b}$$

as $|z| \to \infty$, and

$$U(a, b, z) \sim z^{-a},$$

as $\Re(z) \to \infty$ (cf. [1], pg. 504). In particular, ${}_1F_1$ does not decay to zero as $x \to \infty$ and thus we do not expect that this function will play a part in the solution to the problem. The other Kummer function U, however, has the asymptotic property

$$U(1 + \frac{b\alpha}{2g}, \frac{3}{2}, \frac{gx^2}{2}) \sim C' x^{-(1+b\alpha/2g)}$$

as $x \to \infty$, where

$$C' = \left(\frac{g}{2}\right)^{-(1+b\alpha/2g)}.$$

We know that ${}_1F_1$ does not have a Mellin transform defined at s=1, and this leads us to consider the solutions

$$y_u = A_2 x U\left(1 + \frac{b\alpha}{2g}, \frac{3}{2}, \frac{gx^2}{2\Upsilon}\right).$$

The asymptotic behaviour of U as $x \to \infty$ indicates that

$$y_u \sim \mathcal{O}(x^{-(1+b\alpha/2g)})$$

as $x \to \infty$, and this relation signals that solutions to the functional equation will probably decay slowly as $x \to \infty$.

Applying the Mellin transform to equation (3.3.14) gives

$$\begin{aligned} x^{s-1}y'(x)|_0^\infty - (s-1)x^{s-2}y(x)|_0^\infty - \frac{g}{\Upsilon}x^s y(x)|_0^\infty + (s-1)(s-2)M(s-2) \\ + \left(\frac{g}{\Upsilon}s - \frac{g+b\alpha}{\Upsilon} + \frac{b/\Upsilon}{\alpha^{s-2}}\right)M(s) = 0. \end{aligned}$$

Assume that

$$x^{s-1}\left(\Upsilon y'(x) - gxy(x)\right)\Big|_0^\infty - (s-1)x^{s-2}y(x)\Big|_0^\infty = 0. \tag{3.3.26}$$

Then,

$$(s-1)(s-2)M(s-2) + \left(\frac{g}{\Upsilon}s - \frac{(b\alpha+g)}{\Upsilon} + \frac{b/\Upsilon}{\alpha^{s-2}}\right)M(s) = 0. \tag{3.3.27}$$

The assumption (3.3.26) indicates that if the *No Flux Conditions 3* are satisfied,

$$(s-1)x^{s-2}y(x)|_0^\infty = 0. \tag{3.3.28}$$

Evidently, the term (3.3.28) is satisfied for all $\Re(s) > 0$ if $y'(0) = 0$, $y''(0) = 0$ and

$$y(0) = 0. \tag{3.3.29}$$

Therefore, if any solution to equation (3.3.27) exists then this solution belongs to a special class that satisfies the boundary condition (3.3.29).

Applying the Mellin transform to equation (3.3.25) gives

$$F(s) = \frac{(s-1)(s-2)}{\frac{(b\alpha+g-gs)}{\Upsilon}} F(s-2),$$

that has a solution of the form

$$F(s) = \left(\frac{2g}{\Upsilon}\right)^{-s/2} \Gamma(s)\Gamma\left(\frac{b\alpha}{2g} + \frac{1}{2} - \frac{s}{2}\right). \tag{3.3.30}$$

Note that the equation (3.3.30) can also be obtained by taking the Mellin transform of y_u [cf. [78], pg. 49]. A solution to the Mellin equation (3.3.27) is

$$M(s) = \left(\frac{2g}{\Upsilon}\right)^{-s/2} \Gamma(s)\Gamma\left(\frac{b\alpha}{2g} + \frac{1}{2} - \frac{s}{2}\right) \prod_{k=0}^{\infty} \left(1 + \frac{b/g}{\alpha^{s+2k}\left(s + 2k + 1 - \frac{b\alpha}{g}\right)}\right). \tag{3.3.31}$$

There is no conspicuous conversion formula to write the infinite product as a series. Nonetheless, we can deduce the existence of an inverse M using a version of the Paley Weiner theorem [60], [82] for the Mellin transform.

Theorem 3.3.1 (The Paley Wiener theorem). *There exists a unique function $f(x)$ that has the Mellin transform $\tilde{M}(s)$ provided*

$$|\tilde{M}(s)| \leq C|s|^{-2}$$

for some constant C, and $\tilde{M}(s)$ is holomorphic in the strip $S(a_1, a_2) = \{s \in \mathbb{C} : a_1 < \Re(s) < a_2\}$.

In order to apply the Paley Weiner theorem we first determine a bound for M.

Lemma 3.3.2. *Let $s = \kappa + i\delta$ then*

$$|M(s)| \sim \mathcal{O}(e^{\frac{-3\delta\pi}{4}}),$$

as $\delta \to \infty$.

Proof: Now,

$$|M(s)| = \left|(2g/\Upsilon)^{-s/2}\Gamma(s)\Gamma\left(\frac{b\alpha}{2g}+\frac{1}{2}-\frac{s}{2}\right)\prod_{k=0}^{\infty}\left(1+\frac{b/g}{\alpha^{s+2k}\left(s+2k+1-\frac{b\alpha}{g}\right)}\right)\right|,$$
$$\leq |(2g/\Upsilon)^{-s/2}||\Gamma(s)|\left|\Gamma\left(\frac{b\alpha}{2g}+\frac{1}{2}-\frac{s}{2}\right)\right|\prod_{k=0}^{\infty}\left(1+\frac{b/g}{|\alpha^{s+2k}\left(s+2k+1-\frac{b\alpha}{g}\right)|}\right).$$

Let $s=\kappa+\iota\delta$. For $\delta\to\infty$, we have

(i) $|(2g/\Upsilon)^{-s/2}| = e^{\frac{-\kappa\log(\frac{2g}{\Upsilon})}{2}}$,

(ii) $|\Gamma(s)| \sim \mathcal{O}(e^{\frac{-\delta\pi}{2}})$,

(iii) $|\Gamma(\frac{b\alpha}{2g}+\frac{1}{2}-\frac{s}{2})| \sim \mathcal{O}(e^{\frac{-\delta\pi}{4}})$,

(iv) $\prod_{k=0}^{\infty}\left|\left(1+\frac{b/g}{\alpha^{s+2k}(s+2k+1-\frac{b\alpha}{g})}\right)\right| \sim \mathcal{O}(1)$,

which implies

$$|M(s)| \sim \mathcal{O}(e^{\frac{-3\delta\pi}{4}}),$$

as $\delta\to\infty$. ■

Lemma 3.3.3. *If the No Flux Conditions 3 are satisfied then the strip of holomorphy for $M(s)$ does not include $s=1$.*

Proof: Determination of the strip of holomorphy of M relies on the term $b\alpha/g$. We first consider the case when $b\alpha/g$ is an integer and then discuss the case when $b\alpha/g$ is a non integer.

An integer case: For an integer case we first suppose that $\frac{b\alpha}{g}$ is an odd integer that is $\frac{b\alpha}{g}=2m+1$, where $m=0,1,2,...$, then

$$M_{\frac{b\alpha}{g}=2m+1}(s) = \left(\frac{2g}{\Upsilon}\right)^{-s/2}\Gamma(s)\Gamma\left(\frac{2m+1}{2}+\frac{1}{2}-\frac{s}{2}\right)\prod_{k=0}^{\infty}\left(1+\frac{b/g}{\alpha^{s+2k}\left(s+2(k-m)\right)}\right).$$

Conspicuously, for an odd positive integer, the infinite product induces a pole of order 1 at $s=0$ which leads to a solution y that does not satisfy the condition (3.3.29). For a second case when $\frac{b\alpha}{g}$ is an even integer that is $\frac{b\alpha}{g}=2m$, where $m=0,1,2,...$, the infinite product becomes

$$\prod_{k=0}^{\infty}\left(1+\frac{b/g}{\alpha^{s+2k}\left(s+1+2(k-m)\right)}\right).$$

The function $M_{\frac{b\alpha}{g}=2m}(s)$ has a pole at $s = 1$ from the above infinite product factor for $k = m - 1$. Since the existence of a pole at $s = 1$ leads to a non integrable y, therefore the inverse Mellin transform y of M is non integrable for this case.

Non integer case: For this case when $0 < \frac{b\alpha}{g} < 1$, the infinite product has no pole for all $\Re(s) \geq 0$; however $M_{\frac{b\alpha}{g}=noninteger}(s)$ has a pole at $s = 0$ due to the factor $\Gamma(s)$ which contradicts the assumption (3.3.29). For all positive non integer $\frac{b\alpha}{g} > 1$, the infinite product either induces a first order pole at $s = 0$ or $s = 1$.

A particular choice when

$$\frac{b\alpha}{g} = \frac{(2m+1)\alpha^{2m+1}}{\alpha^{2m+1} - 1},$$

where $m = 0, 1, 2, ...,$ is an interesting case. For simplicity, consider $m = 0$ then

$$\frac{b\alpha}{g} = \frac{\alpha}{\alpha - 1},$$

and consequently

$$M_{\frac{b\alpha}{g}=\frac{\alpha}{\alpha-1}}(s) = \left(\frac{2g}{\Upsilon}\right)^{-s/2} \Gamma(s)\Gamma\left(\frac{\alpha}{2(\alpha-1)} + \frac{1}{2} - \frac{s}{2}\right) \prod_{k=0}^{\infty} \left(1 + \frac{\alpha/(\alpha-1)}{\alpha^{s+2k+1}\left(s + 2k + 1 - \frac{\alpha}{(\alpha-1)}\right)}\right).$$

The infinite product has first order zeros at $s = -j$, where $j = 0, 1, 2, ...$ These first order zeroes cancel all the first order poles emerging from $\Gamma(s)$ at $s = -j$. However, the infinite product induces first order poles at $s = \frac{1}{(\alpha-1)} - 2j$, where $0 < \frac{1}{(\alpha-1)} \leq 1$. The strip of holomorpy for this particular choice of $\frac{b\alpha}{g}$ also does not include $s = 1$. ■

Evidently, $M(s)$ is not satisfied at $s = 1$ which leads to the inverse Mellin transform y that is not integrable. Using the Paley Weiner theorem we conclude the following result

Theorem 3.3.4. *There exists a non trivial solution to the Hermite Problem that satisfies the No Flux Conditions 3. This solution is not integrable.*

In the previous problems with dispersion, when the Mellin transform is applied, we obtained M that involved an infinite product of the form $\prod_{k=0}^{\infty}\left(1 - \frac{1}{\alpha^{k+s}}\right)$ that could readily be converted into a series form by using the *Euler identity.* We therefore obtained the solution by inverting the transform. However the transform for the *Hermite Problem* involves an infinite product that is different from the previous forms due to the presence of poles. In this case, the Euler identity cannot be used, and there is no obvious partition relation to convert the product to a series. We have seen that solutions of all the second order functional differential equations derived from the modified Fokker-Planck equation, studied earlier decay exponentially: in contrast, the solution to this problem decays slowly. We have shown that it is possible to find the Mellin transform for this problem. The technique for inverting the transform, however, is different from the earlier problems owing to the form of the infinite product. It is shown that there do not exist pdf solutions to the problem for a range of parameters α, b and g. In addition, we show that there exists non trivial solutions to the *Hermite Problem* that satisfy the *No Flux Conditions 3* if the integrability and the positivity conditions are dropped. The non trivial solutions are non integrable.

Chapter 4

Exponential decay for the fragmentation or cell division equation

In this chapter we discuss a method developed by Perthame & Ryzhik [64]. Perthame & Ryzhik considered the *Pde Problem without Dispersion* for a general non constant division and a constant growth rate. Under some assumptions on the division rate Perthame & Ryzhik established the existence of a unique eigenvalue λ and a corresponding eigenfunction $y(x)$. They showed that in the $L^1[0,\infty)$ norm, $e^{-\lambda t}n(x,t) \to y(x)$ as $t \to \infty$.

The experimental observation in [32] showed that for a certain cell types, the shape of the curve defined by the number density of cells $n(x,t)$ approaches the separable solution as $t \to \infty$. These separable solutions give rise to the pantograph equations in the *Pde Problem with Dispersion*. In the previous chapter we studied pantograph equations for certain types of division and growth rates. Based on the observations in [32], Hall & Wake [30],[31] discussed the SSD solution to the *Pde Problem without Dispersion*. However, they did not establish that the the solution to the *Pde Problem without Dispersion* converges to a separable solution as $t \to \infty$.

In order to show that $n(x,t) \to e^{\lambda t}y(x)$ as $t \to \infty$, Perthame & Ryzhik used the concept of general relative entropy (cf. [54] & [63]) and introduced an adjoint equation. The solution to the adjoint equation then served as a weight to establish the convergence of the solution of the *Pde Problem without Dispersion* to a separable solution as $t \to \infty$. The adjoint equation is also known as dual equation or backward equation. The idea of relative entropy technique has been used in many fields such as system of conservation laws [62] and partial differential equations of general type [21]. Perthame

& Ryzhik applied this technique to the *Pde Problem without Dispersion* for $\alpha = 2$ and constant growth rate. In this chapter we discuss their analysis for $\alpha > 1$.

We have structured the chapter as follows. We begin with a brief introduction and then show a proof concerning the long time behaviour of the solution to the *Pde Problem without Dispersion* for a constant division rate. We then consider a non constant division rate and in order to discuss the long time behaviour we discuss their strategy to show the existence of a unique eigenvalue, corresponding eigenfunction and the existence of a unique and bounded solution to the dual equation. The main focus of the chapter is on the non constant division rate case.

4.1 Introduction

Consider the differential equation

$$\frac{\partial}{\partial t}n(x,t) + \frac{\partial}{\partial x}\left(n(x,t)\right) + B(x)n(x,t) = \alpha^2 B(\alpha x)n(\alpha x,t), \quad t > 0, \quad x \geq 0, \tag{4.1.1}$$

along with the boundary condition

$$n(0,t) = 0, \quad t > 0, \tag{4.1.2}$$

and the *Initial Condition*

$$n(x,0) = n_0(x) \in L^1[0,\infty). \tag{4.1.3}$$

Equation (4.1.1) is a special case of the growth-fragmentation that is discussed in chapter 1, where division conserves mass and when a particle divides, it divides into particles of equal mass. We refer equation (4.1.1) along with the conditions (4.1.2) and the *Initial Condition* as *Perthame & Ryzhik Problem*.

Let

$$n(x,t) = y(x)N(t). \tag{4.1.4}$$

Substituting solution form (4.1.4) into the equation (4.1.1) yields

$$\frac{d}{dx}y(x) + \left(\lambda + B(x)\right)y(x) = \alpha^2 B(\alpha x)y(\alpha x), \quad x \geq 0, \tag{4.1.5}$$

where λ is a constant. For simplicity, we refer equation (4.1.5) along with the *Boundary Conditions* and the *Pdf Conditions* as *y Problem*.

Let $\Psi(x)$ be the adjoint function and $\mathcal{L}^*$ denotes the adjoint operator. Then the condition for adjoint operators that is

$$\int_0^\infty \Psi(x)\mathcal{L}(y)dx = \int_0^\infty y(x)\mathcal{L}^*(\Psi)dx, \tag{4.1.6}$$

gives the adjoint equation

$$\frac{d}{dx}\Psi(x) - (\lambda + B(x))\,\Psi(x) = -\alpha B(x)\Psi\left(\frac{x}{\alpha}\right), \quad x \geq 0, \tag{4.1.7}$$

note that $\mathcal{L}$ denotes the operator for y. The Ψ *Problem* consists of finding a solution to equation (4.1.7) such that

$$\Psi(x) > 0 \quad \text{for} \quad x \geq 0, \quad \text{and} \quad \int_0^\infty y(x)\Psi(x)dx = 1. \tag{4.1.8}$$

4.2 Constant division rate

Suppose $B(x) = b$, where $b > 0$. Integrating the y *Problem* from 0 to ∞ yields

$$\lambda = b(\alpha - 1). \tag{4.2.1}$$

Using (4.2.1) in equation (4.1.5) and (4.1.7) gives

$$\frac{d}{dx}y(x) + b\alpha y(x) = b\alpha^2 y(\alpha x), \quad x \geq 0, \tag{4.2.2}$$

and

$$\frac{d}{dx}\Psi(x) - b\alpha\Psi(x) = -b\alpha\Psi\left(\frac{x}{\alpha}\right), \quad x \geq 0. \tag{4.2.3}$$

We refer equation (4.2.2) along with the *Boundary Conditions* and the *Pdf Conditions* as *y Problem with Constant Coefficients* and equation (4.2.3) along with the conditions (4.1.8) as Ψ *Problem with Constant Coefficients*.

The function $\Psi = 1$ satisfies the Ψ *Problem with Constant Coefficients*; hence, $\Psi = 1$ is the solution to the Ψ *Problem with Constant Coefficients*. The equation (4.2.2) is a pantograph equation. Wake [29] and van Brunt [88] solved the *y Problem with Constant Coefficients* explicitly by applying the Laplace transform and the Mellin transform respectively. The solution form and the uniqueness of the *y Problem with Constant Coefficients* is derived in chapter 2. Perthame & Rhyzik used an approach involving signum function to show the uniqueness of *y Problem with Constant Coefficients*. Here we show a part of that proof in which they showed

$$\operatorname{sgn}(y(x)) = \operatorname{sgn}(y(x/\alpha)), \tag{4.2.4}$$

for all $x > 0$. Above result will be used in the proof of Theorems 4.2.1 to obtain the inequality (4.2.16) from inequality (4.2.15).

Suppose $\operatorname{sgn}(\xi)$ denotes the signum function. Then multiplying equation (4.2.2) by $\operatorname{sgn}(y(x))$ yields

$$\operatorname{sgn}(y(x))\frac{d}{dx}y(x) + b\alpha y(x)\operatorname{sgn}(y(x)) = b\alpha^2 y(\alpha x)\operatorname{sgn}(y(x)).$$

Since $|\xi| = \xi\operatorname{sgn}(\xi)$

$$\frac{d}{dx}|y(x)| + b\alpha|y(x)| = b\alpha^2 y(\alpha x)\operatorname{sgn}(y(x)), \tag{4.2.5}$$

for all $x \in \mathbb{R}$. Integrating (4.2.5) from 0 to ∞ gives

$$b\alpha\int_0^\infty |y(x)|dx = b\alpha^2\int_0^\infty y(\alpha x)\operatorname{sgn}(y(x))dx. \tag{4.2.6}$$

Setting $\alpha x = u$ in the second integral and then dividing equation (4.2.6) by $b\alpha$ yields

$$\int_0^\infty |y(x)|dx = \int_0^\infty y(x)\operatorname{sgn}(y(x/\alpha))dx.$$

which leads to (4.2.4).

The next theorem shows that any solution to the *Perthame & Ryzhik Problem* converges in the $L^1[0,\infty)$ norm to the solution of the *y Problem with Constant Coefficients* as $t \to \infty$.

Theorem 4.2.1. *For $B(x) = b$, where $b > 0$, all solutions to the Perthame & Ryzhik Problem satisfy*

$$\begin{aligned}&||n(x,t)e^{-b(\alpha-1)t} - \langle n_0\rangle y(x)||_{L^1[0,\infty)}\\ &\qquad \le e^{-b(\alpha-1)t}\left[||n_0(x) - \langle n_0\rangle y(x)||_{L^1[0,\infty)} + 2b(\alpha+1)||H^0||_{L^1[0,\infty)}\right],\end{aligned}$$

where

$$\langle n_0\rangle = \int_0^\infty n_0(x)dx,$$

and

$$H^0(x) = \int_0^x [n_0(\xi) - \langle n_0 \rangle y(\xi)]\, d\xi \to 0 \quad as \quad x \to \infty. \tag{4.2.7}$$

Proof: Multiplying equation (4.1.1) by $e^{-b(\alpha-1)t}$ yields

$$\begin{aligned} e^{-b(\alpha-1)t}\frac{\partial}{\partial t}n(x,t) + \frac{\partial}{\partial x}[e^{-b(\alpha-1)t}n(x,t)] \\ + b[e^{-b(\alpha-1)t}n(x,t)] = b\alpha^2[e^{-b(\alpha-1)t}n(\alpha x,t)]. \end{aligned} \tag{4.2.8}$$

Since

$$e^{-b(\alpha-1)t}\frac{\partial}{\partial t}n(x,t) = \frac{\partial}{\partial t}[e^{-b(\alpha-1)t}n(x,t)] + b(\alpha-1)[e^{-b(\alpha-1)t}n(x,t)],$$

equation (4.2.8) becomes

$$\begin{aligned} \frac{\partial}{\partial t}[e^{-b(\alpha-1)t}n(x,t)] + \frac{\partial}{\partial x}[e^{-b(\alpha-1)t}n(x,t)] \\ + b\alpha[e^{-b(\alpha-1)t}n(x,t)] = b\alpha^2[e^{-b(\alpha-1)t}n(\alpha x,t)]. \end{aligned} \tag{4.2.9}$$

Let

$$h(x,t) = n(x,t)e^{-b(\alpha-1)t} - \langle n_0 \rangle y(x),$$

then equation (4.2.9) becomes

$$\begin{aligned} \frac{\partial}{\partial t}h(x,t) + \frac{\partial}{\partial x}h(x,t) + b\alpha h(x,t) - b\alpha^2 h(\alpha x,t) + \frac{\partial}{\partial t}\langle n_0 \rangle y(x) \\ = -\langle n_0 \rangle \left(\frac{\partial}{\partial x}y(x) + b\alpha y(x) - b\alpha^2 y(\alpha x) \right), \end{aligned}$$

where $\frac{\partial}{\partial t}\langle n_0 \rangle y(x) = 0$. Using equation (4.2.2) in above equation gives

$$\frac{\partial}{\partial t}h(x,t) + \frac{\partial}{\partial x}h(x,t) + b\alpha h(x,t) = b\alpha^2 h(\alpha x,t), \tag{4.2.10}$$

where $t > 0$, $x \geq 0$. Equation (4.2.10) satisfies

$$h(0,t) = 0, \quad \int_0^\infty h(x,t)dx = 0, \quad \forall \quad t > 0.$$

Integrating equation (4.2.10) from 0 to x yields

$$\frac{\partial}{\partial t}\int_0^x h(\xi,t)d\xi + \frac{\partial}{\partial x}\int_0^x h(\xi,t)d\xi + b\alpha\int_0^x h(\xi,t)d\xi = b\alpha\int_0^{\alpha x} h(\xi,t)d\xi. \tag{4.2.11}$$

Let

$$H(x,t)=\int_0^x h(\xi,t)d\xi, \tag{4.2.12}$$

then equation (4.2.11) can be written as

$$\begin{cases}\frac{\partial}{\partial x}H(x,t)+\frac{\partial}{\partial t}H(x,t)+b\alpha H(x,t)=b\alpha H(\alpha x,t), & t>0, x\geq 0,\\ H(0,t)=0, \quad H(\infty,t)=0, \quad \forall \ \ t>0.\end{cases} \tag{4.2.13}$$

Using (4.2.12) in equation (4.2.13) gives

$$h(x,t)=-\frac{\partial}{\partial t}H(x,t)-b\alpha H(x,t)+b\alpha H(\alpha x,t).$$

To study the decay rate of h, the determination of $\frac{\partial}{\partial t}H(x,t)$ is required. In order to determine $\frac{\partial}{\partial t}H(x,t)$, we first find $\int_0^\infty |H(x,t)|dx$.

Multiplying equation (4.2.13) by $e^{b(\alpha-1)t}$ implies

$$\frac{\partial}{\partial t}[e^{b(\alpha-1)t}H(x,t)]+\frac{\partial}{\partial x}[e^{b(\alpha-1)t}H(x,t)] \\ +b[e^{b(\alpha-1)t}H(x,t)]=b\alpha[e^{b(\alpha-1)t}H(\alpha x,t)].$$

Multiplying sgn$(H(x,t)$ on both sides of the above equation gives

$$\frac{\partial}{\partial t}|e^{b(\alpha-1)t}H(x,t)|+\frac{\partial}{\partial x}|e^{b(\alpha-1)t}H(x,t)| \\ +b|e^{b(\alpha-1)t}H(x,t)|=b\alpha|e^{b(\alpha-1)t}H(\alpha x,t)\,\mathrm{sgn}(H(x,t)). \tag{4.2.14}$$

Simplifying above equation and then integrating with respect to x from 0 to ∞ and using the boundary conditions from equation (4.2.13) gives

$$\frac{\partial}{\partial t}\int_0^\infty |e^{b(\alpha-1)}H(x,t)|dx+b\int_0^\infty |e^{b(\alpha-1)}H(x,t)|dx \\ \leq b\alpha\int_0^\infty e^{b(\alpha-1)}H(\alpha x,t)\,\mathrm{sgn}(H(x,t))dx. \tag{4.2.15}$$

Using (4.2.4) in the inequality (4.2.15) implies

$$\frac{\partial}{\partial t}\int_0^\infty |e^{b(\alpha-1)t}H(x,t)|dx+b\int_0^\infty |e^{b(\alpha-1)t}H(x,t)|dx\leq b\alpha\int_0^\infty |e^{b(\alpha-1)t}H(\alpha x,t)|dx. \tag{4.2.16}$$

Using a simple substitution $u=\alpha x$ in the first integral on the right reduces above equation to

$$\frac{\partial}{\partial t}\int_0^\infty |e^{b(\alpha-1)t}H(x,t)|dx \leq 0,$$

i.e.,

$$\frac{\partial}{\partial t}\int_0^\infty |H(x,t)|dx \leq e^{-b(\alpha-1)t}\int_0^\infty |H^0(x)|dx, \tag{4.2.17}$$

where $H^0(x)$ is given in (4.2.7).

We are now ready to determine $\frac{\partial}{\partial t}H(x,t)$. Suppose

$$K(x,t) = \frac{\partial}{\partial t}H(x,t). \tag{4.2.18}$$

Using (4.2.18) in equation (4.2.13) and then differentiating the equation with respect to t gives

$$\begin{cases} \frac{\partial}{\partial x}K(x,t) + \frac{\partial}{\partial t}K(x,t) + b\alpha K(x,t) = b\alpha K(\alpha x,t), \quad t > 0, x \geq 0, \\ K(0,t) = 0, \quad K(t,\infty) = 0, \quad \forall \;\; t > 0. \end{cases}$$

Using the definition of K in (4.2.13) gives

$$K(x,t) = b\alpha H(\alpha x,t) - b\alpha H(x,t) - \frac{\partial}{\partial x}H(x,t). \tag{4.2.19}$$

For $t = 0$, equation (4.2.19) yields

$$K_0(x) = b\alpha H^0(\alpha x) - b\alpha H^0(x) - h_0(x), \tag{4.2.20}$$

where $h_0(x) = h(x,0)$ and $K_0(x) = K(x,0)$. Following the steps that are used to obtain (4.2.17) we get

$$\int_0^\infty |K(x,t)|dx \leq e^{-b(\alpha-1)t}\int_0^\infty |K_0(x)|dx. \tag{4.2.21}$$

Using (4.2.20) in equation (4.2.21) gives

$$\int_0^\infty |K(x,t)|dx \leq e^{-b(\alpha-1)t}\left(\int_0^\infty [|b(\alpha+1)H^0(x)| + |h_0(x)|]dx\right). \tag{4.2.22}$$

Substituting $\frac{\partial}{\partial t}H(x,t) = K(x,t)$ in equation (4.2.13) and integrating from 0 to ∞ gives

$$\int_0^\infty |h(x,t)|dx \leq e^{-b(\alpha-1)t}\left(\int_0^\infty |K(x,t)|dx + b(\alpha+1)\int_0^\infty |H^0(x)|dx\right),$$

where $|H(x,0)| = |H^0(x)|$. Using the bounds in (4.2.22) therefore implies

$$\int_0^\infty |h(x,t)|dx \le e^{-b(\alpha-1)t}\left(\int_0^\infty |h_0(x)|dx + 2b(\alpha+1)\int_0^\infty |H^0(x)|dx\right).$$

Theorem (4.2.1) is hence proved. ■

4.3 The variable division case

In this section we first show that in the $L^1[0,\infty)$ norm the solution to the *Perthame & Ryzhik Problem* converges to the solution to the *y Problem* as $t \to \infty$. In order to show the proof we initially assume that the Ψ *Problem* has a unique eigenvalue and corresponding unique and bounded eigenfunction $\Psi(x)$.

We now show the proof of an important theorem in which Perthame & Ryzhik have shown the large time asymptotics of the solution to the *Perthame & Ryzhik Problem.*

Theorem 4.3.1. *Assume that:*

1. $B \in \mathcal{C}(\mathbb{R}^+)$*, and there are positive numbers* b_m*,* b_M *and* b_∞ *such that*

$$(minB(x))_{x\in\mathbb{R}^+} = b_m, \tag{4.3.1}$$

$$(maxB(x))_{x\in\mathbb{R}^+} = b_M, \tag{4.3.2}$$

$$\lim_{x\to\infty} B(x) = b_\infty; \tag{4.3.3}$$

2. *there exists a unique solution* (λ, y, Ψ) *to the y Problem and the* Ψ *Problem with* ψ*,* $y \in \mathcal{C}^1(\mathbb{R})$*;*

3. *the renormalized division rate* $\tilde{B}$ *defined by*

$$\tilde{B}(x) = B(x)\frac{\Psi(x/\alpha)}{\Psi(x)}, \tag{4.3.4}$$

satisfies

$$0 < \tilde{b}_m \leq \tilde{B}(x) \leq \tilde{b}_M < \infty, \tag{4.3.5}$$

for some constants $\tilde{b}_m$ *and* $\tilde{b}_M$*. In addition,* Ψ *satisfies*

$$\tilde{c}(1 + x^{k_0}) \leq \Psi(x) \leq \tilde{C}(1 + x^{k_0}), \tag{4.3.6}$$

where $\tilde{c}$*,* $\tilde{C}$ *and* k_0 *are positive constants such that* $\alpha^{k_0} = \frac{\alpha b_\infty}{\lambda + b_\infty}$*;*

4. $y(x)$ *decays rapidly, that is for all* $p > 0$*,*

$$\int_0^\infty x^p y(x) dx < \infty;$$

and

5. *the following bounds are satisfied*

$$b_m(\alpha - 1) \leq \lambda \leq b_M(\alpha - 1).$$

Then there exists a constant $E > 0$ *such that*

$$\gamma := ||\tilde{B}(x) - E||_{L^\infty[0,\infty)} < \frac{E}{\alpha^2(\alpha + E)}, \tag{4.3.7}$$

and the solution to the Perthame & Ryzhik Problem satisfies

$$\begin{aligned} &|| \left(n(x,t)e^{-\lambda t} - \langle n_0 \rangle y(x)\right) \Psi(x)||_{L^1[0,\infty)} \\ &\leq e^{-\mu t} \left[\alpha|| \left(n_0(x) - \langle n_0 \rangle y(x)\right) \Psi(x)||_{L^1[0,\infty)} + \alpha^2 E ||H^0||_{L^1[0,\infty)}\right], \end{aligned}$$

where $\mu = (\alpha-1)E - \alpha^3\gamma - \alpha^2 E\gamma$*, and* $H^0(x) = \int_0^x [n_0(\xi) - \langle n_0 \rangle y(\xi)]\, d\xi \to 0$ *as* $x \to \infty$*.*

Proof: Multiplying equation (4.1.1) by $e^{-\lambda t}$ yields

$$e^{-\lambda t}\frac{\partial}{\partial t}n(t,x) + \frac{\partial}{\partial x}[e^{-\lambda t}n(x,t)] + B(x)[e^{-\lambda t}n(x,t)] = \alpha^2 B(\alpha x)[e^{-\lambda t}n(\alpha x,t)]. \tag{4.3.8}$$

Using

$$e^{-\lambda t}\frac{\partial}{\partial t}n(x,t) = \frac{\partial}{\partial t}[e^{-\lambda t}n(x,t)] + \lambda[e^{-\lambda t}n(x,t)],$$

in equation (4.3.8) gives

$$\frac{\partial}{\partial t}[e^{-\lambda t}n(x,t)] + \frac{\partial}{\partial x}[e^{-\lambda t}n(x,t)] + (B(x)+\lambda)[e^{-\lambda t}n(x,t)] \\ = \alpha^2 B(\alpha x)[e^{-\lambda t}n(\alpha x,t)].$$

Let

$$h(x,t) = n(x,t)e^{-\lambda t} - \langle n_0\rangle y(x),$$

then equation (4.3.8) yields

$$\frac{\partial}{\partial t}h(x,t) + \frac{\partial}{\partial x}h(x,t) + (B(x)+\lambda)h(x,t) - \alpha^2 B(\alpha x)h(\alpha x,t) + \frac{\partial}{\partial t}\langle n_0\rangle y(x) \\ = -\langle n_0\rangle \left(\frac{\partial}{\partial x}y(x) + (B(x)+\lambda)\alpha y(x) - \alpha^2 B(\alpha x)y(\alpha x)\right),$$

where $\frac{\partial}{\partial t}\langle n_0\rangle y(x) = 0$. Equation (4.1.5) implies

$$\frac{\partial}{\partial t}h(x,t) + \frac{\partial}{\partial x}h(x,t) + (B(x)+\lambda)h(x,t) = \alpha^2 B(\alpha x)h(\alpha x,t), \tag{4.3.9}$$

where $t > 0$, $x \geq 0$ and $h(0,t) = 0$. Multiplying equation (4.3.9) by $\Psi(x)$ gives

$$\frac{\partial}{\partial t}\Psi(x)h(x,t) + \frac{\partial}{\partial x}\Psi(x)h(x,t) + (B(x)+\lambda)h(x,t)\Psi(x) \\ - h(x,t)\frac{\partial}{\partial x}\Psi(x) = \alpha^2 B(\alpha x)h(\alpha x,t)\Psi(x), \tag{4.3.10}$$

using equation (4.1.7) in 4.3.10 gives

$$\frac{\partial}{\partial t}\Psi(x)h(x,t) + \frac{\partial}{\partial x}\Psi(x)h(x,t) + \alpha\Psi\left(\frac{x}{\alpha}\right)B(x)h(x,t) = \alpha^2 B(\alpha x)h(\alpha x,t)\Psi(x). \tag{4.3.11}$$

Dividing both sides of equation (4.1.7) by $\Psi(x)$ and using (4.3.4) gives

$$\lambda + B(x) = \frac{\Psi'(x)}{\Psi(x)} + \alpha\tilde{B}(x).$$

Let

$$q(x,t) = \Psi(x)h(x,t). \tag{4.3.12}$$

Then equation (4.3.11) implies

$$\frac{\partial}{\partial t}q(x,t)+\frac{\partial}{\partial x}q(x,t)+\alpha B(x)\frac{\Psi\left(\frac{x}{\alpha}\right)}{\Psi(x)}q(x,t)=\alpha^2 B(\alpha x)\frac{\Psi(x)}{\Psi(\alpha x)}q(\alpha x,t),$$

i.e,

$$\frac{\partial}{\partial t}q(x,t)+\frac{\partial}{\partial x}q(x,t)+\alpha\tilde{B}(x)q(x,t)=\alpha^2\tilde{B}(\alpha x)q(\alpha x,t). \tag{4.3.13}$$

Note that $q(0,t)=0$, since $h(0,t)=0$.

To prove the bounds (4.3.7), we first show the exponential decay of the anti derivative of $q(x,t)$. Integrating equation (4.3.13) from 0 to x yields

$$\frac{\partial}{\partial t}\int_0^x q(\xi,t)d\xi+\frac{\partial}{\partial x}\int_0^x q(\xi,t)d\xi+\alpha\int_0^x\tilde{B}(\xi)q(\xi,t)d\xi=\alpha\int_0^{\alpha x}\tilde{B}(\xi)q(\xi,t)d\xi.$$

Let

$$H(x,t)=\int_0^x q(\xi,t)d\xi, \tag{4.3.14}$$

then recasting equation (4.2.11) in terms of H gives

$$\begin{cases}\frac{\partial}{\partial x}H(x,t)+\frac{\partial}{\partial t}H(x,t)+\alpha\int_0^x\tilde{B}(\xi)q(\xi,t)d\xi=\alpha\int_0^{\alpha x}\tilde{B}(\xi)q(\xi,t)d\xi, & t>0, x\geq 0,\\ H(0,t)=0,\quad H(\infty,t)=0,\quad \forall\ \ t>0.\end{cases}$$

Adding and subtracting $EH(x,t)-EH(\alpha x,t)$ in the above equation gives

$$\begin{aligned}\frac{\partial}{\partial t}H(x,t)+\frac{\partial}{\partial x}H(x,t)&+\alpha EH(x,t)-\alpha EH(\alpha x,t)\\ &=\alpha\int_x^{\alpha x}(\tilde{B}(\xi)-E)q(\xi,t)d\xi.\end{aligned}$$

Multiplying the above equation by $\operatorname{sgn}(H(x,t))$ gives

$$\begin{aligned}\frac{\partial}{\partial t}|H(x,t)|+\frac{\partial}{\partial x}|H(x,t)|&+\alpha E|H(x,t)|-\alpha E|H(\alpha x,t)|\\ &\leq\alpha\int_x^{\alpha x}|(\tilde{B}(\xi)-E)q(\xi,t)|d\xi.\end{aligned}$$

Integrating this inequality with respect to x from 0 to ∞ and using the boundary conditions for H yields

$$\begin{aligned}\frac{\partial}{\partial t}||H(t)||_{L^1[0,\infty)}+E(\alpha-1)||H(t)||_{L^1[0,\infty)}&\leq\alpha\int_{x=0}^{\infty}\int_{\xi=x}^{\alpha x}|(\tilde{B}(\xi)-E)q(\xi,t)|d\xi dx,\\ &\leq\alpha\int_{x=0}^{\infty}\int_{\xi=0}^{\infty}|(\tilde{B}(\xi)-E)q(\xi,t)|d\xi dx,\end{aligned}$$

where $||H(t)||_{L^1[0,\infty)}$ denotes the $L^1[0,\infty)$ norm of H with respect to x. The above inequality can be written as

$$\frac{\partial}{\partial t}||H(t)||_{L^1[0,\infty)} + E(\alpha-1)||H(t)||_{L^1[0,\infty)} \leq \gamma||q(t)||_{L^1[0,\infty)}. \tag{4.3.15}$$

Multiplying (4.3.15) by $e^{E(\alpha-1)t}$ and then integrating with respect to t from 0 to ∞ gives

$$||H(t)||_{L^1[0,\infty)} \leq e^{-E(\alpha-1)t}||H^0||_{L^1[0,\infty)} + \gamma\int_0^t e^{-E(\alpha-1)(t-s)}||q(s)||_{L^1[0,\infty)}ds, \tag{4.3.16}$$

where H^0 is the value of $H(t)$ for $t=0$ and γ is given in (4.3.7). Inequality (4.3.16) shows the exponential decay of the anti-derivative of the function $q(x,t)$. Due to the form of integrand in (4.3.16), the steps given in constant coefficient case cannot be followed. Therefore a function $\phi(x,t)$ is introduced. Let

$$\phi(x,t) = q(x,t)e^{\alpha Et}. \tag{4.3.17}$$

Then multiplying the equation (4.3.13) by $e^{E\alpha t}$ yields

$$\frac{\partial}{\partial t}\phi(x,t) + \frac{\partial}{\partial x}\phi(x,t) = \alpha^2 E\phi(\alpha x,t) + R(x,t), \tag{4.3.18}$$

where $R(x,t) = e^{\alpha Et}[\alpha^2(\tilde{B}(\alpha x) - E)q(\alpha x,t) - \alpha(\tilde{B}(x) - E)q(x,t)]$. Let ξ and η be the characteristic variables, then integrating equation (4.3.18) along the characteristics projections gives

$$\frac{dt}{d\xi} = 1, \quad t(0,\eta) = 0, \frac{dx}{d\xi} = 1, \quad x(0,\eta) = \eta,$$

so that $x = \xi + \eta$ and $t = \xi$; hence,

$$\frac{d\phi}{d\xi} = \alpha^2 E\phi(\alpha(\xi+\eta),\xi) + R((\xi+\eta),\xi), \quad \phi(\eta,0) = \phi_0(\eta). \tag{4.3.19}$$

Integrating (4.3.19) with respect to ξ yields

$$\phi(\eta,\xi) = \phi_0(\eta) + E\alpha^2\int_0^\xi \phi(\alpha\sigma+\alpha\eta,\sigma)d\sigma + \int_0^\xi R(\sigma+\eta,\sigma)d\sigma,$$

and recasting ϕ in terms of t and x gives

$$\phi(x,t) = \phi_0(x-t) + E\alpha^2\int_0^t \phi(\alpha\sigma+\alpha(x-t),\sigma)d\sigma + \int_0^t R(\sigma+(x-t),\sigma)d\sigma. \tag{4.3.20}$$

Let $\sigma = t - u$, then equation (4.3.20) implies

$$\phi(x,t) = \phi_0(x-t) - E\alpha^2 \int_t^0 \phi(\alpha(x-u), t-u)du - \int_t^0 R((x-u), t-u)du,$$

i.e.,

$$\phi(x,t) = \phi_0(x-t) + E\alpha^2 \int_0^t \phi(\alpha(x-u), t-u)du + \int_0^t R((x-u), t-u)du. \quad (4.3.21)$$

Iterating the formula for ϕ, first integral on the right hand side in the equation (4.3.21) can be written as

$$\begin{aligned} \int_0^t \phi(\alpha(x-u), t-u)du = & \int_0^t \phi_0(\alpha x - t - (\alpha-1)u)du \quad (4.3.22) \\ & + E\alpha^2 \int_0^t \int_{s=0}^{t-u} \phi((\alpha^2(x-u) - \alpha s), t-u-s)dsdu \\ & + \int_0^t \int_{s=0}^{t-u} R((\alpha x - \alpha u - s), t-u-s)dsdu. \end{aligned}$$

Substituting (4.3.22) in (4.3.21) gives

$$\begin{aligned} \phi(x,t) = & \phi_0(x-t) + \int_0^t R((x-u), t-u)du + E\alpha^2 \int_0^t \phi_0(\alpha x - t - (\alpha-1)u)du \\ & + E\alpha^2 \int_0^t \int_{s=0}^{t-u} R((\alpha x - \alpha u - s), t-u-s)dsdu \\ & + (E\alpha^2)^2 \int_0^t \int_{s=0}^{t-u} \phi((\alpha^2(x-u) - \alpha s), t-u-s)dsdu. \quad (4.3.23) \end{aligned}$$

Let $s + u = v$, then (4.3.23) implies

$$\begin{aligned} \phi(x,t) = & \phi_0(x-t) + \int_0^t R((x-u), t-u)du + E\alpha^2 \int_0^t \phi_0(\alpha x - t - (\alpha-1)u)du \\ & + E\alpha^2 \int_0^t \int_{u=0}^{v} R((\alpha x - v - (\alpha-1)u), t-v)dudv \\ & + (E\alpha^2)^2 \int_0^t \int_{u=0}^{v} \phi((\alpha^2 x - \alpha v - \alpha(\alpha-1)u), t-v)dudv; \end{aligned}$$

thus,

$$\begin{aligned}||\phi(t)||_{L^1[0,\infty)} \leq &\int_0^\infty |\phi_0(x-t)|dx \\ &+ \int_{x=0}^\infty \int_{u=0}^t |R((x-u), t-u))|dudx + E\alpha^2 \int_{x=0}^\infty \int_{u=0}^t |\phi_0(\alpha x - t - (\alpha-1)u)|dudx \\ &+ E\alpha^2 \int_{x=0}^\infty \int_{v=0}^t \int_{u=0}^v |R((\alpha x - v - (\alpha-1)u), t-v)|dudvdx \\ &+ (E\alpha^2)^2 \int_{x=0}^\infty \int_{v=0}^t \int_{u=0}^v |\phi((\alpha^2 x - \alpha v - \alpha(\alpha-1)u), t-v)|dudvdx,\end{aligned} \tag{4.3.24}$$

where ϕ_0, R and ϕ are known to be non negative. We first estimate the bound for the last integral in equation (4.3.24). Let I_L denote the last integral of equation (4.3.24). Then substituting (4.3.17) in I_L yields

$$\begin{aligned}I_L &= (E\alpha^2)^2 \int_{x=0}^\infty \int_{v=0}^t \int_{u=0}^v e^{\alpha E(t-v)}|q((\alpha^2 x - \alpha v - \alpha(\alpha-1)u), t-v)|dudvdx, \\ &= (E\alpha^2)^2 \int_{x=0}^\infty \int_{v=0}^t e^{\alpha E(t-v)} \int_{u=0}^v |q((\alpha^2 x - \alpha v - \alpha(\alpha-1)u), t-v)|dudvdx.\end{aligned}$$

Using the substitution (4.3.14) in the above integral gives

$$\begin{aligned}I_L &= (E\alpha^2)^2 \int_{x=0}^\infty \int_{v=0}^t e^{\alpha E(t-v)}|H((\alpha^2 x - \alpha^2 v), t-v)|dvdx, \\ &= (E\alpha^2)^2 \int_{v=0}^t \int_{x=0}^\infty e^{\alpha E(t-v)}|H((\alpha^2 x - \alpha^2 v), t-v)|dxdv.\end{aligned} \tag{4.3.25}$$

Let $\alpha^2 x = z$. Then equation (4.3.25) becomes

$$\begin{aligned}I_L &= E^2\alpha^2 \int_0^t e^{\alpha E(t-v)} \int_{z=0}^\infty |H(z - \alpha^2 v, t-v)|dzdv \\ &\leq E^2\alpha^2 \int_0^t e^{\alpha E(t-v)}||H(t-v)||_{L^1[0,\infty)}dv.\end{aligned}$$

Let $p = (t-v)$ then

$$I_L \leq E^2\alpha^2 \int_0^t e^{\alpha E(p)}||H(p)||_{L^1[0,\infty)}dp.$$

In the similar fashion we can find the bounds for every term in the equation (4.3.24), and this gives

$$
\begin{aligned}
||\phi(t)||_{L^1[0,\infty)} \leq & (1+\alpha E t)||\phi_0||_{L^1[0,\infty)} + \alpha^2\gamma \int_0^t ||\phi(u)||_{L^1[0,\infty)} du \\
& + \alpha^3 E\gamma \int_0^t (t-u)||\phi(u)||_{L^1[0,\infty)} du + \alpha^2 E^2 \int_0^t e^{\alpha E v}||H(v)||_{L^1[0,\infty)} dv.
\end{aligned}
\tag{4.3.26}
$$

Using inequality (4.3.16), the last integral on the right hand side of inequality (4.3.26) becomes

$$
\begin{aligned}
\alpha^2 E^2 \int_0^t e^{\alpha E v}||H(v)||_{L^1[0,\infty)} dv = & \alpha^2 E^2 ||H^0||_{L^1[0,\infty)} \int_0^t e^{\alpha E v} e^{-E(\alpha-1)v} dv \\
& + \alpha^2 b^2 \gamma \int_0^t e^{E\alpha v} \int_0^v e^{-E(\alpha-1)(v-s)}||q(s)||_{L^1[0,\infty)} ds dv.
\end{aligned}
$$

In terms of ϕ, the above equation is

$$
\begin{aligned}
\alpha^2 E^2 \int_0^t e^{\alpha E v}||H(v)||_{L^1[0,\infty)} dv = & \alpha^2 E \left(e^{Et}-1\right)||H^0||_{L^1[0,\infty)} \\
& + \alpha^2 E^2 \gamma \int_0^t e^{Ev} \int_0^v e^{-Es}||\phi(s)||_{L^1[0,\infty)} ds dv.
\end{aligned}
\tag{4.3.27}
$$

Changing the order of integration in the second term on the right hand side of above inequality yields

$$
\alpha^2 E^2 \gamma \int_0^v e^{-Es}||\phi(s)||_{L^1[0,\infty)} \int_0^t e^{Ev} dv ds \leq \alpha^2 E\gamma \int_0^v e^{-E(t-s)}||\phi(s)||_{L^1[0,\infty)} ds,
$$

since $0 \leq v \leq t$

$$
\alpha^2 E^2 \gamma \int_0^v e^{-Es}||\phi(s)||_{L^1[0,\infty)} \int_0^t e^{Ev} dv ds \leq \alpha^2 E\gamma \int_0^t e^{-E(t-s)}||\phi(s)||_{L^1[0,\infty)} ds.
$$

Using above bound and equation (4.3.27) in (4.3.26) implies

$$
\begin{aligned}
||\phi(t)||_{L^1[0,\infty)} \leq & (1+\alpha E t)||\phi_0||_{L^1[0,\infty)} + \alpha^2\gamma \int_0^t \left(1+E\alpha(t-u)\right)||\phi(u)||_{L^1[0,\infty)} du \\
& + \alpha^2 E(e^{Et}-1)||H^0||_{L^1[0,\infty)} + \alpha^2 E\gamma \int_0^t e^{E(t-v)}||\phi(v)||_{L^1[0,\infty)} dv.
\end{aligned}
\tag{4.3.28}
$$

Because $(1+\alpha E t) \leq \alpha e^{Et}$ and $\alpha^2 E(e^{Et}-1) \leq \alpha^2 E e^{Et}$, inequality (4.3.28) reduces to

$$
\begin{aligned}
||\phi(t)||_{L^1[0,\infty)} \leq & \alpha e^{Et}||\phi_0||_{L^1[0,\infty)} + \alpha^2 E e^{Et}||H^0||_{L^1[0,\infty)} \\
& + \alpha^2(\alpha+E)\gamma \int_0^t e^{E(t-v)}||\phi(v)||_{L^1[0,\infty)} dv,
\end{aligned}
$$

that is

$$e^{-Et}||\phi(t)||_{L^1[0,\infty)} \le \alpha||\phi_0||_{L^1[0,\infty)} + \alpha^2 E||H^0||_{L^1[0,\infty)} + \alpha^2(\alpha+E)\gamma\int_0^t e^{-Ev}||\phi(v)||_{L^1[0,\infty))}dv. \quad (4.3.29)$$

Let

$$p(t) = \int_0^t e^{-Es}||\phi(s)||_{L^1[0,\infty)}ds,$$

then

$$p^{'}(t) = e^{-Et}||\phi(t)||_{L^1[0,\infty)}, \quad (4.3.30)$$

where $p^{'}(t) = \frac{dp(t)}{dt}$. Recasting inequality (4.3.29) in terms of p gives

$$p^{'}(t) \le \alpha||\phi_0||_{L^1[0,\infty)} + \alpha^2 E||H^0||_{L^1[0,\infty)} + \alpha^2(\alpha+E)\gamma E p(t), \quad (4.3.31)$$

with $p(0) = 0$.

Let A_1, A_2 and f be real-valued functions defined on the interval I. Assume that A_2 and f are continuous and that the negative part of A_1 is integrable on every closed and bounded subinterval of I. Gronwall's lemma tells that if

$$f(\tau) \le A_1(\tau) + \int_o^\tau A_2(t)f(t)dt,$$

then for a non decreasing A_1,

$$f(\tau) \le A_1(\tau)e^{\int_o^\tau A_2(t)f(t)dt}.$$

For this problem $f(\tau) = p^{'}(t)$, $A_1(\tau) = \alpha||\phi_0||_{L^1[0,\infty)} + \alpha^2 E||H^0||_{L^1[0,\infty)}$, $A_2(\tau) = \alpha^2(\alpha+E)\gamma$ and $f(t) = p(t)$. Applying Gronwall's lemma to the inequality (4.3.31) gives

$$p(t) \le \frac{e^{\alpha^2(\alpha+E)\gamma t}}{\alpha^2(\alpha+E)\gamma}[\alpha||\phi_0||_{L^1[0,\infty)} + \alpha^2 E||H^0||_{L^1[0,\infty)}]. \quad (4.3.32)$$

Differentiating above equation with respect to t we get

$$p^{'}(t) \le e^{\alpha^2(\alpha+E)\gamma t}[\alpha||\phi_0||_{L^1[0,\infty)} + \alpha^2 E||H^0||_{L^1[0,\infty)}], \quad (4.3.33)$$

and using (4.3.30) in the inequality (4.3.33) gives

$$||\phi(t)||_{L^1[0,\infty)} = e^{Et}p^{'}(t) \le [\alpha||\phi_0||_{L^1[0,\infty)} + \alpha^2 E||H^0||_{L^1[0,\infty)}](e^{\alpha^2(\alpha+E)\gamma t})e^{Et}.$$

In terms of h and Ψ, above inequality can be written as

$$||h(t)\Psi(x)||_{L^1[0,\infty)} \leq \Big(\alpha||\phi_0||_{L^1[0,\infty)} + \alpha^2 E||H^0||_{L^1[0,\infty)}\Big)\, e^{(\alpha^2(\alpha+E)\gamma-(\alpha-1)E)t}.$$

Theorem 4.3.1 is hence proved. ∎

We now discuss the proof of Theorem 4.3.2 that gives a consolidated overview of the construction of Perthame & Ryzhik's analysis.

Theorem 4.3.2. *Suppose that,*

$B(x) \in \mathcal{C}(\mathbb{R}^+)$, *and there are positive numbers* b_m *and* b_M *such that*

$$(minB(x))_{x\in\mathbb{R}^+} = b_m, \tag{4.3.34}$$

$$(maxB(x))_{x\in\mathbb{R}^+} = b_M. \tag{4.3.35}$$

Then:

1. *there exists a unique solution* (λ, y, Ψ) *to the* y *Problem and the* Ψ *Problem with* Ψ, $y \in \mathcal{C}^1$;

2. $y(x)$ *decays rapidly for all* $p > 0$

$$\int_0^\infty x^p y(x)dx < \infty;$$

and

3. *the following bounds are satisfied*

$$b_m(\alpha-1) \leq \lambda \leq b_M(\alpha-1),$$

$$\frac{c}{(1+x)^k} \leq \Psi(x) \leq C(1+x^k),$$

where c, C *and* k *are positive constants and* k *satisfies* $\alpha^k > \frac{b_M}{b_m}$.

In order to prove Theorem (4.3.2), following Perthame & Ryzhik, we first consider the *y Problem* and the Ψ *Problem* in a bounded domain and show the existence of a positive eigenvalue and associated eigenfunction for the *y Problem* and the Ψ *Problem.* Perthame & Ryzhik then pass on the limit and showed the uniform convergence.

Perthame & Ryzhik also showed that Ψ is a bounded function.

Let $\Xi = [0, L]$, $L > 0$ be the truncated domain. Then the *y Problem* on Ξ is

$$\begin{cases} \frac{d}{dx} y_L(x) + (\lambda_L + B(x))\, y_L(x) = \alpha^2 B(\alpha x) y_L(\alpha x), \;\; 0 < x < L/\alpha, \\ \\ \frac{d}{dx} y_L(x) + (\lambda_L + B(x))\, y_L(x) = 0, \;\; L/\alpha \le x < L, \\ \\ y_L(0) = y_L(L) = 0. \end{cases} \tag{4.3.36}$$

We refer equations (4.3.36) along with the conditions

$$y_L(x) > 0 \quad \text{for} \quad x > 0, \quad \text{and} \quad \int_0^L y_L(x) dx = 1. \tag{4.3.37}$$

as y_L *Problem*. In similar fashion, the Ψ *Problem* on Ξ becomes

$$\begin{cases} \frac{d}{dx} \Psi_L(x) - (\lambda_L + B(x))\, \Psi_L(x) = -\alpha B(x) \Psi_L(\frac{x}{\alpha}), \quad x \ge 0, \\ \\ \Psi_L(L) = 0, \end{cases} \tag{4.3.38}$$

that along with the conditions

$$\Psi_L(x) > 0 \quad \text{for} \quad x > 0, \quad \text{and} \quad \int_0^L y_L(x) \Psi_L(x) dx = 1. \tag{4.3.39}$$

will be referred as Ψ_L *Problem.*

4.3.1 Uniqueness of λ

In order to show the uniqueness of λ, Perthame & Ryzhik considered the *y Problem* and the Ψ *Problem* in a truncated domain. In the following theorem they used the finite difference scheme and the Perron Frobenuis theorem to show the existence of the unique eigenvalue and corresponding eigenfunction to the y_L *Problem* and the Ψ_L *Problem.*

Lemma 4.3.3. *Suppose B satisfies (4.3.1), (4.3.2) and let $L > 0$. Then there exists a unique solution (λ_L, y_L, Ψ_L) to the y_L Problem and the Ψ_L Problem such that y_L is Lipschitz continuous and $\Psi_L \in C^1$.*

Proof: The finite difference approximation for a function y_i is defined as

$$\frac{d}{dx_i} y_i = \frac{y_i - y_{i-1}}{h}, \tag{4.3.40}$$

where $y_i = y(x_i)$, $x_{i-1} = x_i - h$, h is the interval size and $1 \leq i \leq L$. Dividing the interval $(0, L)$ into n subintervals, choosing the interval size equal to $\frac{L}{n}$ and applying the finite difference method to the y_L *Problem* gives a discretized equation

$$\begin{cases} \frac{y_i - y_{(i-1)}}{L/n} + (\lambda + b_i)\, y_i = \alpha^2 b_{i\alpha} y_{i\alpha}, \quad 1 \leq i \leq L/h, \\ \\ \frac{y_i - y_{(i-1)}}{L/n} + (\lambda + b_i)\, y_i = 0, \quad L/h \leq i < L, \\ \\ y_0 = y_L = 0, \end{cases} \tag{4.3.41}$$

where $y_i = y(x_i)$, $y_{i\alpha} = y(\alpha x_i)$, $b_{i\alpha} = b(\alpha x_i)$. To evaluate $y_{i\alpha}$ at any grid point x_i, set $y(\alpha x_i) = 0$ for $\alpha x_i \geq L$. It is straightforward from the definition (4.3.40) that y_i' is bounded by M, where $M > 0$ for all $i \geq 1$. A set of functions $\{y_k | \; 0 \leq k \leq L\}$ with a common Lipschitz constant is therefore equicontinuous. For an integer value of α, discretized equation (4.3.41) immediately generates a square matrix. But for a non integer α, the discretized system (4.3.41) does not generate a square matrix straight away. However a linear interpolation technique can be employed to generate a square matrix for such α. To make more sense to later case we generate a square matrix for a non integer α. It can easily be shown for an integer α.

For $i = 1$, $x_1 = \frac{L}{n}$ and equation (4.3.41) becomes

$$\left(\frac{n}{L} + (\lambda + b_1)\right) y_1 - \alpha^2 b_{1\alpha} y_{1\alpha} = 0. \tag{4.3.42}$$

Let $j < \alpha < j + 1$, where $j \geq 1$ and is a positive integer. Applying linear interpolation to the functional term in the equation (4.3.42) gives $y_\alpha\left(\frac{\alpha L}{n}\right) = m_{11} y_j + m_{12} y_{j+1}$, where m_{11} and m_{12} are positive constants. Equation (4.3.42) therefore can be written as

$$\left(\frac{n}{L} + (\lambda + b_1)\right) y_1 - \alpha^2 m_{11} b_i y_j - \alpha^2 m_{12} b_{j+1} y_{j+1} = 0.$$

For $x = \frac{2L}{n}$, equation (4.3.41) yields

$$-\frac{n}{L} y_1 + \left(\frac{n}{L} + (\lambda + b_2)\right) y_2 - \alpha^2 b_{2\alpha} y_{2\alpha} = 0, \tag{4.3.43}$$

applying linear interpolation to equation (4.3.43) gives

$$-\frac{n}{L}y_1 + \left(\frac{n}{L} + (\lambda + b_2)\right) y_2 - \alpha^2 m_{21} b_{2j} y_{2j} - \alpha^2 m_{22} b_{2(j+1)} y_{2(j+1)} = 0.$$

This can be continued to obtain the equations for the rest of the intervals. Note that the functional term vanishes when argument of the functional term is greater or equal to n. A square matrix A generated by the equation (4.3.41) is thus

$$\begin{pmatrix}
(\frac{n}{L} + \lambda_L + b_1)) & 0 & \dots & -\alpha^2 m_{11} b_j & -\alpha^2 m_{12} b_{j+1} & 0 & 0 & 0 & 0 \\
-\frac{n}{2L} & (\frac{n}{L} + \lambda_L + b_2) & 0 & \dots & -\alpha^2 m_{21} b_{2j} & -\alpha^2 m_{22} b_{2(j+1)} & 0 & 0 & 0 \\
0 & -\frac{n}{2L} & (\frac{n}{L} + \lambda_L + b_3)) & 0 & \dots & -\alpha^2 m_{31} b_{3j} & -\alpha^2 m_{32} b_{3(j+1)} & \dots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \dots & (-\frac{n}{2L}) & (\frac{n}{L} + \lambda_L + b_k)) & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & (-\frac{n}{2L}) & (\frac{n}{L} + \lambda_L + b_1))
\end{pmatrix}$$

In matrix $A = a_{ij}$, all the off diagonal entries are negative that is $a_{ij} < 0$, where $i \neq j$. Such a matrix is an M matrix (cf. [80]). Perthame & Ryzhik [74] applied the Perron Frobenius theorem to show that there exists a first eigenvalue λ_{disc} to matrix A associated with a positive eigenvector y_{disc}.

Since y_{disc} is eqicontinuous, it can be shown that the solution of this discrete model $(\lambda_{disc}, y_{disc})$ uniformly converges to (λ_L, y_L) by applying Ascoli theorem to the problem. Construction of a dual eigenfunction Ψ_L and eigenvalue follows the similar arguments. ■

4.3.2 Bounds for λ_L

We now determine eigenvalues. Determination of λ_L is the only place where Perthame & Ryzhik used the condition (4.3.34). In chapter 5 we use another technique to determine λ that does not require positivity of B at the origin.

Lemma 4.3.4. *If B satisfies (4.3.34) and (4.3.35) then the following bounds are satisfied*

$$b_m(\alpha-1)-\frac{1}{L}\leq\lambda_L\leq b_M(\alpha-1).$$

Proof: Integrating equation (4.3.36) from 0 to L and using the boundary condition (**??**) gives

$$y_L(L)+\lambda_L\int_0^L y_L(x)dx=(\alpha-1)\int_0^L B(\xi)y_L(\xi)d\xi. \tag{4.3.44}$$

Condition (4.3.34) yields

$$y_L(L)+\lambda_L\int_0^L y_L(x)dx\leq b_M(\alpha-1)\int_0^L y_L(\xi)d\xi, \tag{4.3.45}$$

which implies

$$\lambda_L\leq b_M(\alpha-1). \tag{4.3.46}$$

In order to obtain the lower bound multiplying the equation (4.3.36) by x and then integrating with respect to x and using the boundary condition (**??**) gives

$$Ly_L(L)-\int_0^L y_L(x)dx+\int_0^L x(\lambda_L+B(x))y_L=\int_0^{\alpha L}\xi B(\xi)y_L(\xi)d\xi. \tag{4.3.47}$$

Since $\int_0^{\alpha L}\xi B(\xi)y_L(\xi)d\xi=\int_0^L\xi B(\xi)y_L(\xi)d\xi$,

$$Ly_L(L)-\int_0^L y_L(x)dx+\lambda_L\int_0^L xy_L(x)dx=0. \tag{4.3.48}$$

Multiplying equation (4.3.44) by L and then subtracting from equation (4.3.48) gives

$$\lambda_L\geq b_m(\alpha-1)-\frac{1}{L}. \tag{4.3.49}$$

Combining (4.3.49) and (4.3.46) yields

$$b_m(\alpha-1)-\frac{1}{L}\leq\lambda_L\leq b_M(\alpha-1). \tag{4.3.50}$$

■

It was shown that y_L is compact and bounded in $L^1[0, \infty)$. The Bolzano–Weierstrass theorem can be used to extract a convergent subsequence λ and y from λ_L and y_L.

In order to show the positivity of the solution they applied the method of the characteristics to the equation (4.1.5) which gives

$$y(x) = \alpha^2 e^{\lambda + B(x)} \int_0^x B(\alpha\xi) y(\alpha\xi) e^{\lambda + B(\xi)} d\xi.$$

The positivity of $y(x)$ depends on the positivity of $y(\alpha x)$. Let $m = \inf\{x \mid y(x) > 0\}$. For the positivity of $y(\alpha x)$, x needs to be greater than m/α. Continuity of y in the given interval therefore implies $m = 0$. Consequently $y(x) > 0$ for all $x > 0$.

4.3.3 Bounds for Ψ_L

Lemma 4.3.5. *Let Ψ_L be a solution to the Ψ_L Problem, then*

$$\Psi_L(x) \leq C(1 + x^k),$$

for all $x > 0$, C and k are positive constants and k satisfies

$$\alpha^k > \frac{b_M}{b_m}. \tag{4.3.51}$$

Proof: In order to prove bounds for Ψ_L, Perthame & Ryzhik first established

$$\sup_{0 \leq x \leq A} \Psi_L(x) \leq K(A), \tag{4.3.52}$$

for all $x \in [0, A]$, where $A \leq L$. Here, we show their proof for (4.3.52).

Recasting equation (**??**) gives

$$\frac{\partial}{\partial x}[\Psi_L(x) e^{-\int_0^x (\lambda_L + B(\xi)) d\xi}] = -\alpha B(x) e^{-\int_0^x (\lambda_L + B(\xi)) d\xi} \Psi_L(x/\alpha). \tag{4.3.53}$$

Let $x_1 \leq x \leq x_2$. Then integrating equation (4.3.53) from x_1/α to x_2/α gives

$$\begin{aligned} \frac{\Psi_L(x_2/\alpha)}{J(x_2/\alpha)} &= \frac{\Psi_L(x_1/\alpha)}{J(x_1/\alpha)} - \alpha \int_{x_1/\alpha}^{x_2/\alpha} \frac{B(\xi)}{J(\xi)} \Psi(\xi/\alpha) d\xi \\ &\leq \frac{\Psi_L(x_1/\alpha)}{J(x_1/\alpha)}, \end{aligned} \tag{4.3.54}$$

where $J(x) = e^{\int_0^x (\lambda_L + B(\tau))d\tau}$. Evidently, $J(x) \geq 1$ and hence

$$\frac{\Psi_L(x_1/\alpha)}{J(x_1/\alpha)} \leq \Psi_L(x_1/\alpha), \tag{4.3.55}$$

so that

$$\frac{\Psi_L(x_2/\alpha)}{J(x_2/\alpha)} \leq \Psi_L(x_1/\alpha). \tag{4.3.56}$$

For any $x \in (x_1, x_2)$, integrating equation (4.3.53) from x_1 to x, and using the inequality (4.3.55) implies that

$$\Psi_L(x_1) \leq \Psi_L(x) + b_M \alpha \Psi_L(x_1/\alpha) \int_{x_1}^{x} J(\xi/\alpha) d\xi.$$

Taking the supremum norm on both the sides of above inequality implies

$$\sup_{0 \leq x_1 \leq x} \Psi_L(x_1) \leq \Psi_L(x) + b_M \alpha \sup_{0 \leq x_1 \leq x} (x - x_1) \Psi_L(x_1/\alpha) J(x/\alpha),$$

$$\leq \Psi_L(x) + b_M \alpha \sup_{0 \leq x_1 \leq x} x \Psi_L(x_1) J(x/\alpha).$$

Choosing $x = x_L$ so that $b_M \alpha x_L J(x_L/\alpha) = 1/\alpha$ gives

$$\sup_{0 \leq x_1 \leq x} \Psi_L(x_1) \leq \left(\frac{\alpha}{\alpha - 1}\right) \Psi_L(x_L).$$

For any $x_L > x$, inequality (4.3.56) gives

$$\sup_{0 \leq x_1 \leq x} \Psi_L(x_1) \leq \left(\frac{\alpha}{\alpha - 1}\right) \Psi_L(x) J(x_L).$$

For any $A > x_L$,

$$\sup_{0 \leq x_1 \leq x} \Psi_L(x_1) \leq \left(\frac{\alpha}{\alpha - 1}\right) \Psi_L(A) J(A),$$
$$= K(A),$$

where $K(A)$ is a constant value that depends on A. Thus for all $A \geq x$,

$$\sup_{0 \leq x \leq A} \Psi_L(x) \leq K(A).$$

This proves the bound (4.3.52) on $[0, A]$ for some fixed A.

Let

$$v(x) = C(1 + x^k). \tag{4.3.57}$$

Then v satisfies the adjoint equation (4.1.7). Following Perthame & Ryzhik, we show that

$$\Psi_L(x) \leq v(x). \tag{4.3.58}$$

Let

$$z = L - x, \tag{4.3.59}$$

and denote

$$\underline{\Psi}_L(z) = \Psi_L(x), \tag{4.3.60}$$

$$\underline{v}(z) = v(x), \tag{4.3.61}$$

and $\underline{B}(z) = B(x)$. Differentiating $\underline{\Psi}_L(z)$ with respect to z yields

$$\frac{\partial}{\partial z}\underline{\Psi}_L(z) = -\frac{\partial}{\partial x}\Psi_L(x). \tag{4.3.62}$$

Equation (4.3.59) implies

$$\Psi\left(\frac{x}{\alpha}\right) = \underline{\Psi}(\frac{L+z}{\alpha}). \tag{4.3.63}$$

Now, using (4.3.62), (4.3.60) and (4.3.63) in the equation (**??**) gives

$$\frac{\partial}{\partial z}\underline{\Psi}_L(z) + (\lambda_L + \underline{B}(z))\underline{\Psi}_L(z) = \alpha\underline{B}(z)\underline{\Psi}_L\left(\frac{L+z}{\alpha}\right). \tag{4.3.64}$$

Since solution v is a solution to the equation (4.3.64)

$$\frac{\partial}{\partial z}\underline{v}(z) + (\lambda_L + \underline{B}(z))\underline{v}(z) = \alpha\underline{B}(z)\underline{v}\left(\frac{L+z}{\alpha}\right). \tag{4.3.65}$$

In terms of z, we have

$$\underline{v}(z) = C_1(L - z)^k, \tag{4.3.66}$$

where $C_1(L-z)^k = C(1+(L-z)^k)$. Using (4.3.66) in equation (4.3.65) yields

$$
\begin{aligned}
-C_1 k(L-z)^{k-1} + C_1(\underline{B}(z)+\lambda_L)(L-z)^k &= \alpha C_1 \underline{B}(z)\left(\frac{(\alpha-1)L-z}{\alpha}\right)^k,\\
C_1(b_m+\lambda_L)(L-z)^k &\geq \alpha C_1 b_m\left(\frac{(\alpha-1)L-z}{\alpha}\right)^k,\\
C_1 b_m \alpha (L-z)^k &\geq \alpha C_1 b_m\left(\frac{(\alpha-1)L-z}{\alpha}\right)^k,\\
\underline{v}(z) &\geq C_1\left(\frac{(\alpha-1)L-z}{\alpha}\right)^k.
\end{aligned}
$$

Let $0 \leq z \leq L-A$. Then

$$
\begin{aligned}
\underline{v}(z) &\geq C_1\alpha^{-k}\left(((\alpha-1)L-((\alpha-1)L+A)\right)^k\\
&= C_1\alpha^{-k}\left(A\right)^k,
\end{aligned}
$$

for all $0 \leq z \leq L-A$. Choosing C_1 and k in such manner that $C_1\alpha^{-k}\left(A\right)^k \geq K(A)$ implies

$$
\underline{\Psi}_L(z) \leq \underline{v}(z),
$$

for all $0 \leq y \leq L-A$. Using the relation (4.3.60) and (4.3.61) we have,

$$
\Psi_L(x) \leq v(x), \tag{4.3.67}
$$

for all $0 \leq x \leq L$. ■

The upper bound $v(x)$ given in (4.3.67) is independent of L and depends only on x, therefore Ψ_L converges to Ψ and satisfies the same bounds as $\Psi_L(x)$. Using a similar argument it can be shown that $\Psi(x) \geq \frac{c}{(1+x^k)}$.

Above Lemmas thus prove Theorem 4.3.2. The uniqueness of y and Ψ can be proved using the arguments that are used to show the uniqueness of y for a constant coefficient case (cf. [64]).

In addition, Perthame & Ryzhik also proved the following theorem:

Theorem 4.3.6. *Under the assumptions $B(x) > 0$ and $B(x) \to b_\infty$ as $x \to \infty$, there are two constants such that*

$$
\tilde{c}(1+x^{k_0}) \leq \Psi(x) \leq \tilde{C}(1+x^{k_0}),
$$

with $\alpha^{k_0} = \frac{\alpha b_\infty}{\lambda + b_\infty}$.

In this chapter, we discussed an approach, developed by Perthame & Ryzhik [64], to study the *Pde Problem without Dispersion*. Under certain conditions on division rate, Perthame & Ryzhik [64] developed an analysis to study the behaviour of solutions for a non constant division rate. They used the solution of an adjoint equation, which served as a weight, to show that in the weighted $L^1[0,\infty)$ norm, the asymptotic solution to the *Perthame & Ryzhik Problem* approaches the separable solution. In addition they proved that a first order pantograph equation with constant growth and a general non constant division rate, under certain conditions on B, has a unique solution. Among the conditions on B, the use of the condition 4.3.1 appears in the determination of a non zero eigenvalue only; however in chapter 5 we use a different approach to obtain the eigenvalue for the *Pde Problem without Dispersion* for a specific choice of non constant division rate. This approach does not require the condition 4.3.1. We adapt the analysis discussed in this chapter to show the uniqueness of eigenvalue and the existence of a SSD solution to the problem considered in chapter 5.

Chapter 5

A cell growth model adapted for minimum cell size division

5.1 Introduction

This chapter is a part of our published work [85]. In this chapter we consider the case in which cells divide only when they reach a certain minimum size and after which they divide at a constant rate. We note that Diekmann *et al.* [14] studied a closely related problem, where division occurs after a minimum size and there is an upper bound on the maximum cell size. Their analysis established the existence and uniqueness of solutions to the associated initial boundary value problem. Moreover, they established the existence and the uniqueness of positive eigenvalue and corresponding eigenfunction such that the solution to initial value problem converges to the eigenfunction (the "pdf solution") as $t \to \infty$. The model we study is a limiting case where there is no upper bound on the size. Here, we adapt Perthame & Ryzhik's analysis discussed in the previous chapter to show the uniqueness of the eigenvalue and to show that the solution of the problem follows the separable solution as $t \to \infty$.

Without loss of generality, we can always scale the size variable and use $g = 1$. The division rate can be modelled by

$$B(x) = bH(x - c),$$

where b is some positive constant, c represents the minimum size at which a cell will divide, and H is the Heaviside function. We thus consider the equation

$$\frac{\partial}{\partial t}n(x,t) + \frac{\partial}{\partial x}n(x,t) + bH(x-c)n(x,t) = \alpha^2 bH(\alpha x - c)n(\alpha x, t), \qquad (5.1.1)$$

$t > 0$, $x \geq 0$. We refer equation (5.1.1) along with the boundary conditions (1.3.5), (1.3.10) and the *Initial Condition* such that $n_0(x) \in L^1[0, \infty)$ as the *Heaviside Problem.*

Using the separable solution (1.4.1) in the *Heaviside Problem* yields

$$y'(x) + (bH(x-c) + \Lambda)\, y(x) = b\alpha^2 H(\alpha x - c) y(\alpha x). \tag{5.1.2}$$

We seek a pdf solution to the equation (5.1.2) along with the *Pdf Conditions* and the *Boundary Conditions*. For this problem Λ cannot be found by (1.4.11) or (1.4.12) and the division rate does not fall directly into the class considered by [10] or [64]. Nonetheless, it is sufficiently simple that a solution can be constructed.

Equation (5.1.2) can be regarded as three equations for the pdf y in three size intervals. In particular we have

$$y_1'(x) + \Lambda y_1(x) = 0, \quad 0 \leq x < c/\alpha, \tag{5.1.3}$$

$$y_2'(x) + \Lambda y_2(x) = b\alpha^2 y_3(\alpha x), \quad c/\alpha \leq x < c, \tag{5.1.4}$$

$$y_3'(x) + (b + \Lambda)\, y_3(x) = b\alpha^2 y_3(\alpha x), \quad c \leq x. \tag{5.1.5}$$

Equation (5.1.3) is simply an ode which, with the initial value (**??**), gives the unique solution $y_1(x) = 0$ for all $x \in [0, c/\alpha)$. If y_3 is known, then equation (5.1.4) is an ode for y_2 that can readily be solved. Equation (5.1.5) is the only equation that is functional in character. This equation is an example of the well-known pantograph equation, which has been studied extensively by several researchers. Detailed analytical accounts can be found in preceding chapters. We seek a continuous non negative solution to (5.1.2); thus, equations (5.1.3),(5.1.4) and (5.1.5) are supplemented with the *Continuity Conditions*

$$y_1(c/\alpha) = y_2(c/\alpha) = 0, \tag{5.1.6}$$

$$y_2(c) = y_3(c). \tag{5.1.7}$$

Note that, in general, the derivative of y will not be continuous at c/α or c.

Although Λ cannot be determined directly, it is possible to get upper and lower bounds on the eigenvalue. Specifically, equation (1.4.11) implies

$$\Lambda = b(\alpha - 1) \int_c^\infty y(x)\, dx. \tag{5.1.8}$$

Given that y is a pdf, we know the value of the above integral lies between 0 and 1. If $\Lambda = 0$, then the integral would have to be zero, and since y is non negative this means that the solution to equation (5.1.5) would have to be the trivial solution. Equation (5.1.4) and the *Continuity Conditions* then force y_2 to also be trivial. Since the only solution to (5.1.3) is the trivial solution we conclude that $y(x) = 0$ for all $x \geq 0$. But y is a pdf, so this cannot be the case; hence,

$$0 < \Lambda \leq b(\alpha - 1). \tag{5.1.9}$$

We establish the existence of an eigenvalue in section 5.3 where we construct a solution. It is well-known that a class of rapidly decaying solutions to pantograph equations such as (5.1.5) can be expressed as Dirichlet series. In the next section we turn to the Dirichlet series and study certain properties that are needed to establish the eigenvalue. In particular, we will see in section 5.3 that the *Continuity Condition* (5.1.6) determines any eigenvalues Λ as the zeros of a function involving a Dirichlet series and its integral. It is not obvious that there is a zero that satisfies inequality (5.1.9), and if there is such a zero, it is not clear that the resulting solution is non negative. In order to resolve these problems a closer study of the Dirichlet series as a function of both x and the eigenvalue parameter is required.

5.2 A Class of Dirichlet Series

Recall the Dirichlet series D which is defined as

$$D(x, \lambda) = e^{-\lambda x} + \sum_{k=1}^{\infty} \frac{(-1)^k (b\alpha^2)^k}{\lambda^k \prod_{m=1}^{k} (\alpha^m - 1)} e^{-\lambda \alpha^k x}, \tag{5.2.1}$$

for $x \geq 0$ and $\lambda > 0$. It can be verified directly that this series is a solution to the pantograph equation

$$D_x(x, \lambda) + \lambda D(x, \lambda) = b\alpha^2 D(\alpha x, \lambda), \tag{5.2.2}$$

where $D_x = \partial D / \partial x$. This series has been studied in detail for the special value $\lambda = b\alpha$ by Hall and Wake [30]. Suitably normalized the function $D(x, b\alpha)$ is the pdf solution to their cell division model. The series also appears in the analysis of the pantograph equation by Kato and McLeod [35] and Iserles [34] among others. Dirichlet series for second order versions of the pantograph equations have been studied in [92] , and a double series version of D can be found in [79] in connexion with an asymmetric cell division model. The focus in most of these studies was on particular values of λ. The interpretation of λ as an eigenvalue parameter, aside from its rôle in cell division models as the value that produces a pdf solution, was made for a related problem in [87] and [86]. We also note that an eigenvalue problem for a second order version was studied in [89].

The pantograph equation (5.1.5) has a solution of the form (5.2.1) with $\lambda = b + \Lambda$. Inequality (5.1.9) shows that for any pdf solution

$$b < \lambda \leq b\alpha, \tag{5.2.3}$$

and it is in this interval that we wish to study D. In order to do this, however, we need to study D for larger values of λ.

A key property of the Dirichlet series is

$$D_x(x, \lambda) = -\lambda D(\alpha x, \lambda/\alpha). \tag{5.2.4}$$

Theorem 5.2.1. *If $\lambda \geq b\alpha^2$, then $D(x, \lambda) > 0$ for all $x \geq 0$. Moreover, $D(x, \lambda)$ is monotonic strictly decreasing with respect to x on the interval* $(0, \infty)$.

Proof: For any $\lambda > 0$,

$$\begin{aligned} D(0, \lambda) &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k (b\alpha^2)^k}{\lambda^k \prod_{m=1}^{k} (\alpha^m - 1)} \\ &= 1 + \sum_{k=1}^{\infty} \frac{(-1)^k (b\alpha/\lambda)^k}{\alpha^{k(k-1)/2} \prod_{m=1}^{k} (1 - \alpha^{-m})}. \end{aligned}$$

The above series can be recast as an infinite product using the *Euler identity* (2.4.12) with $z = -b\alpha/\lambda$ and $q = 1/\alpha$ to get

$$D(0, \lambda) = \prod_{k=0}^{\infty} \left(1 - \frac{b\alpha}{\lambda \alpha^k}\right), \tag{5.2.5}$$

which immediately gives all the zeros of $D(0, \lambda)$, *viz.*, $\lambda_n = b\alpha^{1-n}$ for $n = 0, 1, 2 \ldots$. These values correspond to the eigenvalues determined in [87]. Here, we simply note that $D(0, \lambda) > 0$ for all $\lambda > b\alpha$.

We now show that $D(x, \lambda)$ cannot have any local extrema if $\lambda \geq b\alpha^2$. First note that for any λ

$$\lim_{x \to \infty} D(x, \lambda) = 0. \tag{5.2.6}$$

Suppose that D has a positive local maximum at $x = m_1$. Then equation (5.2.2) implies

$$\lambda D(m_1, \lambda) = b\alpha^2 D(\alpha m_1, \lambda),$$

and since $\lambda \geq b\alpha^2$ the above equation along with equation (5.2.6) imply the existence of another positive local maximum at some point $m_2 \geq \alpha m_1$ at which $D(m_2, \lambda) \geq D(m_1, \lambda)$. We can continue this argument and thereby construct a sequence $\{m_k\}$ of local positive maxima such that $m_k \to \infty$ as $k \to \infty$, and $D(m_k, \lambda) \geq D(m_1, \lambda) > 0$ for all $k \geq 1$. This contradicts equation (5.2.6) and we thus conclude that D cannot have a positive local maximum. The same argument can be applied to $-D$ to preclude the possibility of a negative local minimum. Evidently, D cannot have a positive local minimum or a negative local maximum and also satisfy (5.2.6).

We know that $D(0, \lambda) > 0$. Suppose that D has a zero at $\xi \in (0, \infty)$. Then, D must be identically zero on the interval $[\xi, \infty)$ or otherwise it would require a non zero local extremum to satisfy (5.2.6). For any $\lambda > 0$, however, $D(z, \lambda)$, regarded as a function of the complex variable z, is holomorphic in the right half plane. Since D is not identically zero in the half plane the identity theorem rules out the possibility of this function being zero on any such interval. ■

We thus see that D must be positive for all $x \geq 0$ whenever $\lambda \geq b\alpha^2$. Finally we note that $D_x(x, \lambda) < 0$ for all $x > 0$, since $D_x(x, \lambda) = 0$ would induce a local positive maximum.

Corollary 5.2.2. *If $b\alpha \leq \lambda < b\alpha^2$, then $D(x, \lambda) > 0$ for all $x > 0$, and $D(x, \lambda)$ has precisely one local maximum in $(0, \infty)$.*

Proof: Equation (5.2.4) can be recast

$$D_x(x, \alpha\lambda) = -\alpha\lambda D(\alpha x, \lambda),$$

and because $\alpha\lambda \geq b\alpha^2$, the positivity of D follows immediately from Theorem 5.2.1.

Concerning the existence of the maximum, the result has already been established for the case $\lambda = b\alpha$ (cf. [10]). Suppose that $b\alpha < \lambda < b\alpha^2$. Equation (5.2.2) implies

$$D_x(0, \lambda) = (b\alpha^2 - \lambda)D(0, \lambda).$$

We know that $D(0, \lambda) > 0$ by equation (5.2.5); hence, $D_x(0, \lambda) > 0$. Condition (5.2.6) indicates that D must have at least one local positive maximum. In fact, the positivity of the derivative at $x = 0$ precludes the possibility that the global maximum occurs at $x = 0$. Now,

$$D_x(x, \lambda) = -\lambda e^{-\lambda x} + O(e^{-\lambda\alpha x}) \tag{5.2.7}$$

as $x \to \infty$, so there cannot be a sequence $\{x_k\}$ such that $x_k \to \infty$ as $k \to \infty$ with $D_x(x_k, \lambda) = 0$. The function D can thus have only a finite number of local maxima. Suppose that D has more than one local maximum and let M denote the largest value of x at which D has a local maximum. Since D has more than one maximum, there must be at least one local minimum. Let m denote the largest value of x at which D has a local minimum. The positivity of D implies that $m < M$. Equation (5.2.2) implies that $\lambda D(m, \lambda) = b\alpha^2 D(\alpha m, \lambda)$, so that in particular $D(\alpha m, \lambda) < D(m, \lambda)$ and hence $M < \alpha m$. Moreover, differentiating equation (5.2.2) and noting that $D_{xx}(m, \lambda) \geq 0$ we see that D_x must be non negative at $x = \alpha m$. The derivative cannot be positive since the last local maximum occurred at M; hence, the derivative must be zero at αm and this means that D_{xx} must also be zero there because there are no more extrema beyond M. We can now apply the same argument at the point αm to assert that D_x and D_{xx} must both vanish at $x = \alpha^2 m$. It is clear in this manner that we could construct a sequence $\{x_k\}$ such that with $D_x(x_k, \lambda) = 0$ and $x_k \to \infty$ as $k \to \infty$. But equation (5.2.7) indicates that $D_x < 0$, as $x \to \infty$. This contradiction shows that D can have only one local maximum. We finally reach the interval with which we are most concerned. Equation (5.2.5) shows that $D(0, \lambda) < 0$ for all $\lambda \in (b, b\alpha)$. ■

Although the Dirichlet series starts out negative, it is clear that it is positive for x sufficiently large. Since this Dirichlet series will form part of a pdf solution, it is important to identify the zeros of the function and where it is positive.

Corollary 5.2.3. *If $b \leq \lambda < b\alpha$, then $D(x, \lambda)$ has only one zero $z \in (0, \infty)$; moreover,*

$$z \leq \frac{\alpha}{b(\alpha - 1)}. \tag{5.2.8}$$

Proof: The existence and uniqueness of z follows immediately from Corollary 5.2.2 and relation (5.2.4). Here, z/α corresponds to the maximum for $D(x, \alpha\lambda)$. Equation (5.2.2) and the positivity of $D(x, \alpha\lambda)$ for $\lambda \geq b$ imply

$$\begin{aligned} D(z/\alpha, \alpha\lambda) &= \alpha\lambda \int_{z/\alpha}^{z} D(\xi, \alpha\lambda)\, d\xi + (\alpha\lambda - b\alpha) \int_{z/\alpha}^{\infty} D(\xi, \alpha\lambda)\, d\xi \\ &\geq \alpha\lambda \int_{z/\alpha}^{z} D(\xi, \alpha\lambda)\, d\xi \\ &\geq \alpha\lambda z(1 - 1/\alpha) D(z, \alpha\lambda), \end{aligned}$$

where we have used the fact that $D(x, \alpha\lambda)$ is monotonic decreasing after the local maximum at z/α. Equation (5.2.2) also shows that

$$\alpha\lambda D(z/\alpha, \alpha\lambda) = b\alpha^2 D(z, \alpha\lambda);$$

hence,

$$z \leq \frac{b\alpha}{\lambda^2(\alpha-1)} \leq \frac{\alpha}{b(\alpha-1)}.$$

Note that equation (5.2.5) shows that $D(0,\lambda) < 0$, and it is clear for x large that $D(x,\lambda) > 0$. We thus see that $D(x,\lambda)$ changes sign from negative to positive as x increases through z. The function $D(x,\lambda)$ is graphed for different values of λ in figure 5.2.1.

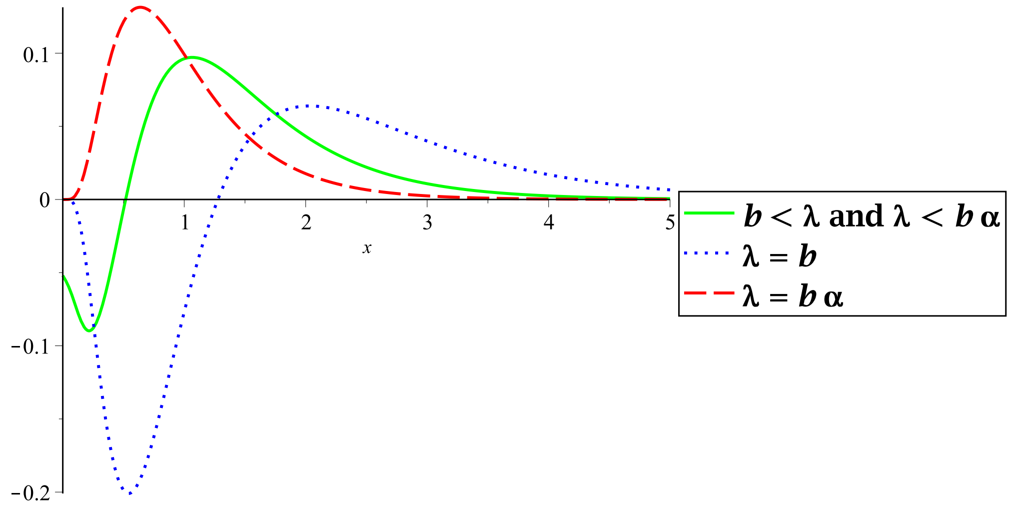

Figure 5.2.1: $D(x,\lambda)$ for different values of λ with $\alpha = 2$, $b = 1$.

■

5.3 A Pdf Solution

In this section we construct a pdf solution to equation (5.1.2) that satisfies the *Boundary Conditions* and the *Continuity Conditions*. We know from section 5.2 that equation (5.1.5) has a solution of the form

$$y_3(x) = kD(x,\lambda), \tag{5.3.1}$$

where k is a constant and

$$\lambda = \Lambda + b.$$

It is straightforward to determine the solution of (5.1.4), *viz.*

$$y_2(x) = kb\alpha^2 e^{-(\lambda-b)x} \int_{c/\alpha}^{x} e^{(\lambda-b)\xi} D(\alpha\xi,\lambda)\, d\xi, \tag{5.3.2}$$

where we have used (5.1.6). The constant k can be used eventually to normalize y to make it a pdf. The main problem now is to show that there is an eigenvalue that produces a positive eigenfunction. The *Continuity Condition* (5.1.7) places a restriction on λ. In particular, equations (5.3.1), (5.3.2) and (5.1.7) imply

$$F(\lambda) = 0, \tag{5.3.3}$$

where

$$F(\lambda) = b\alpha^2 \int_{c/\alpha}^{c} e^{(\lambda-b)\xi} D(\alpha\xi, \lambda)\, d\xi - e^{(\lambda-b)c} D(c, \lambda).$$

We know from inequality (5.2.3) that λ for any pdf solution must be in the interval $(b, b\alpha]$, but it is not clear that equation (5.3.3) has a solution in this interval. Certainly one complication is that $D(x, \lambda)$ changes sign once along the positive x-axis. Theorem 5.2.1 and Corollary 5.2.2 show, in contrast, that $D(x, \alpha\lambda) > 0$ for all $x > 0$ when $\lambda \in (b, b\alpha]$. We can recast F using equations (5.2.2) and (5.2.4) along with integration by parts to get

$$\begin{aligned} F(\lambda) &= D(c/\alpha, \alpha\lambda) \left(\frac{b\alpha}{\lambda} e^{(\lambda-b)c/\alpha} - e^{(\lambda-b)c} \right) \\ &\quad + \frac{b\alpha}{\lambda}(\lambda - b) \int_{c/\alpha}^{c} e^{(\lambda-b)\xi} D(\xi, \alpha\lambda)\, d\xi. \end{aligned} \tag{5.3.4}$$

Lemma 5.3.1. *For any $c > 0$ there exists a solution to equation (5.3.3) in the interval $(b, b\alpha)$.*

Proof: The function F is continuous on the interval $[b, b\alpha]$. Now, using equation (5.3.4),

$$F(b) = (\alpha - 1) D(c/\alpha, \alpha\lambda) > 0,$$

and

$$\begin{aligned} F(b\alpha) &= D(c/\alpha, b\alpha^2) \left(e^{b(\alpha-1)c/\alpha} - e^{b(\alpha-1)c} \right) \\ &\quad + b(\alpha - 1) \int_{c/\alpha}^{c} e^{b(\alpha-1)\xi} D(\xi, b\alpha^2)\, d\xi. \end{aligned}$$

Theorem 5.2.1 shows that $D(x, \alpha\lambda)$ is monotonic decreasing on $[0, \infty)$; hence,

$$\begin{aligned} F(b\alpha) &< D(c/\alpha, b\alpha^2)\left(e^{b(\alpha-1)c/\alpha} - e^{b(\alpha-1)c}\right) \\ &\quad + b(\alpha-1)D(c/\alpha, b\alpha^2)\int_{c/\alpha}^{c} e^{b(\alpha-1)\xi}\, d\xi \\ &= D(c/\alpha, b\alpha^2)\left(e^{b(\alpha-1)c/\alpha} - e^{b(\alpha-1)c}\right) \\ &\quad + D(c/\alpha, b\alpha^2)\left(e^{b(\alpha-1)c} - e^{b(\alpha-1)c/\alpha}\right) \\ &= 0. \end{aligned}$$

We thus see that F changes sign in the interval and therefore must have a zero in this interval. ■

Although there is a solution to equation (5.3.3) in the interval $(b, b\alpha)$, it is not clear that the resulting solution to equation (5.1.2) is non negative. For instance, it may be, for some choice of $c > 0$, that $D(c, \lambda) < 0$. For each $\lambda \in (b, b\alpha)$ the function $D(x, \lambda)$ has precisely one zero $z(\lambda)$ in $[0, \infty)$ and hence the question is whether equation (5.3.3) precludes the case $c < z(\lambda)$. Corollary 5.2.3 shows that for c sufficiently large, $D(c, \lambda) > 0$ for any $\lambda \in (b, b\alpha)$. The next lemma shows that this is true for all $c > 0$.

Lemma 5.3.2. *Let $c > 0$ and suppose $\lambda \in (b, b\alpha)$ is a solution to equation (5.3.3). Then, $D(c, \lambda) > 0$.*

Proof: Recall from the proof of Corollary 5.2.3 that if $m(\lambda)$ is the maximum for $D(x, \alpha\lambda)$, then $\alpha m(\lambda)$ is the zero for $D(x, \lambda)$. The lemma will be established if it can be shown that $m(\lambda) < c/\alpha$.

The continuous function $D(x, \alpha\lambda)$ has one local maximum and no local minima; consequently, on any compact interval of $[0, \infty)$, the global minimum of the function over this interval must occur at an endpoint of the interval. Specifically, for any $c > 0$, the global minimum of $D(x, \alpha\lambda)$ on the interval $[c/\alpha, c]$ must occur at either c/α or c. Suppose that this minimum occurs at c/α. Then, the second term in (5.3.4) can be bounded as follows

$$\begin{aligned} &\frac{b\alpha}{\lambda}(\lambda - b)\int_{c/\alpha}^{c} e^{(\lambda-b)\xi} D(\xi, \alpha\lambda)\, d\xi \\ > \; &\frac{b\alpha}{\lambda}(\lambda - b) D(c/\alpha, \alpha\lambda)\int_{c/\alpha}^{c} e^{(\lambda-b)\xi}\, d\xi \\ = \; &\frac{b\alpha}{\lambda} D(c/\alpha, \alpha\lambda)\left(e^{(\lambda-b)c} - e^{(\lambda-b)c/\alpha}\right); \end{aligned}$$

hence,

$$F(\lambda) > D(c/\alpha, \alpha\lambda)\left(\frac{b\alpha}{\lambda} - 1\right)e^{(\lambda-b)c} > 0,$$

so that λ cannot be a solution to (5.3.3). We thus conclude that the minimum cannot be achieved at c/α and must therefore be at c. This leads to the inequality

$$\begin{aligned} F(\lambda) &> D(c/\alpha, \alpha\lambda)\left(\frac{b\alpha}{\lambda}e^{(\lambda-b)c/\alpha} - e^{(\lambda-b)c}\right) \\ &\quad + D(c, \alpha\lambda)\frac{b\alpha}{\lambda}\left(e^{(\lambda-b)c} - e^{(\lambda-b)c/\alpha}\right) \\ &= e^{(\lambda-b)c}\left(\frac{b\alpha}{\lambda}D(c, \alpha\lambda) - D(c/\alpha, \alpha\lambda)\right) \\ &\quad + e^{(\lambda-b)c/\alpha}\left(D(c/\alpha, \alpha\lambda) - D(c, \alpha\lambda)\right), \end{aligned}$$

and, using equation (5.2.2), this yields

$$F(\lambda) > \frac{e^{(\lambda-b)c}}{\lambda}D_x(c/\alpha, \alpha\lambda) + e^{(\lambda-b)c/\alpha}\left(D(c/\alpha, \alpha\lambda) - D(c, \alpha\lambda)\right).$$

The last term in the above expression is clearly positive since the minimum is at c. If $c/\alpha < m(\lambda)$, then $D_x(c/\alpha, \alpha\lambda) > 0$, which leads to the contradiction that $F(\lambda) > 0$. We thus conclude that $m(\lambda) < c/\alpha$. ■

The above lemma indicates that $D(x, \lambda) > 0$ for all $x \geq c$, which ensures that a positive solution y_3 exists. It is clear from (5.3.2) that the positivity of y_3 implies that of y_2 and that a suitable positive constant k can be obtained to normalize the solution to satisfy (1.4.10). In summary, we have the following result.

Theorem 5.3.3. *For any $c > 0$ there exists a unique eigenvalue $\lambda \in (b, b\alpha)$ and corresponding positive eigenfunction y that satisfies equation (5.1.2).*

■

5.4 An SSD solution

In this section we adapt the analysis discussed in chapter 4 to prove the uniqueness of eigenvalue λ and the existence of SSD solution to the *Heaviside Problem.* Perthame & Ryzhik [64] considered a case where division rate is bounded away from the origin in the interval $x \in [0, \infty)$; whereas, in this chapter we study a cell growth model in which division rate is such that $B(x) = 0$ for $x \in [0, c]$ and $B(x) > 0$ when $x \in (c, \infty)$.

To embark on the existence of an SSD solution to the *Heaviside Problem*, we first define an adjoint equation for this model that is

$$\frac{d}{dx}\hat{\Psi}(x) - (\lambda - b + bH(x-c))\,\hat{\Psi}(x) = -b\alpha H(x-c)\hat{\Psi}\left(\frac{x}{\alpha}\right), \quad x \geq 0. \tag{5.4.1}$$

Equation (5.4.1) can be written as

$$\frac{d}{dx}\hat{\Psi}_0(x) - (\lambda - b)\,\hat{\Psi}_0(x) = 0, \quad 0 \leq x \leq c, \tag{5.4.2}$$

and for $j \geq 1$, $\alpha^{j-1}c \leq x < \alpha^j c$,

$$\frac{d}{dx}\hat{\Psi}_j(x) - \lambda\hat{\Psi}_j(x) = -b\alpha\hat{\Psi}_{j-1}\left(\frac{x}{\alpha}\right). \tag{5.4.3}$$

The equations (5.4.2) and (5.4.3) satisfy the continuity conditions

$$\hat{\Psi}_j(\alpha^{j-1}c) = \hat{\Psi}_{j-1}(\alpha^{j-1}c).$$

The equation (5.4.2) is a first order differential equation whose solution in the interval $0 \leq x \leq c$ is bounded by two functions, *viz.*, $\hat{\Psi}_0(0) \leq \hat{\Psi}_0(x) \leq \hat{\Psi}_0(0)e^{b(\alpha-1)c}$. Since we are interested in a non trivial and positive solution we assume $\hat{\Psi}_0(0) > 0$ which implies that $\hat{\Psi}(x) > 0$ for $0 \leq x \leq c$. Our next move is to show that $\hat{\Psi}(x) > 0$ for all $x \geq 0$.

Recall equation (5.1.2) that in terms of λ becomes

$$\frac{d}{dx}y(x) + (bH(x-c) + \lambda - b)y(x) = b\alpha^2 H(\alpha x - c)y(\alpha x). \tag{5.4.4}$$

Multiplying equations (5.4.1) and (5.4.4) by $y(x)$ and $\hat{\Psi}(x)$ respectively and then adding them up gives

$$\frac{d}{dx}\hat{\Psi}(x)y(x) = b\alpha^2 H(\alpha x - c)y(\alpha x)\hat{\Psi}\left(x\right) - b\alpha H(x-c)y(x)\hat{\Psi}\left(\frac{x}{\alpha}\right). \tag{5.4.5}$$

Let $x_0 > 0$, then integrating equation (5.4.5) from x_0 to x gives

$$\begin{aligned}\hat{\Psi}(x)y(x) = \hat{\Psi}\left(x_0\right)y\left(x_0\right) + b\alpha^2 \int_{x_0}^{x} H(\alpha\xi - c)y(\alpha\xi)\hat{\Psi}(\xi)d\xi \\ - b\alpha\int_{x_0}^{x} H(\xi - c)y(\xi)\hat{\Psi}\left(\frac{\xi}{\alpha}\right)d\xi.\end{aligned}$$

Since $y(x) = 0$ for $x \in (0, \frac{c}{\alpha})$, above equation can be written as

$$\begin{aligned}\hat{\Psi}(x)y(x) &= \hat{\Psi}\left(x_0\right)y\left(x_0\right) + b\alpha^2\int_{\frac{c}{\alpha}}^{x} y(\alpha\xi)\hat{\Psi}(\xi)d\xi - b\alpha\int_{c}^{x} y(\xi)\hat{\Psi}\left(\frac{\xi}{\alpha}\right)d\xi, \\ &= \hat{\Psi}\left(x_0\right)y\left(x_0\right) + b\alpha\int_{c}^{\alpha x} y(\xi)\hat{\Psi}\left(\frac{\xi}{\alpha}\right)d\xi - b\alpha\int_{c}^{x} y(\xi)\hat{\Psi}\left(\frac{\xi}{\alpha}\right)d\xi, \\ &= \hat{\Psi}\left(x_0\right)y\left(x_0\right) + b\alpha\int_{x}^{\alpha x} y(\xi)\hat{\Psi}\left(\frac{\xi}{\alpha}\right)d\xi. \end{aligned} \tag{5.4.6}$$

Recasting equation (5.4.6) in terms of $\hat{\Psi}_j$ gives

$$\hat{\Psi}_j(x)y(x) = \hat{\Psi}\left(x_0\right)y\left(x_0\right) + b\alpha\int_{x}^{\alpha x} y(\xi)\hat{\Psi}_{j-1}\left(\frac{\xi}{\alpha}\right)d\xi, \tag{5.4.7}$$

where $x \in (\alpha^{j-1}c, \alpha^j c)$. For $j = 1$, equation (5.4.7) becomes

$$\hat{\Psi}_1(x)y(x) = \hat{\Psi}\left(x_0\right)y\left(x_0\right) + b\alpha\int_{x}^{\alpha x} y(\xi)\hat{\Psi}_0\left(\frac{\xi}{\alpha}\right)d\xi, \tag{5.4.8}$$

where $x \in (c, \alpha c)$. If there exists a pdf solution y to the *Heaviside Problem* then equation (5.4.8) implies $\hat{\Psi}(x) > 0$ for all $x \in (c, \alpha c)$. Repeating this step for $j = 2$ and assuming that y is a pdf in equation (5.4.7) implies $\hat{\Psi}(x) > 0$ for all $x \in (\alpha c, \alpha^2 c)$. Continuing this process leads to the result

$$\hat{\Psi}x) > 0, \tag{5.4.9}$$

for all $x \geq 0$.

Integrating the equation (5.4.5) from 0 to x gives $y(\xi)\hat{\Psi}(\xi) = 0$, which on further integration from 0 to ∞ implies $\int_0^\infty y(\xi)\hat{\Psi}(\xi)d\xi = \hat{a}$, where $\hat{a}$ is a constant. Let $\hat{a} = 1$, then

$$\int_0^\infty y(\xi)\hat{\Psi}(\xi)d\xi = 1. \tag{5.4.10}$$

We refer equation (5.4.1) along with the condition (5.4.9) and (5.4.10) as the *Adjoint Problem.*

We now mimic Perthame and Ryzhik to show the convergence of a solution to the *Heaviside Problem* as $t \to \infty$. Suppose,

A1. There exists a positive eigenvalue and corresponding eigenfunction;

A2. The eigenvalue is bounded by $b \leq \lambda \leq b\alpha$; and

A3. there are constants $\hat{c}$, $\hat{C}$, $0 < \hat{c} \leq \hat{C}$ such that

$$\hat{c}(1 + x^k) \leq \hat{\Psi}(x) \leq \hat{C}(1 + x^k), \tag{5.4.11}$$

where $k > 1$, which can be obtained by the condition (4.3.51). Then the solution to the *Heaviside Problem* follows the separable solution y as $t \to \infty$.

For the *Heaviside Problem*, the matrix of the discretized version of the *Adjoint Problem* has all the off diagonal entries, non negative. Applying a similar approach given in the proof of the Lemma 4.3.3, it is straightforward that eigenvalue λ and the corresponding eigenvector are unique. In order to show the bounds given in the condition *A2*, Perthame & Ryzhik used the condition $B(x) > 0$ for all $x \geq 0$. The division rate for the *Heaviside Problem* does not satisfy the positivity condition for B at origin; however early in this chapter we have used another approach to prove the bounds *A2* and the bounds for λ are given by the equation (5.2.3). Recall Theorem 4.3.5 in which Perthame & Ryzhik first proved an inequality (4.3.52) that led to the proof of the bounds for Ψ. Moreover, for the *Adjoint Problem*, the inequality similar to (4.3.52) is

$$\sup_{0 \leq x \leq A} \hat{\Psi}_L(x) \leq \tilde{K}(A), \tag{5.4.12}$$

for all $x \in [0, A]$, where A is some large fixed value of x. Following the approach used in the Theorems 4.3.5 we can establish the bounds (5.4.12), which then leads to the bounds (5.4.11).

In addition, mimicking Perthame & Ryzhik, we define a renormalized division rate

$$\tilde{\hat{B}}(x) = bH(x - c)\frac{\hat{\Psi}(x/\alpha)}{\hat{\Psi}(x)},$$

where $\hat{\Psi}$ is the solution of the *Adjoint Problem*. Bounds for the $\hat{\Psi}$ given in (5.4.11) implies

$$\tilde{\hat{B}}(x) \to E,$$

as $x \to \infty$, where E is a constant. Following Perthame & Ryzhik study it is thus straightforward to conclude the following result:

Theorem 5.4.1. *There exists a constant $E > 0$ such that*

$$\gamma := ||\tilde{\hat{B}}(x) - E||_{L^\infty[0,\infty)} < \frac{E}{\alpha^2(\alpha + E)},$$

and the solution to the Heaviside Problem satisfies

$$\begin{aligned} &|| \left(n(x,t)e^{-(\lambda-b)t} - y(x)\right) \hat{\Psi}(x)||_{L^1[0,\infty)} \\ &\leq e^{-\mu t} \left[\alpha|| \left(n_0(x) - y(x)\right) \hat{\Psi}(x)||_{L^1[0,\infty)} + \alpha^2 E||H^0||_{L^1[0,\infty)}\right], \end{aligned}$$

where $\mu =, (\alpha - 1)E - \alpha^3\gamma - \alpha^2 E\gamma$, $\langle n_0 \rangle = \int_0^\infty n_0 dx = 1$ is a conserved quantity and $H^0(x) = \int_0^x [n_0(\xi) - y(\xi)]\, d\xi \to 0$ as $x \to \infty$.

In this chapter we studied the cell growth model of Hall and Wake for a cell division function that models cells that divide only after they reach a minimum size. Unlike the earlier models of Hall and Wake, the determination of the SSD solution involved an eigenvalue that could not be found explicitly. This problem brought to the fore a class of Dirichlet series that solve a pantograph equation. Using the analysis of Perthame & Ryzhik, it is shown that the eigenvalue is unique and in a weighted $L^1[0,\infty)$ norm the solution to the *Heaviside Problem* converges to the separable solution as $t \to \infty$. Adaptation of the Perthame & Ryzhik analysis to the *Heaviside Problem* indicates that using an alternative approach to find the bounds for λ, the condition (4.3.1) in their analysis can be relaxed.

Chapter 6

An extension to second order Pdes

In this chapter we discuss an extension of the cell growth model to include a dispersion term. This leads to a second order Pde. Analysis and techniques studied in chapter 2 cannot be adapted to study this case. Begg *et al.* [6] used the relative entropy structure (cf. [53], [52]) to study the *Pde Problem* for a constant growth, constant dispersion rate and a specific non constant division rate. Begg *et al.* established the existence of an SSD solution but they did not give an analytic solution to the problem. Their analysis can be adapted to the Pdes for more general division rate. In this chapter we focus on the approach developed by Efendiev *et al.* [18], which considered the constant coefficient case of the *Pde Problem*. Efendiev *et al.* first converted the problem in terms of a cumulative function and then applied the Laplace transform to the spatial variable to show that there exists a unique solution to the transformed problem in the form of the Neumann series. The construction of the Neumann series crucially relied on obtaining an appropriate Green's function. They exploited the Green's function to deduce the solution to the original problem and used that solution form to study the behaviour of the solution as $t \to \infty$. It was shown that there exists an SSD solution to the problem. For a non constant growth rate, the solution of the Laplace transformed equation is formidable. We show, however, that this step can be bypassed and a fundamental solution can be obtained using the Fourier transform. We first illustrate this approach on a *Pde Problem* with constant coefficients and then solve the *Pde Problem* for a linear growth rate along with constant dispersion and constant division rate.

6.1 Second order Pde with constant coefficients

Let $D(x,t) = \Upsilon$, $G(x,t) = g$ and $B(x,t) = b$, then using the substitution $n(x,t) = e^{-bt}\tilde{n}(x,t)$ in (1.3.4) yields

$$-\Upsilon \tilde{n}_{xx}(x,t) + \tilde{n}_t(x,t) + g\tilde{n}_x(x,t) = b\alpha^2\tilde{n}(\alpha x,t). \qquad (6.1.1)$$

We call equation (6.1.1) along with the the *No Flux Conditions 1*, the *Initial Condition* and the condition $n(0,t) = 0$ as *Pde Problem with Constant Coefficient*. We first look at the case when $\Upsilon = 0$.

6.1.1 First order case

Zaidi *et al.* [96] considered the first order analogue to the *Pde Problem with Constant Coefficient*. They constructed an explicit solution and showed that the problem has an SSD solution. Specifically, they considered the equation

$$\hat{n}_t(\hat{x},t) + g\hat{n}_{\hat{x}}(\hat{x},t)) + b\hat{n}(\hat{x},t) = b\alpha^2\hat{n}(\alpha\hat{x},t), \qquad (6.1.2)$$

$b > 0$, $g > 0$. Using the substitution

$$n(\hat{x},t) = e^{-bt}\hat{n}(\hat{x},t), \qquad (6.1.3)$$

they converted the equation (6.1.2) to

$$n_t(\hat{x},t) + gn_{\hat{x}}(\hat{x},t) = b\alpha^2 n(\alpha\hat{x},t).$$

Substituting $\hat{x} = g\tilde{x}$, they further simplified above equation to

$$n_t(x,t) + n_x(x,t) = b\alpha^2 n(\alpha x,t), \qquad (6.1.4)$$

where $n(x,t) = n(g\tilde{x},t)$. The equation (6.1.4) satisfies the *Initial Condition* and the *No Flux Conditions 2*. Using the transformation

$$\bar{m}(x,t) = \int_x^\infty n(\xi,t)d\xi,$$

they transformed equation (6.1.4) to,

$$\bar{m}_t(x,t) + \bar{m}_x(x,t) = b\alpha\bar{m}(\alpha x,t), \qquad (6.1.5)$$

that satisfies

$$\bar{m}_0(x) = \int_x^\infty n_0(\xi)d\xi, \tag{6.1.6}$$

and

$$\bar{m}(0,t) = e^{b\alpha t}.$$

They exploited the hyperbolic character of the differential operator involved in the model and the advanced nature of the functional argument to convert the problem to a sequence of simple *Cauchy Problems*. Specifically, they constructed a sequence of functions $\{N_k(x,t)\}$ such that

$$Q(x,t) = \sum_{k=0}^{\infty} N_k(x,t),$$

is a solution to the equation (6.1.5) that satisfies the initial condition (6.1.6) and is valid for the interval $x \geq t$. Here,

$$(N_k)_t + (N_k)_x = b\alpha^2 N_{k-1}(\alpha x, t), \quad k \geq 1, \tag{6.1.7}$$

and

$$(N_0)_t + (N_0)_x = 0, \quad N_0(x,0) = m_0(x). \tag{6.1.8}$$

The function $N_0(x,t) = m_0(x-t)$ satisfies the *Cauchy Problem* (6.1.8). Zaidi *et al.* solved equations (6.1.7) using the method of characteristics. It was then shown that the series $\sum_{k=0}^{\infty} N_k$ converges uniformly in any set $\{(x,t)|0 \leq t \leq x\}$. They established the existence of a non negative solution and continued the solution through the sequence of wedges "W_n" which is defined as

$$W_n = \{(x,t) : \frac{t}{\alpha^n} \leq x \leq \frac{t}{\alpha^{n-1}}\},$$

where $n \geq 1$. Using the solution form for $x \geq t$, they obtained the solution for $0 < x < t$ and then studied the limiting case. The solution involved an arbitrary function G_0 that was determined through a continuity condition. They also proved the uniqueness and smoothness of the solution.

Unfortunately, it is not straightforward to generalize this analysis either for non constant rates or for a dispersion term ($\Upsilon > 0$). The arguments leading to the solution rely heavily on the linear geometry of characteristic projections and on certain combinatorial identities that are not preserved if any rate is non constant.

6.1.2 The second order case

We now consider the case when $\Upsilon > 0$. As noted, the analysis for the first order case is of limited value. The problem has been solved in [18] via the Laplace transform in the spatial variable. This approach requires the inversion of a Green's function in the Laplace variable. The solution to the Pde, however, requires only the fundamental solution to the *Cauchy Problem*

$$\begin{aligned} -\Upsilon\tilde{u}_{xx}(x,t) + \tilde{u}_t(x,t) + g\tilde{u}_x(x,t) &= 0, \\ u(x,0) &= u_0(x), \end{aligned} \tag{6.1.9}$$

along with the *No Flux Conditions 1.* It is possible to glean the fundamental solution directly using the Fourier transform and the method of images. This approach is a well-known technique for solving certain Fokker-Planck equations (cf. [68], pg.99-100). Following the analysis in [18] , we first reduce the equation to a form that makes the transform analysis straightforward. We then illustrate how the fundamental solution can be obtained via the Fourier transform, which avoids directly finding a Green's function.

Suppose $\tilde{n}(x,t)$ is integrable with respect to x for all $t \geq 0$ and the order of integration and differentiation can be interchanged. Let

$$m(x,t) = \int_x^\infty \tilde{n}(\xi,t)d\xi,$$

then integrating equation (6.1.1) from x to ∞ and using the *No Flux Conditions 1* imply

$$-\Upsilon m_{xx}(x,t) + m_t(x,t) + gm_x(x,t) = b\alpha m(\alpha x,t). \tag{6.1.10}$$

At $x = 0$, we have

$$m_t(0,t) = b\alpha m(0,t),$$

so that the boundary condition at $x = 0$ is

$$m(0,t) = ke^{b\alpha t}, \tag{6.1.11}$$

where k is a constant. In terms of m the *Initial Condition* is

$$m(x,0) = m_0(x) = \int_x^\infty n_0(\xi)d\xi. \tag{6.1.12}$$

Since n_0 is a pdf, $m_0(x) \leq 1$ for all $x \geq 0$.

We refer to equation (6.1.10) along with the boundary condition (6.1.11) and the initial condition (6.1.12) as *m Problem*. We investigate the behaviour of solutions to the *m Problem* as $t \to \infty$.

Let $H(t)h(x)$, where $h(x)$ is a pdf, be the separable solution of the *Pde Problem with Constant Coefficient*; in addition, suppose that the *m Problem* has a separable solution,

$$\hat{m}(x,t) = N(t)y(x). \tag{6.1.13}$$

Then

$$N(t) = Ke^{b\alpha t},$$

where K is some constant and

$$y(x) = \int_x^\infty h(\xi)d\xi,$$

which yields

$$y(0) = 1. \tag{6.1.14}$$

Substituting the separable solution (6.1.13) in equation (6.1.10) gives

$$-\Upsilon y''(x) + gy'(x) + b\alpha y(x) = b\alpha y(\alpha x). \tag{6.1.15}$$

Equation (6.1.15) is a second order pantograph equation that has a Dirichlet series solution of the form (3.2.2). This solution form has already been derived in chapter 3.

Let m be the solution to the *m Problem*. If there is no SSD solution to the *m Problem*, a prime candidate is the separable solution $\hat{m}$. Motivated by this, let

$$\begin{aligned} v(x,t) &= m(x,t) - \hat{m}(x,t), \\ &= m(x,t) - Ke^{b\alpha t}y(x). \end{aligned} \tag{6.1.16}$$

Then,

$$v_t(x,t) - \Upsilon v_{xx}(x,t) + g v_x(x,t) = b\alpha v(\alpha x, t). \tag{6.1.17}$$

Using conditions (6.1.11) and (6.1.14) in (6.1.16) imply

$$v(0,t) = 0, \tag{6.1.18}$$

and condition (6.1.12) gives

$$\begin{aligned} v(x,0) &= m_0(x) - Ky(x), \\ &= w_0(x). \end{aligned} \tag{6.1.19}$$

We refer equation (6.1.17) along with conditions (6.1.18) and (6.1.19) as v_m *Problem.*

The fundamental solution in the domain of the Fourier transform

Equation (6.1.16) indicates that $\hat{m}$ is a steady state solution to the m *Problem* if $||v||_{L^1[0,\infty)} \to 0$ as $t \to \infty$. In order to show the convergence of the solution to m *Problem* to the separable solution, we design a function $\chi(x,\xi,t)$ so that the solution to the *Cauchy Problem* (6.1.9) can be expressed as

$$u(x,t) = \int_{-\infty}^{\infty} \chi(x,\xi,t) u_0(\xi) d\xi,$$

and

$$\lim_{t \to 0^+} \int_0^{\infty} \chi(x,\xi,t) u_0(\xi) d\xi = u_0(x). \tag{6.1.20}$$

In addition,

$$\chi_t(x,\xi,t) - \Upsilon \chi_{xx}(x,\xi,t) + g\chi_x(x,\xi,t) = 0, \tag{6.1.21}$$

along with the condition

$$\chi(x,\xi,0) = \delta(x-\xi), \tag{6.1.22}$$

where δ is the Dirac delta function.

The solution to equation (6.1.21) can be found by taking the Fourier transform with respect to x, (cf. [68]). The Fourier transform of function f is defined as

$$\mathcal{F}(s) = \int_{-\infty}^{\infty} f(x)e^{-isx}dx.$$

We will regard χ as a function defined for all $x \in \mathbb{R}$. Moreover, we will assume that $\chi(x,\xi,t) \to 0$ as $x \to -\infty$ for all $t \geq 0$ and $\chi(x,\xi,t)$ satisfies the *No Flux Conditions 1*. Applying the Fourier transform to equation (6.1.21) gives

$$\int_{-\infty}^{\infty} e^{-isx}\chi_t(x,\xi,t)dx - \Upsilon \int_{-\infty}^{\infty} e^{-isx}\chi_{xx}(x,\xi,t)dx \\ +g\int_{-\infty}^{\infty} e^{-isx}\chi_x(x,\xi,t)dx = 0. \tag{6.1.23}$$

Now,

$$\int_{-\infty}^{\infty} e^{-isx}\chi_{xx}(x,\xi,t)dx = -s^2\int_{-\infty}^{\infty}\left(e^{-isx}\chi(x,\xi,t)\right)dx + \frac{\partial}{\partial x}\left(e^{-isx}\chi(x,\xi,t)\right)\Big|_{-\infty}^{\infty},$$

and

$$\int_{-\infty}^{\infty} e^{-isx}\chi_x(x,\xi,t)dx = (is)\int_{-\infty}^{\infty}\chi(x,\xi,t)e^{-isx}dx + \left(e^{-isx}\chi(x,\xi,t)\right)\Big|_{-\infty}^{\infty},$$

so that equation (6.1.23) yields

$$\frac{\partial}{\partial t}\int_{-\infty}^{\infty} e^{-isx}\chi(x,\xi,t)dx + \Upsilon s^2\int_{-\infty}^{\infty} e^{-isx}\chi(x,\xi,t)dx \\ + g\,(is)\int_{-\infty}^{\infty}\chi(x,\xi,t)e^{-isx}dx = 0. \tag{6.1.24}$$

Let

$$\mathcal{F}(s,\xi,t) = \int_{-\infty}^{\infty} e^{-isx}\chi(x,\xi,t)dx,$$

then equation (6.1.24) becomes

$$\left(\frac{d}{dt} + \Upsilon s^2 + gis\right)\mathcal{F}(s,\xi,t) = 0, \tag{6.1.25}$$

and the condition (6.1.22) gives

$$\mathcal{F}(s,\xi,0) = e^{-is\xi}. \tag{6.1.26}$$

We refer equation (6.1.25) together with initial condition (6.1.26) as $\mathcal{F}_m$ *Problem.* The $\mathcal{F}_m$ *Problem* is a first order linear differential equation together with an initial condition whose solution is

$$\mathcal{F}(s,\xi,t) = e^{-\Upsilon s^2 t} e^{is(\xi+gt)}.$$

The inverse Fourier transform of above function is

$$\begin{aligned}\chi_1(x,\xi,t) &= \int_{-\infty}^{\infty} e^{-\Upsilon s^2 t} e^{is(x-(\xi+gt)} ds \\ &= \frac{1}{2\sqrt{\pi}\sqrt{\Upsilon t}} e^{\frac{-((x-\xi)-gt)^2}{4\Upsilon t}}, \\ &= \frac{e^{\frac{-g^2 t}{4\Upsilon}} e^{\frac{-(\xi-x)g}{2\Upsilon}}}{2\sqrt{\pi}\sqrt{\Upsilon t}} e^{\frac{-(x-\xi)^2}{4\Upsilon t}}.\end{aligned}$$

Another solution to equation (6.1.21) that satisfies the conditions (6.1.22) and (6.1.20) is

$$\chi_2(x,\xi,t) = \frac{e^{\frac{-g^2 t}{4\Upsilon}} e^{\frac{-(\xi-x)g}{2\Upsilon}}}{2\sqrt{\pi}\sqrt{\Upsilon t}} e^{\frac{-(x+\xi)^2}{4\Upsilon t}}.$$

Solution to the v_m *Problem*

To find the solution to the v_m *Problem* we first make sure that the fundamental solution satisfies the initial condition

$$\chi(0,\xi,t) = 0; \tag{6.1.27}$$

as evident from (6.1.18). We therefore form the fundamental solution

$$\begin{aligned}\chi(x,\xi,t) &= \chi_1(x,\xi,t) - \chi_2(x,\xi,t), \\ &= \frac{e^{\frac{-g^2 t}{4\Upsilon}} e^{\frac{-(\xi-x)g}{2\Upsilon}}}{2\sqrt{\pi}\sqrt{\Upsilon t}} \left(e^{\frac{-(x-\xi)^2}{4\Upsilon t}} - e^{\frac{-(x+\xi)^2}{4\Upsilon t}} \right),\end{aligned}$$

that satisfies the equation (6.1.21), the conditions (6.1.22), (6.1.27) and the *No Flux Conditions 1.* For large t, the term $e^{\frac{-(x+\xi)^2}{4\Upsilon t}}$ vanishes and the first term is dominant for x near ξ. It can be verified that for $t > 0$, the function $\chi(x,\xi,t)$ satisfies the equation (6.1.21).

We seek a solution to the v_m *Problem* of the form

$$v(x,t) = \sum_{k=0}^{\infty} v_k(x,t), \tag{6.1.28}$$

where

$$v_0(x,t) = \int_0^\infty \chi(x,\xi,t) w_0(\xi) d\xi,$$
$$v_0(x,0) = w_0(x),$$

and

$$v_{k+1}(x,t) = b\alpha \int_0^t \int_0^\infty \chi(x,\xi,t-\tau) v_k(\alpha\xi,\tau) d\xi d\tau, \quad k \geq 0, \tag{6.1.29}$$
$$v_k(x,0) = 0.$$

Let $\mathcal{L} = \frac{\partial}{\partial t} - \Upsilon \frac{\partial^2}{\partial^2 x} + g\frac{\partial}{\partial x}$, then applying operator $\mathcal{L}$ to both sides of equation (6.1.29) gives

$$\mathcal{L}v_{k+1}(x,t) = \mathcal{L}\left(b\alpha \int_0^t \int_0^\infty \chi(x,\xi,t-\tau) v_k(\alpha\xi,\tau) d\xi d\tau\right), \tag{6.1.30}$$

using Leibniz's rule, condition (6.1.20) and the equation (6.1.21), it can be shown from (6.1.30) that

$$\mathcal{L}v_{k+1}(x,t) = b\alpha \int_0^t \int_0^\infty \mathcal{L}\chi(x,\xi,t-\tau) v_k(\alpha\xi,\tau) d\xi d\tau + b\alpha \lim_{\tau\to t} \int_0^\infty \chi(x,\xi,t-\tau) v_k(\alpha\xi,\tau) d\xi,$$
$$= b\alpha v_k(\alpha x, t),$$

for all $k \geq 0$.

Convergence of the series

Using the bounds for $\int_{-\infty}^{\infty} \chi(x,\xi,t) d\xi$ we show the uniform convergence of the series (6.1.28). This approach is given in [18]. Evidently,

$$0 \leq \chi(x,\xi,t) \leq \frac{e^{\frac{-g^2 t}{4\Upsilon}} e^{\frac{-(\xi-x)g}{2\Upsilon}}}{2\sqrt{\pi}\sqrt{\Upsilon t}} e^{\frac{-(x-\xi)^2}{4\Upsilon t}},$$

so that

$$\int_0^\infty \chi(x,\xi,t) d\xi \leq \int_0^\infty \frac{e^{\frac{-g^2 t}{4\Upsilon}} e^{\frac{-(\xi-x)g}{2\Upsilon}}}{2\sqrt{\pi}\sqrt{\Upsilon t}} e^{\frac{-(x-\xi)^2}{4\Upsilon t}} d\xi.$$

Let $\omega = (\xi - x)$, then

$$\int_0^\infty \chi(x,\xi,t) d\xi \leq \frac{e^{\frac{-g^2 t}{4\Upsilon}}}{2\sqrt{\pi}\sqrt{\Upsilon t}} \int_0^\infty e^{\frac{-g}{2\Upsilon}\left(\frac{\omega}{\sqrt{2gt}} + \sqrt{\frac{gt}{2}}\right)^2} d\omega.$$

Let $\sigma = \sqrt{\frac{g}{2\Upsilon}}\left(\frac{\omega}{\sqrt{2gt}} + \sqrt{\frac{gt}{2}}\right)$ then,

$$\begin{aligned}\int_0^\infty \chi(x,\xi,t)d\xi &\leq \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-\sigma^2}d\sigma,\\ &\leq \frac{1}{\sqrt{\pi}}\int_{-\infty}^\infty e^{-\sigma^2}d\sigma.\end{aligned}$$

The final integral is a Gaussian integral whose value is 1; hence,

$$\int_0^\infty \chi(x,\xi,t)d\xi \leq 1. \tag{6.1.31}$$

We are now ready to show the uniform convergence of v. Now,

$$\begin{aligned}|v_0(x,t)| &= |\int_0^\infty \chi(x,\xi,t)w_0(\xi)d\xi|,\\ &\leq ||w_0||_\infty \int_0^\infty \chi(x,\xi,t)d\xi,\end{aligned}$$

where $||w_0||_\infty$ denotes the supremum norm. Using (6.1.31) in the above inequality implies

$$|v_0(x,t)| \leq ||w_0||_\infty.$$

Similarly,

$$\begin{aligned}|v_1(x,t)| &= b\alpha\int_0^t\int_0^\infty |\chi(x,\xi,\tau)v_0(\alpha\xi,\tau)|d\xi d\tau,\\ &\leq b\alpha||w_0||_\infty t,\end{aligned}$$

$$|v_2(x,t)| \leq (b\alpha)^2||w_0||_\infty\frac{t^2}{2!},$$

$$|v_3(x,t)| \leq (b\alpha)^3||w_0||_\infty\frac{t^3}{3!},$$

$$\vdots$$

$$|v_k(x,t)| \leq (b\alpha)^k||w_0||_\infty\frac{t^k}{k!}.$$

We conclude that

$$|v(x,t)| \leq ||w_0||_\infty \sum_{k=0}^{\infty} \frac{(b\alpha t)^k}{k!},$$
$$= e^{b\alpha t}||w_0||_\infty.$$

Hence, the series is uniformly convergent for all $x \geq 0$ and $0 \leq t \leq T$.

Large time asymptotic solution

Recall equation (6.1.16) that is

$$v(x,t) = m(x,t) - Ke^{b\alpha t}y(x),$$

that can be written as

$$e^{-b\alpha t}v(x,t) = e^{-b\alpha t}m - Ky(x).$$

Taking $L^1[0,\infty)$ norm on both sides of above equation implies

$$e^{-b\alpha t}||v(x,t)||_{L^1[0,\infty)} = ||e^{-b\alpha t}m - Ky(x)||_{L^1[0,\infty)} \tag{6.1.32}$$

where $||.||_{L^1[0,\infty)}$ denotes the $L^1[0,\infty)$ norm with respect to x. Using the arguments in [18] we now show that in the $L^1[0,\infty)$ norm, the solution to the *m Problem* converges to the separable solution as $t \to \infty$.

Taking $L^1[0,\infty)$ norm on both sides of the terms v_0, v_1,...v_k,.. of v gives

$$||v_0(x,t)||_{L^1[0,\infty)} = \int_0^\infty | \int_0^\infty \chi(x,\xi,t)w_0(\xi)d\xi|dx,$$
$$\leq ||w_0||_{L^1[0,\infty)};$$

$$
\begin{aligned}
||v_1||_{L^1[0,\infty)} &= b\alpha \int_0^\infty | \int_0^\infty \chi(x,\xi,t-\tau)v_0(\alpha\xi,\tau)d\xi|dx, \\
&= b\alpha \int_0^\infty | \int_0^t \int_0^\infty \chi(x,u/\alpha,t-\tau)v_0(u,\tau)d\tau du|dx, \\
&= b\alpha \int_0^t \int_0^\infty | \int_0^\infty \chi(x,u/\alpha,t-\tau)v_0(u,\tau)du|dxd\tau, \\
&\le b||w_0||_{L^1[0,\infty)} \int_0^t d\tau, \\
&= bt||w_0||_{L^1[0,\infty)};
\end{aligned}
$$

$$
||v_2||_{L^1[0,\infty)} \le \frac{(bt)^2}{2!}||w_{01}||_{L^1[0,\infty)};
$$

$$
\vdots
$$

$$
||v_k||_{L^1[0,\infty)} \le \frac{(bt)^k}{(k)!}||w_0||_{L^1[0,\infty)}.
$$

Equation (6.1.32) thus implies

$$
||e^{-b\alpha t}m - y(x)||_{L^1[0,\infty)} \le e^{-b(\alpha-1)t}||w_0||_{L^1[0,\infty)}.
$$

Hence $m \to \hat{m}$ as $t \to \infty$.

6.2 Second order Pde with Linear growth rate

Let $D(x,t) = \Upsilon$, $G(x,t) = gx$ and $B(x,t) = b$ then the equation (1.3.4) implies

$$
-\Upsilon n_{xx}(x,t) + n_t(x,t) + g(xn(x,t))_x + bn(x,t) = b\alpha^2 n(\alpha x,t). \tag{6.2.1}
$$

We call equation (6.2.1) together with the *No Flux Conditions 1*, the *Initial Condition* and the boundary condition (1.3.10) as *Pde Problem with Linear Growth Rate*.

In this section we begin with a brief discussion of a first order case of the *Pde Problem with Linear Growth Rate* studied by Doumic & van Brunt [17].

6.2.1 The first order case

Doumic & van Brunt [17] considered a first order analogue of the *Pde Problem with Linear Growth Rate* that is

$$n_t(x,t) + (gxn(x,t))_x + bn(x,t) = b\alpha^2 n(\alpha x, t). \tag{6.2.2}$$

They showed that the the problem does not have a steady state solution. Here, we first derive the explicit solution form given in [17]. Let

$$n(x,t) = \frac{w(x,t)}{x}. \tag{6.2.3}$$

Then equation (6.2.2) implies

$$w_t(x,t) + gxw_x(x,t) + bw(x,t) = b\alpha w(\alpha x, t). \tag{6.2.4}$$

Consider the homogeneous equation of (6.2.4)

$$w_{0t}(x,t) + gxw_{0x}(x,t) + bw_0(x,t) = 0.$$

Let η_1 and ξ_1 be the characteristic projections, then applying the method of characteristics gives $t = \xi_1$, $x = \eta_1 e^{g\xi_1}$, and

$$\frac{dw_0}{d\xi_1} = -bw_0. \tag{6.2.5}$$

A solution to equation (6.2.5) is

$$w_0(\eta_1, \xi_1) = w_0(\eta_1)e^{-b\xi_1},$$

which in terms of x and t becomes

$$w_0(x,t) = w_0(xe^{-gt})e^{-bt}.$$

Equation (6.2.4) can be written as

$$w_{0t}(x,t) + gxw_{0x}(x,t) + bw_0(x,t) = 0,$$

and

$$w_{kt}(x,t) + gxw_{kx}(x,t) + bw_k(x,t) = b\alpha w_{k-1}(\alpha x, t), \tag{6.2.6}$$

where $k \in \mathbb{N}$. For $k = 1$, equation (6.2.6) yields

$$w_{1t}(x,t) + gxw_{1x}(x,t) + bw_1(x,t) = b\alpha w_0(\alpha x e^{-gt})e^{-bt}. \tag{6.2.7}$$

Let ξ_2 and η_2 be the characteristic projections. Then applying the characteristic method to the equation (6.2.7) gives $t = \xi_2$, $x = \eta_2 e^{g\xi_2}$ and

$$\frac{dw_1(\xi_2, \eta_2)}{d\xi_2} = -bw_1(\xi_2, \eta_2) + b\alpha w_0(\alpha\eta_2)e^{-b\xi_2}. \tag{6.2.8}$$

Equation (6.2.8) is a linear equation with respect to ξ_2 whose solution is

$$w_1(\xi_2, \eta_2) = e^{-b\xi_2} b\alpha w_0(\alpha\eta_2)\xi_2 + e^{-b\xi_2} w_0(\eta_2).$$

Continuing for higher values of k, it is straightforward that

$$w(\sigma, \gamma) = e^{-b\sigma} \sum_{k=0}^{\infty} w_0(\alpha^k \gamma) \frac{(b\alpha\sigma)^k}{k!},$$

where σ and γ are the characteristic projections. In terms of x and t above solution becomes

$$w(x,t) = e^{-bt} \sum_{k=0}^{\infty} w_0\left(\alpha^k x e^{-gt}\right) \frac{(b\alpha t)^k}{k!}.$$

In terms of $n(x,t)$, we have

$$n(x,t) = e^{-(b+g)t} \sum_{k=0}^{\infty} x n_0\left(\alpha^k x e^{-gt}\right) \frac{(b\alpha^2 t)^k}{k!}. \tag{6.2.9}$$

The solution does not lead to a simple asymptotic behaviour. Doumic & van Brunt applied the Mellin transform to the solution (6.2.9) and determine the asymptotics using the formula (cf. [16], Theorem 1). Despite the resemblance to the the solution (6.2.9), these asymptotics provided no further information. They found that the result of (cf. [8], Theorem 1) contradicted the oscillatory asymptotic behaviour. Doumic & van Brunt showed the weak convergence but they knew that a weak convergence may appear even for a point wise oscillatory solution. Therefore they further investigated and by changing the variable and using the Poisson formula to the asymptotics showed that the solution exhibits some specific periodicity in time.

6.2.2 The second order case

The first order case, studied by Doumic & van Brunt [17] does not have any SSD solutions as $t \to \infty$. We therefore anticipate that the *Pde Problem with Linear Growth Rate* does not converge to the separable solution as time goes to ∞. In order to identify the actual behaviour of the solution to the *Pde Problem with Linear Growth Rate* as $t \to \infty$, we first determine the solution form using the approach discussed in the previous section and then determine the long time behaviour of the solution.

Using the substitution (6.1.3) in the *Pde Problem with Linear Growth Rate* gives

$$-\Upsilon \tilde{n}_{xx}(x,t) + \tilde{n}_t(x,t) + gx\tilde{n}_x(x,t) + g\tilde{n}(x,t) = b\alpha^2 \tilde{n}(\alpha x, t).$$

In terms of cumulative function

$$q(x,t) = \int_x^\infty \tilde{n}(\xi, t) d\xi,$$

the equation (6.2.1) becomes

$$-\Upsilon q_{xx}(x,t) + q_t(x,t) + gxq_x(x,t) = b\alpha q(\alpha x, t). \tag{6.2.10}$$

For $x = 0$,

$$q(0,t) = K_0 e^{b\alpha t}, \tag{6.2.11}$$

where K_0 is a constant and for $t = 0$,

$$q(x,0) = q_0(x) = \int_x^\infty n_0(\xi) d\xi, \tag{6.2.12}$$

where $q_0(x) \leq 1$ for all $x \geq 0$.

We refer to equation (6.2.10) along with the boundary condition (6.2.11) and the initial condition (6.2.12) as *q Problem*.

Let $\hat{H}(t)\hat{h}(x)$, where $\hat{h}(x)$ is a pdf, be the separable solution to the *Pde Problem with Linear Growth Rate* and

$$\hat{q}(x,t) = \hat{N}(t)\hat{y}(x) \tag{6.2.13}$$

be the separable solution to the q *Problem.* Evidently,

$$\hat{y}(0) = 1,$$

and

$$\hat{N}(t) = K_1 e^{b\alpha t},$$

where K_1 is some constant.

Substituting the separable solution (6.2.13) in (6.2.10) gives

$$-\Upsilon \hat{y}_{xx}(x) + b\alpha \hat{y}(x) + gx\hat{y}_x(x) = b\alpha \hat{y}(\alpha x), \tag{6.2.14}$$

above equation is a type of the *Hermite Problem* that has been discussed in chapter 3.

It is evident that the function $\hat{y}(x) = 1$ satisfies the equation (6.2.14), hence;

$$\hat{q}(x,t) = K_1 e^{b\alpha t}. \tag{6.2.15}$$

Let

$$\bar{v}(x,t) = q(x,t) - K_1 e^{b\alpha t}. \tag{6.2.16}$$

Then,

$$\bar{v}_t(x,t) - \Upsilon \bar{v}_{xx}(x,t) + gx\bar{v}_x(x,t) = b\alpha \bar{v}(\alpha x, t), \tag{6.2.17}$$

$$\bar{v}(0,t) = 0, \tag{6.2.18}$$

and

$$\begin{aligned} \bar{v}(x,0) &= q_0(x) - K_1 \\ &= \bar{w}_0(x). \end{aligned} \tag{6.2.19}$$

We refer equation (6.2.17) along with the conditions (6.2.18) and (6.2.19) as v_q *Problem.*

If there is no SSD solution to the q *Problem* then in the $L^1[0,\infty)$ norm, $\bar{v}(x,t) \not\to 0$ as $t \to \infty$.

Following the analysis developed for the *Pde Problem with Constant Coefficients* we find the solution to the v_q *Problem* in terms of the fundamental solution.

We design a function $\hat{\chi}(x,\xi,t)$ so that solution to the *Cauchy Problem*

$$\hat{u}_t(x,t) - \Upsilon\hat{u}_{xx}(x,t) + gx\hat{u}_x(x,t) = 0,$$
$$\hat{u}(x,0) = \hat{u}_0(x), \quad \hat{u}(0,t) = 0,$$

can be expressed as

$$\hat{u}(x,t) = \int_{-\infty}^{\infty} \hat{\chi}(x,\xi,t)\hat{u}_0(\xi)d\xi,$$

and

$$\lim_{t\to 0^+} \int_0^{\infty} \hat{\chi}(x,\xi,t)u_0(\xi)d\xi = u_0(x). \tag{6.2.20}$$

Furthermore $\hat{\chi}$ satisfies

$$\hat{\chi}_t(x,\xi,t) - \Upsilon\hat{\chi}_{xx}(x,\xi,t) + gx\hat{\chi}_x(x,\xi,t) = 0, \tag{6.2.21}$$

along with the condition

$$\hat{\chi}(x,\xi,0) = \delta(x-\xi), \tag{6.2.22}$$

where δ is the Dirac delta function. We regard $\hat{\chi}$ as a function defined for all $x \in \mathbb{R}$. It is assumed that $\hat{\chi}(x,\xi,t) \to 0$ as $x \to -\infty$ for all $t \geq 0$ and $\hat{\chi}(x,\xi,t)$ satisfies the condition (1.3.8).

In terms of the Fourier transform, equation (6.2.21) becomes

$$\frac{\partial}{\partial t}\hat{\mathcal{F}}(s,\xi,t) + \Upsilon s^2\hat{\mathcal{F}}(s,\xi,t) - g\left(1 + s\frac{\partial}{\partial s}\right)\hat{\mathcal{F}}(s,\xi,t) = 0, \tag{6.2.23}$$

where

$$\hat{\mathcal{F}}(s,\xi,t) = \int_0^{\infty} e^{-isx}\hat{\chi}(x,\xi,t)dx.$$

Condition (6.2.22) implies

$$\hat{\mathcal{F}}(s,\xi,0) = e^{-is\xi}. \tag{6.2.24}$$

We refer equation (6.2.23) along with the initial condition (6.2.24) as $\mathcal{F}_q$ *Problem*. The $\mathcal{F}_q$ *Problem* is a first order Pde that we solve using the method of characteristic. Let η and σ be the characteristic projection then

$$t = \sigma + c_1(\eta),$$
$$s = c_2(\eta)e^{-g\sigma}.$$

Suppose $c_1(\eta) = 0$ and $c_2(\eta) = \eta$ then

$$t(\eta, \sigma) = \sigma,$$
$$s(\eta, \sigma) = \eta e^{-g\sigma}.$$

Evidently,

$$s(\eta, 0) = s_0 = \eta,$$

which gives

$$\hat{\mathcal{F}}_0(s_0, \xi, 0) = e^{-i\eta\xi}. \tag{6.2.25}$$

In addition,

$$\frac{d\hat{\mathcal{F}}}{\hat{\mathcal{F}}} = \left(g - \Upsilon s^2\right) d\sigma,$$

which in terms of σ is

$$\frac{d\hat{\mathcal{F}}}{\hat{\mathcal{F}}} = \left(g - \Upsilon e^{-2g\sigma}\right) d\sigma.$$

Integrating above equation from 0 to σ, using the initial condition (6.2.25) and substituting value of σ gives

$$\hat{\mathcal{F}}\left(s, \xi, t\right) = e^{gt} e^{-\frac{\Upsilon}{2g}s^2(e^{2gt}-1)} e^{-ise^{gt}\xi},$$

which is the solution to the $\mathcal{F}_q$ *Problem*. The Fourier inverse of the above solution is

$$\hat{\chi_1}(x, \xi, t) = \frac{e^{gt}\sqrt{g}}{\sqrt{2}\sqrt{\pi}\sqrt{\Upsilon(e^{2gt}-1)}} e^{\frac{-g(x-e^{gt}\xi)^2}{2\Upsilon(e^{2gt}-1)}}.$$

Following the *Pde Problem with Constant Coefficients* discussed in the previous section we construct

$$\hat{\chi}(x,\xi,t) = \frac{e^{gt}\sqrt{g}}{\sqrt{2}\sqrt{\pi}\sqrt{\Upsilon(e^{2gt}-1)}}\left(e^{\frac{-g(x-e^{gt}\xi)^2}{2\Upsilon(e^{2gt}-1)}} - e^{\frac{-g(x+e^{gt}\xi)^2}{2\Upsilon(e^{2gt}-1)}}\right).$$

It can be verified that $\hat{\chi}$ satisfies the equation (6.2.21), the initial condition (6.2.22) and the boundary condition (6.2.18).

Thus the solution to the v_q *Problem* that we seek as a series solution is

$$\bar{v}(x,t) = \sum_{k=0}^{\infty} \bar{v}_k(x,t),$$

where,

$$\bar{v}_0(x,t) = \int_0^{\infty} \hat{\chi}(x-\xi,\xi,t)\bar{w}_0(\xi)d\xi,$$

and

$$\bar{v}_{k+1}(x,t) = b\alpha \int_0^t \int_0^{\infty} \hat{\chi}(x,\xi,t-\tau)\bar{v}_k(\alpha\xi,\tau)d\xi d\tau.$$

It can be shown that for all $k \geq 0$, above solution satisfies the equation (6.2.17).

Convergence of the series

Evidently,

$$0 \leq \hat{\chi}(x,\xi,t) \leq \frac{e^{gt}\sqrt{g}}{\sqrt{2}\sqrt{\pi}\sqrt{\Upsilon(e^{2gt}-1)}} e^{\frac{-g(x-e^{gt}\xi)^2}{2\Upsilon(e^{2gt}-1)}},$$

therefore,

$$\int_0^{\infty} \hat{\chi}(x,\xi,t)d\xi \leq \int_0^{\infty} \frac{e^{gt}\sqrt{g}}{\sqrt{2}\sqrt{\pi}\sqrt{\Upsilon(e^{2gt}-1)}} e^{\frac{-g(x-e^{gt}\xi)^2}{2\Upsilon(e^{2gt}-1)}} d\xi.$$

Let $w = \frac{\sqrt{g}(x-e^{gt}\xi)}{\sqrt{2\Upsilon(e^{2gt}-1)}}$ then,

$$\begin{aligned}
\int_0^\infty \hat{\chi}(x,\xi,t)d\xi &\leq -\frac{1}{\sqrt{\pi}}\int_{\frac{\sqrt{g}x}{\sqrt{2\Upsilon(e^{2gt}-1)}}}^{-\infty} e^{-w^2}dw, \\
&= \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\frac{\sqrt{g}x}{\sqrt{2\Upsilon(e^{2gt}-1)}}} e^{-w^2}dw, \\
&\leq \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-w^2}dw, \\
&\leq 1. \qquad (6.2.26)
\end{aligned}$$

Now,

$$\begin{aligned}
|\bar{v}_0(x,t)| &= |\int_0^\infty \hat{\chi}(x,\xi,t)\hat{w}_0(\xi)d\xi|, \\
&\leq ||\hat{w}_0||_\infty \int_0^\infty \hat{\chi}(x,\xi,t)d\xi, \\
&\leq ||\hat{w}_0||_\infty, \qquad \text{(using (6.2.26)}
\end{aligned}$$

$$\begin{aligned}
|\bar{v}_1(x,t)| &= b\alpha\int_0^t\int_0^\infty |\hat{\chi}(x,\xi,t-\tau)v_0(\alpha\xi,\tau)|d\xi d\tau, \\
&\leq ||\hat{w}_0||_\infty b\alpha\int_0^t d\tau, \\
&\leq ||\hat{w}_0||_\infty b\alpha t,
\end{aligned}$$

and

$$\begin{aligned}
|\bar{v}_2(x,t)| &= b\alpha\int_0^t\int_0^\infty |\hat{\chi}(x,\xi,\tau)v_1(\alpha\xi,\tau)|d\xi d\tau, \\
&\leq ||\hat{w}_0||_\infty b\alpha^2\int_0^t (\tau)d\tau, \\
&\leq ||\hat{w}_0||_\infty (b\alpha)^2\frac{t^2}{2}.
\end{aligned}$$

Continuing this process gives

$$|\bar{v}_k(x,t)| \leq ||\hat{w}_0||_\infty \left(1 + b\alpha t + \frac{(b\alpha t)^2}{2!} + ... + \frac{(b\alpha t)^k}{k!}\right).$$

Hence,

$$|\bar{v}(x,t)| \leq \sum_{k=0}^{\infty} |\bar{v}_k|,$$
$$= e^{b\alpha t}.$$

Substituting the separable solution (6.2.15) in the equation (6.2.16) gives

$$e^{-b\alpha t}\bar{v}(x,t) = e^{-b\alpha t}q(x,t) - K_1.$$

Taking $L^1[0,\infty)$ norm to both sides of the above equation gives

$$||e^{-b\alpha t}\bar{v}(x,t)||_{L^1[0,\infty)} = ||e^{-b\alpha t}q(x,t) - K_1||_{L^1[0,\infty)}.$$

Mimicking the steps used in the previous section, it can easily be shown that solution to the *q Problem* approaches a constant as $t \to \infty$. We therefore conclude that the *Pde Problem with Linear Growth Rate* does not follow SSD solution as $t \to \infty$.

6.3 Pde with linear growth rate and non constant division rate

Let $D(x,t) = \Upsilon$, $G(x,t) = gx$ then the equation (1.3.4) implies,

$$-\Upsilon n_{xx}(x,t) + n_t(x,t) + g(xn(x,t))_x + B(x)n(x,t) = \alpha^2 B(\alpha x)n(\alpha x,t). \tag{6.3.1}$$

van Brunt *et al.* [84] studied above equation for $\Upsilon = 0$ and $B(x) = bx^r$. Using the substitution

$$n(x,t) = \frac{e^{gt}}{x^2}\theta(x,t),$$

in the equation

$$n_t(x,t) + g(xn(x,t))_x + bx^r n(x,t) = b\alpha^{2+r}x^r n(x,t), \tag{6.3.2}$$

where $g, b, r > 0$, they simplified the equation (6.3.2) to

$$\theta_t(x,t) + gx\theta_x(x,t) + bx^r\theta(x,t) = b\alpha^r x^r \theta(\alpha x,t). \tag{6.3.3}$$

They further used the substitution $z = \frac{x^r}{r}$ and $\hat{\theta}(z,t) = \theta(x,t)$, to transform equation (6.3.3) to

$$\hat{\theta}_t(z,t) + \hat{g}z\hat{\theta}_z(z,t) + \hat{b}z\hat{\theta}(z,t) = \hat{b}z\beta\hat{\theta}(\beta z,t),$$

where $\hat{g} = rg$, $\hat{b} = rb$, $\theta(\alpha x,t) = \hat{\theta}(\beta^r z,t)$ and $\beta = \alpha^r > 1$. van Brunt *et al.* thus studied an equation of the form

$$\psi_t(x,t) + gx\psi_x(x,t) + bx\psi(x,t) = b\alpha x\psi(\alpha x,t), \tag{6.3.4}$$

along with the conditions

$$\lim_{x\to 0^+} \frac{\psi(x,t)}{x} = 0,$$

$$\lim_{x\to\infty} \frac{\psi(x,t)}{x} = 0, \quad \text{for all} \quad t \geq 0,$$

and

$$\psi(x,0) = \psi_0(x).$$

van Brunt *et al.* applied the Mellin transform to the problem (6.3.4) and obtained the solution

$$\psi(x,t) = w_0(xe^{-gt})e^{-\frac{b}{g}x(1-e^{-gt})} + \sum_{k=1}^{\infty} c_k w_0(\alpha^k xe^{-gt})e^{-\frac{b}{g}\alpha^k x(1-e^{-gt})}.$$

The solution contained an arbitrary function w_0 that was determined through the condition ψ_0. They showed that the solution is unique and non negative provided ψ_0 is non negative. Aside from establishing an analytic solution form, they showed that for this problem there is no dominant eigenvalue and this leads to oscillatory terms in the long time asymptotic behaviour.

The determination of analytical solutions to the second order Pdes for $B(x) = x^r$, where $r > 1$ and the behaviour of their solution as $t \to \infty$ represent future work. Determination of analytical solutions to the second order Pdes for a more general non constant division rate is an open question.

In this chapter we have adapted a technique developed by Efendiev *et al* [18] to study the *Pde Problem* for constant and linear growth rates. We have established an analytical solution to the *Pde Problem with Linear Growth Rate* and it is shown that the problem does not have an SSD solution for linear growth rate and constant division rate. Determination of analytical solutions to the second order Pdes for a more general non constant division rate is an open question.

Chapter 7

Conclusion

In this dissertation we have discussed several ordinary and partial functional differential equations that arise in the study of the size structured cell growth/division model.

In order to demonstrate the solutions techniques to solve a pantograph type equations we have utilized a simple first order pantograph equation with constant coefficients. These techniques are largely based on the Mellin and the Laplace transforms. The use of the Laplace and the Mellin transforms to solve pantograph type equation relies on solving the transform equation which in itself can prove formidable. At present, only the transform equations that involve two arguments (e.g., s and $s-1$) can be solved explicitly. These are solved by assuming the transform M is of the form $M(s) = F(s)Q(s)$, where F is the transform of the homogeneous, non functional equation. The other function Q reflects, roughly speaking, the functional character of the equation. For many examples, this function emerges as an infinite product. For some of the simple case we looked at, the *Euler identity* can be used to convert the infinite product into an infinite series and thus allow a term by term inversion of the transform. There are pantograph type equations for which we can find M explicitly in terms of F and an infinite product Q, but there is no prominent identity such as Euler's to convert the product into an infinite series. For example, a solution to the equation

$$y''(x) + xy'(x) - y(\alpha x) = 0,$$

has the Mellin transform

$$M(s) = K\Gamma(s/2)\prod_{k=0}^{\infty}\left(1 - \frac{1}{\{s+2(k+1)\}\alpha^{s+2k+1}}\right).$$

The *Hermite Problem* discussed in chapter 3 is another example. This motivated us to look for some new approach that did not involve converting an infinite product into a series. This led to a technique that involved a sequence of the Mellin convolutions. This technique solves a first order pantograph equation, for a constant coefficients. However, the technique is not robust to solve equation (1.4.8) for non constant coefficients. We also introduced a new approach to prove the uniqueness of a solution based on a proof that the equation for the Mellin transform has a unique solution.

We briefly discussed few first order pantograph equations with non constant coefficients from the literature and then extended our study to the second order pantograph equations with constant and non constant coefficients. These problems are discussed in chapter 2. We discussed a special case of second order pantograph equations, the *Hermite Problem*, in which growth rate is linear. The qualitative study for this problem gave us some insight of the solution. The Mellin transform of this problem is such that we could not find the inverse Mellin transform explicitly. However, using the asymptotics of the Mellin transform it was shown that for a range of parameter values α, b and g, there are no pdf solutions to the *Hermite Problem*. However if we drop the integrability and positivity conditions, then there are non trivial solutions.

In chapter 4, we reviewed an approach developed by Perthame and Ryzhik [64]. Perthame & Ryzhik showed that a first order functional partial differential equation with non constant division rate approaches the separable solution as time goes to ∞. We applied their analysis to the problem discussed in chapter 5 in which we considered a cell growth model where division rate is modelled by a Heaviside function. In contrast to earlier problems, the eigenvalue for this problem cannot be readily obtained. However, the eigenvalue can be determined through a continuity condition. Adapting the analysis by Perthame & Ryzhik it is shown that there exists a unique solution that approaches the separable solution as $t \to \infty$.

Lastly, we studied certain second order Pdes and briefly discussed their corresponding first order analogues. Techniques applied to study the pantograph equations and first order Pdes are of not much help to study the second order partial functional differential equations; however, Efendiev *et al.* [18] recently solved a second order Pde with constant coefficients. Their technique involved the Laplace transforms and the determination of a suitable Green's function. The form of the Laplace transformed equation for a case when the second order Pde has linear growth, is such that the inversion is formidable; we therefore used the Fourier transform instead and obtained the fundamental solution. The determination of the fundamental solution using the Fourier transform is a well-known method (cf. [68]). By adapting the Efendiev *et al.* [18] we established the analytical solution to the second order Pde with a linear growth rate and show that the solution converges to the separable solution as time goes to ∞. Determination of a solution to the second order Pdes with a linear growth rate and a non constant division rate is still an open question.

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Appendices

Appendix A

List of conditions & identities

No Flux Conditions 1

$$\lim_{x \to 0^+} \left(\frac{\partial}{\partial x} \left(D(x) n(x,t) \right) - G(x) n(x,t) \right) = 0,$$
$$\lim_{x \to \infty} \left(\frac{\partial}{\partial x} \left(D(x) n(x,t) \right) - G(x) n(x,t) \right) = 0.$$

No Flux Conditions 2

$$\lim_{x \to 0^+} \left(G(x) n(x,t) \right) = 0,$$
$$\lim_{x \to \infty} \left(G(x) n(x,t) \right) = 0.$$

Initial Condition

$$n(x,0) = n_0(x).$$

No Flux Conditions 3

$$\lim_{x \to 0^+} \left(\frac{d}{dx} \left(D(x) y(x) \right) - G(x) y(x) \right) = 0,$$
$$\lim_{x \to \infty} \left(\frac{d}{dx} \left(D(x) y(x) \right) - G(x) y(x) \right) = 0.$$

No Flux Conditions 4

$$\lim_{x \to 0^+} (G(x)y(x)) = 0,$$
$$\lim_{x \to \infty} (G(x)y(x)) = 0.$$

Boundary Conditions

$$y(0) = 0,$$
$$\lim_{x \to \infty} y(x) = 0.$$

Pdf Conditions

$$y(x) \geq 0, \quad \text{for all} \quad x \geq 0,$$
$$\int_0^\infty y(x)dx = 1.$$

Integrability Condition

$$\int_0^\infty y(x)dx = 1.$$

Euler Identity

$$\prod_{k=0}^{\infty} (1 + zq^k) = 1 + \sum_{k=1}^{\infty} \frac{q^{k(k-1)/2} z^k}{\prod_{j=1}^{k} (1 - q^j)}.$$

Appendix B

List of problems

Chapter 1

Pde Problem

$$-\frac{\partial^2}{\partial^2 x}(D(x)n(x,t)) + \frac{\partial}{\partial t}n(x,t) + \frac{\partial}{\partial x}(G(x)n(x,t)) + B(x)n(x,t) = \alpha^2 B(\alpha x,t)n(\alpha x,t),$$

$$\begin{aligned}
\lim_{x\to 0^+}\left(\frac{\partial}{\partial x}\left(D(x)n(x,t)\right) - G(x)n(x,t)\right) &= 0,\\
\lim_{x\to \infty}\left(\frac{\partial}{\partial x}\left(D(x)n(x,t)\right) - G(x)n(x,t)\right) &= 0,\\
n(x,0) &= n_0(x).
\end{aligned}$$

Pde Problem without Dispersion

$$\frac{\partial}{\partial t}n(x,t) + \frac{\partial}{\partial x}(G(x)n(x,t)) + B(x)n(x,t) = \alpha^2 B(\alpha x,t)n(\alpha x,t),$$

$$\begin{aligned}
\lim_{x\to 0^+} G(x)n(x,t) &= 0,\\
\lim_{x\to \infty} G(x)n(x,t) &= 0,\\
n(x,0) &= n_0(x).
\end{aligned}$$

Chapter 2

Simplified Problem

$$\frac{d}{dx}y(x) + cy(x) = b\alpha c y(\alpha x), \quad \text{where} \quad c = \frac{b\alpha}{g},$$

$$\begin{aligned} y(0) &= 0, \\ \lim_{x\to\infty} y(x) &= 0, \\ y(x) &\geq 0, \quad \text{for all} \quad x \geq 0, \\ \int_0^\infty y(x)dx &= 1. \end{aligned}$$

Chapter 3

Constant Coefficient Problem 1

$$\frac{d^2}{dx^2}y(x) - a\frac{d}{dx}y(x) - by(x) + \lambda y(\alpha x) = 0, \quad a > 0, \quad b > 0,$$

$$y(0) = 0,$$
$$\lim_{x\to\infty} y(x) = 0,$$
$$\int_0^\infty y(x)dx = 1.$$

Constant Coefficient Problem 2

$$\frac{d^2}{dx^2}y(x) - g\frac{d}{dx}y(x) - b\alpha y(x) + b\alpha^2 y(\alpha x) = 0,$$

$$\lim_{x\to 0^+}\left(\frac{d}{dx}y(x) - gy(x)\right) = 0,$$
$$\lim_{x\to\infty}\left(\frac{d}{dx}y(x) - gy(x)\right) = 0,$$
$$\int_0^\infty y(x)dx = 1.$$

w Problem

$$\frac{d^2}{dx^2}w(x) - g\frac{d}{dx}w(x) - b\alpha w(x) + b\alpha w(\alpha x) = 0,$$

$$w(0) = 1,$$
$$\lim_{x\to\infty} w(x) = 0.$$

Modified Bessel Problem

$$x\frac{d^2}{dx^2}y(x) - a\frac{d}{dx}y(x) - b_1 y(x) + b_1\alpha y(\alpha x) = 0,$$

$$\lim_{x\to 0^+}\left(\frac{d}{dx}\left(xy(x)\right)-ay(x)\right)=0,$$
$$\lim_{x\to \infty}\left(\frac{d}{dx}\left(xy(x)\right)-ay(x)\right)=0,$$
$$\int_0^\infty y(x)dx=1.$$

Airy Problem

$$\frac{d^2}{dx^2}y(x)-xy(x)+\lambda xy(\alpha x)=0,$$

$$y(0)=0,$$
$$\lim_{x\to\infty}y(x)=0,$$
$$\int_0^\infty y(x)dx=1.$$

Hermite Problem

$$\frac{d^2}{dx^2}y(x)-\frac{d}{dx}\left(\frac{g}{\Upsilon}xy(x)\right)-\frac{b\alpha}{\Upsilon}y(x)+\frac{b\alpha^2}{\Upsilon}y(\alpha x)=0,$$

$$\lim_{x\to 0^+}\left(\frac{d}{dx}y(x)-\frac{g}{\Upsilon}xy(x)\right)=0,$$
$$\lim_{x\to \infty}\left(\frac{d}{dx}y(x)-\frac{g}{\Upsilon}xy(x)\right)=0,$$
$$\int_0^\infty y(x)dx=1.$$

Chapter 4

Perthame & ***Ryzhik Problem***

$$\frac{\partial}{\partial t}n(x,t) + \frac{\partial}{\partial x}\left(n(x,t)\right) + B(x)n(x,t) = \alpha^2 B(\alpha x)n(\alpha x,t), \quad t>0, \quad x \geq 0,$$

$$\begin{aligned} n(0,t) &= 0, \quad t>0, \\ n(x,0) &= n_0(x) \in L^1[0,\infty). \end{aligned}$$

y Problem

$$\frac{d}{dx}y(x) + \left(\lambda + B(x)\right)y(x) = \alpha^2 B(\alpha x)y(\alpha x), \quad x \geq 0,$$

$$\begin{aligned} \lim_{x\to 0^+} y(x) &= 0, \\ \lim_{x\to\infty} y(x) &= 0, \\ y(x) &\geq 0 \quad \text{for all} \quad x \geq 0, \\ \int_0^\infty y(x)dx &= 1. \end{aligned}$$

Ψ ***Problem***

$$\frac{d}{dx}\Psi(x) - \left(\lambda + B(x)\right)\Psi(x) = -\alpha B(x)\Psi(\frac{x}{\alpha}), \quad x \geq 0,$$

$$\begin{aligned} \Psi(x) &> 0 \quad \text{for} \quad x \geq 0 \\ \int_0^\infty y(x)\Psi(x)dx &= 1, \end{aligned}$$

y Problem with Constant Coefficients

The equation

$$\frac{d}{dx}y(x) + b\alpha y(x) = b\alpha^2 y(\alpha x), \quad x \geq 0,$$

$$\begin{aligned} y(0) &= 0, \\ \lim_{x\to\infty} y(x) &= 0, \\ y(x) &\geq 0, \quad \text{for all} \quad x \geq 0, \\ \int_0^\infty y(x)dx &= 1. \end{aligned}$$

Ψ *Problem with Constant Coefficients*

$$\frac{d}{dx}\Psi(x) - b\alpha\Psi(x) = -b\alpha\Psi\left(\frac{x}{\alpha}\right), \quad x \geq 0,$$

$$\begin{aligned} \Psi(x) &> 0, \quad \text{for} \quad x \geq 0, \\ \int_0^\infty y(x)\Psi(x)dx &= 1. \end{aligned}$$

y_L *Problem*

$$\frac{d}{dx}y_L(x) + (\lambda_L + B(x))\,y_L(x) = \alpha^2 B(\alpha x)y_L(\alpha x), \;\; 0 \leq x \leq \alpha x \leq L,$$

$$\begin{aligned} y_L(0) &= 0, \\ y_L(x) &> 0, \quad \text{for} \quad x > 0, \\ \int_0^L y_L(x)dx &= 1. \end{aligned}$$

Ψ_L *Problem*

$$\frac{d}{dx}\Psi_L(x) - (\lambda_L + B(x))\,\Psi_L(x) = -\alpha B(x)\Psi_L(\frac{x}{\alpha}), \quad x \geq 0,$$

$$\Psi_L(L) = 0,$$
$$\Psi_L(x) > 0, \quad \text{for} \quad x > 0,$$
$$\int_0^L y_L(x)\Psi_L(x)dx = 1.$$

Chapter 5

Heaviside Problem

$$\frac{\partial}{\partial t}n(x,t) + \frac{\partial}{\partial x}n(x,t) + bH(x-c)n(x,t) = b\alpha^2 H(\alpha x - c)n(\alpha x, t), \quad t > 0, x \geq 0,$$

$$\begin{aligned} \lim_{x\to\infty} n(x,t) &= 0, \\ n(0,t) &= 0, \\ n(x,0) &= n_0(x) \in L^1[0,\infty). \end{aligned}$$

Adjoint Problem

$$\frac{d}{dx}\hat{\Psi}(x) - (\lambda - b + bH(x-c))\,\hat{\Psi}(x) = -b\alpha H(x-c)\hat{\Psi}\left(\frac{x}{\alpha}\right), \quad x \geq 0,$$

$$\begin{aligned} \hat{\Psi}(x) > 0, \quad \text{for all} \quad x \geq 0, \\ \int_0^\infty y(\xi)\hat{\Psi}(\xi)d\xi = 1. \end{aligned}$$

Chapter 6

Pde Problem with Constant Coefficients

$$-\Upsilon\frac{\partial^2}{\partial^2 x}\tilde{n}(x,t)+\frac{\partial}{\partial t}\tilde{n}(x,t)+g\frac{\partial}{\partial x}\tilde{n}(x,t)=b\alpha^2\tilde{n}(\alpha x,t),$$

$$\lim_{x\to 0^+}\left(\frac{\partial}{\partial x}n(x,t)-\frac{g}{\Upsilon}n(x,t)\right)=0,$$
$$\lim_{x\to\infty}\left(\frac{\partial}{\partial x}n(x,t)-\frac{g}{\Upsilon}n(x,t)\right)=0,$$
$$n(x,0)=n_0(x).$$

m Problem

$$-\Upsilon\frac{\partial^2}{\partial^2 x}m(x,t)+\frac{\partial}{\partial t}m(x,t)+g\frac{\partial}{\partial x}m(x,t)=b\alpha m(\alpha x,t),$$

$$m(0,t)=ke^{b\alpha t},$$
$$m(x,0)=m_0(x)=\int_x^\infty n_0(\xi)d\xi.$$

v_m Problem

$$-\Upsilon\frac{\partial^2}{\partial^2 x}v(x,t)+\frac{\partial}{\partial t}v(x,t)+g\frac{\partial}{\partial x}v(x,t)=b\alpha v(\alpha x,t),$$

$$v(0,t)=0,$$
$$v(x,0)=w_0(x).$$

$\mathcal{F}_m$ Problem

$$\frac{d}{dt}\mathcal{F}(s,\xi,t)+\Upsilon s^2\mathcal{F}(s,\xi,t)+gis\mathcal{F}(s,\xi,t)=0,$$

$$\mathcal{F}(s,\xi,0)=e^{-is\xi}.$$

Pde Problem with Linear Growth Rate

$$-\Upsilon\frac{\partial^2}{\partial^2 x}n(x,t)+\frac{\partial}{\partial t}n(x,t)+g\frac{\partial}{\partial x}(xn(x,t))+bn(x,t)=b\alpha^2 n(\alpha x,t),$$

$$\lim_{x\to 0^+}\left(\frac{\partial}{\partial x}\left(\Upsilon n(x,t)\right)-gxn(x,t)\right)=0,$$
$$\lim_{x\to \infty}\left(\frac{\partial}{\partial x}\left(\Upsilon n(x,t)\right)-gxn(x,t)\right)=0,$$
$$n(x,0)=n_0(x).$$

q Problem

$$-\Upsilon\frac{\partial^2}{\partial^2 x}q(x,t)+\frac{\partial}{\partial t}q(x,t)+gx\frac{\partial}{\partial x}q(x,t)=b\alpha q(\alpha x,t),$$

$$q(x,0)=q_0(x)=\int_x^\infty n_0(\xi)d\xi,\quad q_0(x)\leq 1\quad \text{for all}\quad x\geq 0,$$
$$q(0,t)=K_0e^{b\alpha t}.$$

v_q Problem

$$-\frac{\partial^2}{\partial^2 x}\Upsilon\bar{v}(x,t)+\frac{\partial}{\partial t}\bar{v}(x,t)+gx\frac{\partial}{\partial x}\bar{v}(x,t)=b\alpha\bar{v}(\alpha x,t),$$

$$\bar{v}(0,t)=0,$$
$$\bar{v}(x,0)=\bar{w}_0(x).$$

$\mathcal{F}_q$ Problem

$$\frac{\partial}{\partial t}\hat{\mathcal{F}}(s,\xi,t)+\Upsilon s^2\hat{\mathcal{F}}(s,\xi,t)-g\left(1+s\frac{\partial}{\partial s}\right)\hat{\mathcal{F}}(s,\xi,t)=0,$$

$$\hat{\mathcal{F}}(s,\xi,0)=e^{-is\xi}.$$