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THE APPLICATION OF MATRIX THEORY

TO OPTIMAL DESIGN OF EXPERIMENTS

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Abstract

A development of the theory of optimum experimental design is presented. The notation and proofs are in terms commonly used by statisticians, rather than in the earlier measure theory terms. The D-optimality equivalence theorem is extended to the singular case, and similar results derived for a number of other criteria. Atwood's theorem for special  $n$ -tic polynomials is extended to the case where not all parameters are of interest. Finally methods of constructing optimal designs are considered and extended to allow deletion of unsatisfactory points, and some numerical examples are included.

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