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An initial-boundary value problem arising in
cell population growth modelling

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Abstract

A partial differential equation modelling cell populations undergoing growth and division characterised by size is studied. This equation is a special case of the fragmentation equation studied by Michel *et al.* [5] with no dispersion term, and the problem is of the initial-boundary value type.

Eigenfunction solutions are derived for separable solutions and related properties such as spectrum, uniqueness and unimodality are investigated. We show that the spectrum is continuous and that the decay of the eigenfunctions is exponential at a critical eigenvalue and algebraic otherwise.

The existence of a fast decay general solution $n(x, t)$ is then established. The problem can be solved analytically, and it is shown that the solution is unique and smooth. The solution properties are illustrated with some numerical simulations.

Finally, the role of exponential decaying eigenfunction solutions is interpreted from the standpoint of the general solution. The asymptotic behaviour as $t \rightarrow \infty$ of the general solution is examined. Slow decay eigenfunction solutions are briefly discussed, but their mathematical role remains to be explored.

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Abbreviations

| | |
|------------|--|
| SSD | S teady S ize D istribution |
| PDE | P artial D ifferential E quation |
| PDF | P robability D ensity F unction |
| ODE | O rdinary D ifferential E quation |

Symbols

| | | |
|-------------|------------------|-------------------|
| $n(x, t)$ | Number density | $\frac{1}{[l]}$ |
| x | Cell size | $[l]$ |
| t | Time | $[s]$ |
| g | Cell growth rate | $\frac{[l]}{[s]}$ |
| μ | Cell death rate | $\frac{1}{[s]}$ |
| $W(\xi, x)$ | Division kernel | $\frac{1}{[ls]}$ |

The notations l and s refer to size unit and second, respectively.

This thesis is dedicated to my parents with all of my love.