Developing 11-13 year old students' conceptual understanding of rational numbers - a case study investigating effective teacher pedagogical actions in a mathematics classroom.

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Abstract

Rational numbers, encompassing fractions, percentages, and decimals, are challenging for many students to understand. Research often focuses on the teaching and learning of these topics in isolation, rather than as part of the broader concept of rational numbers. However, there is limited research on the teaching practices that effectively support students' overall understanding of rational numbers. This study explores the teacher actions that help students develop a conceptual understanding of rational numbers and examines how students demonstrate their growing understanding.

A case study and qualitative methods were chosen for this research. The study involved one teacher and a class of 24 Year 7 and 8 students from an urban school in New Zealand. The teacher facilitated eight lessons focused on rational numbers, which included collaborative mathematical discussions. A variety of data were collected and analysed, including interviews, video recorded classroom observations, and examples of student work.

The findings revealed that when teachers design lessons around real-world mathematical tasks, promote collaborative discourse, and encourage the use of mathematical practices, students develop a deeper understanding of rational numbers. Additionally, using visual representations and explicitly connecting different forms of rational number representations helped enhance students' mathematical understanding.

This study contributes to the literature on how primary school teachers can effectively support students in developing a strong understanding of rational number concepts. It highlights that teacher content knowledge, combined with purposeful teaching strategies, can provide greater opportunities for students to develop a deeper understanding of rational numbers.

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Chapter One: Introduction

1.1 Introduction

This chapter establishes the context of the research and provides the study background. Section 1.2 discusses the complexities of learning rational numbers, and the need for teachers to have secure content knowledge of mathematics. Section 1.3 presents the rationale of the study. Section 1.4 states the specific research questions this case study investigates. Finally, Section 1.5 provides an overview of the subsequent chapters.

1.2 Background to the study

In recent years, the mathematics achievement of students in Aotearoa New Zealand has raised concerns, particularly in relation to inequity and ongoing slippage in the level of students reaching the curriculum level in the senior years of primary schooling. By the end of eight years of schooling, New Zealand primary school students are expected to be working at Level four of the New Zealand Curriculum. However, research from the National Monitoring Study of Student Achievement (NMSSA) revealed that only 42% of Year 8 students met curriculum expectations in mathematics (Ministry of Education, 2023). The study identified that some students particularly struggled with tasks involving fractions and decimal numbers.

The New Zealand mathematics curriculum has been updated and is to be implemented in 2025 (Ministry of Education, 2024). This updated curriculum provides clearer and more detailed expectations for students at each primary school year level. The curriculum outlines specific learning outcomes divided into three phases. Phase one focuses on the foundational understanding of rational numbers, beginning with identifying fractions. By Year three, students are expected to add and subtract unit fractions. Phase two, covering Years four, five and six, includes nine specific learning outcomes related to rational numbers for each year level, building on the skills introduced in Phase one. Phase three encompasses Years seven and eight and introduces an additional nine learning outcomes for each year level, further aiming to deepen students' understanding and application of rational numbers. This structured trajectory focuses on ensuring that students develop the necessary skills and knowledge to succeed in mathematics as they progress through their schooling.

While a curriculum outlines what should be taught, the way the content is delivered and the classroom environment significantly influence student learning outcomes. A substantial body of research (e.g. Chapin & O'Connor, 2007; Hunter et al., 2018; Jacobs & Empson, 2016; Selling, 2016b; Stein et al., 2008) has examined teacher actions that effectively support students in developing mathematical understanding. These studies show that teachers are essential in fostering a supportive learning atmosphere, implementing effective instructional strategies, and encouraging student engagement in mathematical discourse, all of which contribute to improved mathematics achievement.

1.3 Rationale

Rational numbers are a challenging area of mathematics for many students (Kieren, 1993; Mack, 2012; Tian & Siegler, 2017). Research into the teaching and learning of rational numbers is important because the concepts are fundamental to higher-level mathematics and success in secondary school (Siegler et al., 2012). A solid understanding of rational numbers equips students with essential skills that are applicable in both academic settings and everyday life (DeWolf et al., 2015).

Most studies on rational numbers have concentrated primarily on the misunderstandings that students exhibit or on specific forms of rational numbers, such as fractions or decimals (e.g. Jordan et al., 2017; Moss & Case, 1999; Resnick et al., 2019; Siegler et al., 2011). While this research provides valuable insights into common areas of difficulty, it tends to isolate these concepts rather than examining the broader context of how students develop a comprehensive understanding of rational numbers as a whole.

There is a notable gap in the research literature regarding how teachers can effectively influence the development of student understanding of the overarching concepts related to rational numbers. This includes big ideas such as equivalence, ordering, and operations with rational numbers. The current research study examines teacher practices and their impact on student learning of critical rational number understandings. The findings of this study contribute to research related to how teachers can foster deeper understanding of rational numbers, enhance pedagogical approaches, and improve their students' learning outcomes.

1.4 Objectives

The purpose of the study is to investigate the pedagogical actions employed by a teacher to enhance the conceptual understanding of rational numbers among 11-13 year old students.

In particular, the study addresses the following research questions:

1) What pedagogical actions does the teacher enact to support students in developing rational number understanding?
2) How do students demonstrate their understanding of rational numbers?

1.5 Overview

Chapter Two summarises literature that is focused on the teaching and learning of rational numbers. It outlines pedagogical teacher actions that support the development of student reasoning about fractions, decimals, and percentages. It also details the key mathematical concepts of rational numbers and common student miscocnceptions. Chapter Three details the research design and methods employed in the study, including data collection and data analysis processes. It introduces the participants and the research setting, along with the study timeframe. The chapter also discusses ethical considerations, the role of the researcher, and issues related to the validity and reliability of the research. Chapter Four presents the results of the study by integrating the findings and the discussion of the findings. The chapter examines the teacher actions in mathematics lessons in response to student reasoning. It also identifies and analyses student thinking and understanding of rational number concepts. Evidence will be supported by the data discussed in the literacture review. Finally, Chapter Five concludes the research by answering the research questions, summarising the key themes, and offering recommendations for teachers. It also discusses the limitations of the study and suggests potential areas for future research.

Chapter Two: Literature review

2.1 Introduction

This chapter will examine the existing literature surrounding the teaching and learning of rational number concepts, highlighting key pedagogical strategies that can enhance student understanding. Section 2.2 provides the definition of rational numbers. Section 2.3 discusses the teaching and learning of rational numbers.

2.2 Rational numbers

One of the most complex domains of mathematics that students encounter at primary school is rational numbers (Mack, 2012). Rational numbers are an extension of whole numbers and integers. They are defined as a number that can be represented as the quotient, or indicated division, of two integers, as in *a* divided by *b* where b is not 0 (Barnett-Clarke, 2010). Rational number is not a single construct, but a set of related subconstructs including a part-to-whole comparison, a decimal, a quotient, a ratio, a measure of continuous or discrete quantities, and an operator (Behr et al., 1983; Kieren, 2012; Salls, 2014). Partitioning is an essential skill for students to have in order to understand fractions (Kieren, 1993; Sophian & Wood, 1997). The concept of rational number is unified by three different yet related symbolic notations; common fractions, decimal fractions and percentages that represent the same amount or proportion (Vamvakoussi & Vosniadou, 2004). All rational numbers can be written as fractions and fractions with denominators that are positive powers of ten are called decimal fractions (Wu, 2011). To understand rational numbers is to know numbers that are both quotients and ratios and to know them in their many forms at the same time (Kieren, 2012). The complexity of the multiple representations, concepts and coding conventions can make them a problematic construct for students to understand (Behr et al., 1983; Tian & Siegler, 2017).

Student understanding of rational number concepts is vital because proficiency with rational numbers is essential for future mathematics achievement and success beyond high school. Research conducted by Siegler et al. (2012) examined longitudinal data sets from the United States and the United Kingdom. The focus of their study was to test the hypothesis that early knowledge of fractions is uniquely predictive of later knowledge of algebra and overall achievement in mathematics. Students in the American study were tested as 10-12 years old and then again as 15-17 year olds. In the U.K students were tested as 10 years old and again when they were 16 years old. The researchers concluded that a student's understanding of fractions at the first assessment age uniquely predicts later understanding of algebra and overall mathematics achievement in high school. Rational numbers is therefore an area of mathematics requiring specific attention from primary school educators (Winsløw, 2019). Teachers have a significant role in supporting students to develop a conceptual understanding of fractions, decimals, and percentages at primary school to allow for opportunities for lifelong success.

However, despite many efforts to improve instruction in this area of mathematics internationally over the past 30 years, student understanding and knowledge of rational numbers has remained largely stagnant (Tian & Siegler, 2017). In many western countries, changes in curriculum have recognised the need for students to learn about rational numbers alongside and in conjunction with developing whole number sense rather than waiting until whole number understanding has been embedded. Researchers (e.g. Clarke et al., 2008; Jordan et al., 2017; Kilpatrick et al., 2001; Petit et al., 2022; Van Hoof et al., 2017) agree that developing proficiency with rational numbers takes time and is a gradual process across the years of primary schooling. Vamvakoussi and Vosniadou (2004) state that understanding rational numbers is not difficult but is rather part of conceptual change that students move through. Vamvakoussi and Vosniadou conducted research about how prior knowledge of natural numbers can support new rational number understanding. Their study focused on student understanding of the rational number property of density. The research involved 16 students aged 14-15 years. Results were ranked in five categories, naive discreteness, advanced discreteness, discreteness-density, naive density and sophisticated density. They concluded that the understanding of density is constrained by the preconception of discreteness and that developing understanding of the density of rational numbers is a slow and gradual process. The development of rational number understanding is a slow and gradual process because it involves the reorganisation of natural number knowledge concepts. Teachers need to be aware that developing conceptual understanding of rational numbers is a process of student conceptual change that involves the idea of discreteness to density.

2.2.1 *Teaching and learning of rational numbers*

In this section the New Zealand Curriculum expectations are discussed, followed by the rational number learning trajectory. The most common rational number misconceptions are outlined and visual representations that support rational number concepts are defined.

2.2.2 *New Zealand Curriculum*

Rational numbers are taught across all year levels in New Zealand Schools and follow a trajectory of increasingly sophisticated understanding. Recently the curriculum has been revised and Te Mātaiaho The New Zealand Curriculum (Ministry of Education, 2024) outlines that students at all curriculum levels will be thinking mathematically in a range of meaningful contexts. Specifically, they will develop understanding of rational numbers by solving problems and modelling situations that require them to demonstrate the learning outcomes outlined in Table 1 below.

Table 1

Rational Number Learning Outcomes Across Curriculum Phases from The New Zealand Curriculum (Ministry of Education, 2024).

Learning Phase	Year Level	Rational numbers progress outcomes
One	**Year 1**	▪ identify and represent halves and quarters as fractions of sets and regions, using equal parts of the whole ▪ find a half or quarter of a set using equal sharing and grouping.
	Year 2	▪ identify, read, write (using symbols and words), and represent halves, quarters, and eighths as fractions of sets and regions, using equal parts of the whole ▪ directly compare two fractions involving halves, quarters, and eighths ▪ find a half and quarter of a set by identifying groups and patterns (rather than sharing by ones), and identify the whole set or shape when given a half or quarter
	Year 3	▪ identify, read, write, and represent halves, thirds, quarters, fifths, sixths, and eighths as fractions of sets and regions, using equal parts of the whole and by positioning on a number line ▪ compare and order fractions involving halves, quarters, and eighths and identify when two fractions are equivalent ▪ find a unit fraction of a whole number (e.g., 1/3 of 15), and identify the whole set or amount when given a unit fraction (e.g. " 1/4 of the set is 3, what is the whole set?") ▪ add and subtract unit fractions with the same denominator (e.g. 1/8 + 1/8 + 1/8 = 3/8)
Two	**Year 4**	▪ identify, read, write, and represent tenths as fractions and decimals ▪ compare and order tenths as fractions and decimals, and convert decimal tenths to fractions (e.g., 0.3 =3/10) ▪ divide whole numbers by 10 to make decimals ▪ for fractions with related denominators of 2, 4, and 8, 3 and 6, or 5 and 10: compare and order the fractions, identify when two fractions are equivalent by directly comparing them, noticing the simplest form (e.g., 3/6 = 1/2 , which is the simplest form) ▪ convert (using number lines) between mixed numbers and improper fractions with denominators of 2, 3, 4, 5, 6, 8, and 10 ▪ find a unit fraction of a whole number, using multiplication or division facts and where the answer is a whole number (e.g., 1/5 of 40) ▪ identify, from a unit fraction part of a set, the whole set ▪ add and subtract fractions with the same denominators to make up to one whole ▪ add and subtract decimals to one decimal place (e.g., 1.3 + 0.2 = 1.5) ▪ use doubling or halving to scale a quantity (e.g., to double or half a recipe)
	Year 5	▪ identify, read, write, and represent tenths and hundredths as fractions and decimals ▪ compare and order tenths and hundredths as fractions and decimals, and convert decimal tenths and hundredths to fractions ▪ divide whole numbers by 10 and 100 to make decimals ▪ for fractions with denominators of 2, 3, 4, 5, 6, 8, 10, 12, or 100: compare and order the fractions, identify when two fractions are equivalent

		▪ convert between mixed numbers and improper fractions with denominators of up to 10 ▪ find a fraction of a whole number, using multiplication and division facts and where the answer is a whole number (e.g., 2/3 of 24) ▪ identify, from a fractional part of a set, the whole set ▪ add and subtract fractions with the same denominators, including to make more than one whole. ▪ add and subtract whole numbers and decimals to two decimal places (e.g., 32.55 – 21.21 = 11.34) ▪ use known multiplication facts to scale a quantity
	Year 6	▪ identify, read, write, and represent fractions, decimals (to two places), and related percentages ▪ compare and order fractions, decimals (to two places), and percentages, and convert decimals and percentages to fractions ▪ multiply and divide numbers by 10 and 100 to make decimals and whole numbers (e.g., 1.3 × 10 = 13) ▪ for fractions with denominators of 2, 3, 4, 5, 6, 8, 10, 12, or 100: compare and order the fractions, identify when two fractions are equivalent, represent the fractions in their simplest form ▪ convert between mixed numbers and improper fractions ▪ find a fraction or percentage of a whole number where the answer is a whole number (e.g., 3/8 of 48; 30% of $150) ▪ identify, from a fractional part of a set, the whole set. ▪ add and subtract fractions with the same or related denominators (e.g., 1/4 + 1/8) ▪ add and subtract whole numbers and decimals to two decimal places (e.g., 250.11 + 135.29 = 385.4) ▪ use known multiplication and division facts to scale a quantity
Three	**Year 7**	• identify, read, write, and represent fractions, decimals (to three places), and percentages • compare, order, and convert between fractions, decimals (to three places), and percentages • multiply and divide numbers by 10, 100, and 1000 • find equivalent fractions, simplify fractions, and convert between improper fractions and mixed numbers • multiply fractions and decimals by whole numbers • find a percentage of a whole number, and find a whole amount, given a simple fraction or percentage (e.g., “25% is $100, what is the total amount?”) • add and subtract fractions with different denominators of up to a tenth, using equivalent fractions (e.g., 3/4 + 1/3) • add and subtract decimals to three decimal places, with an emphasis on estimating before calculating • use proportional reasoning to explore multiplicative relationships between quantities (e.g., “If there are 3 red for every 7 blue balls, how many balls are there altogether when there are 18 red balls?”)
	Year 8	▪ identify, read, write, and represent fractions, decimals, and percentages ▪ compare, order, and convert between fractions, decimals, and percentages ▪ multiply and divide numbers by powers of 10 ▪ find equivalent fractions, simplify fractions, and convert between improper fractions and mixed numbers ▪ multiply fractions and decimals by whole numbers ▪ find a percentage of a whole number, and find a whole amount, given a simple fraction or percentage (e.g., “75% is $45, what is the total amount?”)

		▪ add and subtract fractions with different denominators, using equivalent fractions. ▪ add, subtract, and multiply decimals, with an emphasis on estimating before calculating ▪ use proportional reasoning to share with unequal proportions (e.g., “We have 100 stickers to share. For every 1 sticker I get, you get 3. How many do we each get?”)

Students in New Zealand begin to learn about rational numbers by developing an understanding of fractions from five years old. As they progress through primary school they learn about decimals and percentages and develop understanding of the equivalent relationships across all three forms of notation. However, a growing body of research that challenges the commonly implemented trajectory of teaching and learning of rational numbers is discussed in the next section.

2.2.3 Conjectures about the rational number learning trajectory

There has been considerable research into the teaching of learning of rational number representations and notations and the effect on developing students conceptual understandings (Durkin & Rittle-Johnson, 2015; Irwin, 2001; Resnick et al., 1989; Tian & Siegler, 2017). Questions have been raised about the sequencing of learning experiences of the different representations. For example, should students be taught about percentages before learning about fractions? Moss and Case (1999) presented a case for teaching percentages before fractions. Their study involved 9–10-year-old students in Canada. Moss and Case developed an experimental curriculum utilising students’ intuitive and original understanding of ratios and designed tasks to allow students to use already known procedures for splitting numbers. Initially, the tasks focused on representing the rational number as a percentage. Moss and Case believed that one symbolic representation understood in depth is better than having little understanding of various representations. As the unit progressed, equivalent representations of decimals and fractions were gradually introduced through benchmarks, and in-depth connections to percentages were explicitly made clear. These researchers concluded that students should begin to learn about percentages before decimals and fractions. The reasons they stated were that every percentage has a fraction and decimal equivalent, percentages have the same denominator, which makes the computation of fractions easier than working with different denominators, and students were familiar with the context of a percentage bar indicating file transfer on a computer. They concluded that using percentages as a starting point for teaching and learning rational numbers supported a deeper understanding of rational numbers and improved proportional number sense. Alternatively, Tian and Siegler (2017) reviewed research into whether decimals are, in fact, easier to understand than fractions and if teaching decimals before fractions could lead to better learning outcomes. They concluded that students had the same conceptual difficulties with decimals and fractions therefore more research is required to provide evidence for the ideal sequence of teaching the three rational number notations. While the ideal rational number learning trajectory is still an area that requires more research, the big mathematical ideas underpinning rational numbers are generally agreed upon. The big ideas of rational numbers are summarised in the next section.

2.2.4 Mathematical big ideas

Teachers making connections to the big mathematical ideas is key for students to develop a robust understanding of rational numbers. Charles and Carmel (2005) define a big idea as, '*a statement of an idea that is central to the learning of mathematics, one that links numerous ideas to a coherent whole'* (p. 10). Big ideas are connected to many other ideas therefore the understanding of big ideas supports a deep understanding of mathematics. The National Council for Teaching Mathematics has outlined four big ideas for rational number and the key understandings for each big idea (Barnett-Clarke, 2010). These are outlined in the Table 2 below:

Table 2

Big Ideas and Key Understandings for Rational Number

Big Idea 1	Extending from whole numbers to rational numbers creates a more powerful and complicated number system.
Big Idea 2	Rational numbers have multiple representations, and making sense of them depends on identifying the unit.
Big Idea 3	Any rational number can be represented in infinitely many equivalent symbolic forms.
Big Idea 4	Computation with rational numbers is an extension of computation with whole numbers but introduces some new ideas and processes.

It is important for teachers to understand the big ideas so they can make informed decisions about next steps in teaching in response to student reasoning. For example, if misconceptions arise in student reasoning, the teacher can ask questions to guide students to change their thinking. Martinie (2014) conducted research with 350 students aged 11-13 years across two schools in the United States. The study had two main components. First, students took a decimal comparison test which highlighted common misconceptions about decimals. Second, they were asked to represent decimal numbers, which allowed researchers to assess their conceptual understanding. The findings identified three teacher actions that support students to overcome misconceptions to better understanding of rational numbers, although the approach for each misconception may differ. The first teacher action is to anticipate student misconceptions in advance. The second is to encourage students to explain their reasoning and to ask guiding questions that prompt students to change their thinking. Finally, teachers should ask students to work with new representations and use their misconceptions as a foundation for deeper understanding. Research into misconceptions about rational numbers are reviewed in the next section.

2.2.5 Common misconceptions

Misconceptions about rational numbers are common when depth and understanding of the number system is not developed thoroughly. Initially, these misconceptions can develop when students apply their whole number knowledge to make sense of non-natural numbers, referred to as the natural number bias (Martinie, 2014; Resnick et al., 1989; Stafylidou & Vosniadou, 2004; Van Hoof et al., 2017). There are three aspects of the natural number bias that relate to density, size and operations (Van Hoof et al., 2018). Natural numbers are discrete and it is relatively easy for students to name a number before or after a given number. For example, when presented with the number 98, students can say the number before is 97 and the number after is 99. However, rational numbers are dense, there are an infinite number of rational numbers between any two numbers. Challenges can arise when students approach a problem with a natural number bias and rely on what they know well. Streefland (2012) called these misconceptions natural number distractors. She described this as a case of failing to regard fractions as a ratio and instead, operating with numerators and denominators independently while Kieren (1993) described rational numbers as being intertwined with and sharing language with whole numbers but being far more complex than a simple system extension. Instead, it has new properties and axioms to understand. Van Hoof et al. (2018) conducted a longitudinal study of student development of rational number understanding over a two year period in Belgium. The study involved 201 participants aged 9 - 12 from 4 schools. Students were administered the same rational number test three times over the research period. The findings outline student understanding of the size of rational numbers forms a necessary first step in learners understanding of rational numbers. They conclude the differences in whole numbers and rational numbers should be highlighted when students learn about rational numbers so that students do not rely on their natural number bias. In a similar longitudinal study, Seethaler et al. (2011) investigated the computational skill of students with rational numbers and how those were similar or different to computational skill with whole-numbers. The study involved 2,023 students (aged 8-9 years at the beginning of the study) over four years in the United States. There were two key findings of the study. First, the greatest predictor of computational skills with rational numbers was computational fluency with whole numbers in earlier years of schooling. Second, oral language uniquely supports the development of rational number computation skill as well as the conceptual understanding of fractions, decimals and percentages. Both studies highlight the critical relationship between prior knowledge and the development of rational number understanding.

The operations aspect of the natural number bias occurs when students assume that operations such as addition and multiplication will lead to a larger result and subtraction and division will lead to a smaller result (Van Hoof et al., 2018). The size aspect of the natural number bias contributes to misconceptions that students demonstrate in rational number tasks in a number of ways. Many students commonly demonstrate that they do not understand that fractions are numbers and may interpret the numerator and the denominator as two separate numbers (Petit et al., 2022). When solving problems about fractions students apply generalisations for whole numbers that do not hold true when working with fractions (Clarke et al., 2008). Students with an inadequate understanding of place value can incorrectly assume that longer decimal fractions are larger or that less numbers after the decimal point is larger (Martinie, 2014). The counting principles of natural numbers are misapplied to fractions and are misinterpreted as the bigger the number, the larger the fraction (Van Hoof et al., 2018). Furthermore, Kilpatrick et al. (2001) describe that one of the significant properties for students to learn about rational numbers is that the two integers that compose a fraction, the numerator and the denominator, are related through multiplication and division, not addition. This bipartite relationship can cause misunderstandings among students when they begin to work with rational numbers (Jordan et al., 2017).

Different sorts of misconceptions can be found in fraction tasks compared to decimal tasks and this was a focus of research conducted by Resnick et al. (1989). The study involved student participants 9-12 years old and took place in the United States, Israel, and France. Resnick et al. focused on examining student errors with decimal fractions as error-based rules. The rules are referred to as whole number rules and fractions rules. They found that students with a poor understanding of the place value system used the whole number rule. They did not know how zero works as a placeholder and, most importantly, did not understand that a decimal represents a fractional part of a whole. In the fraction rule, children understood place value as tenths and hundredths to the right of the decimal point. However, by applying the fraction rule, it is impossible to compare decimals with a different number of places and errors occur. Interestingly, in France, decimals are taught at primary school before fractions and children in France did not demonstrate the fraction rule misconception as American children did (Resnick et al., 1989). In other research, Durkin and Rittle-Johnson (2015) conducted a similar study with 9-11 year olds in the United States. Their study focused on the process of conceptual change where student understanding of the concept of natural numbers, transitions to include rational numbers. It aimed to diagnose student misconceptions and assess changing knowledge of decimals over a period of four weeks. Their findings suggested that misconceptions based on the whole numbers and the role of zero decreased, however, fraction misconceptions increased. They concluded that misconceptions often change into another misconception rather than correct concepts. Both studies illuminate the nuances misconception students hold regarding decimals and fractions.

Teachers noticing student misconceptions and deciding how to address them is an important teacher action. The following section will review literature that describes how teachers can support students to make connections across the domains of rational numbers and develop a flexible understanding.

2.2.6 Visual representations

Number lines are a powerful representation when used to develop rational number understanding. Many researchers have highlighted that student understanding of rational numbers can be best supported by representing relative positions on number lines to show the measurement and density properties of fractions (Bright et al., 1988; Jordan et al., 2017;

Resnick et al., 2019; Siegler et al., 2011; Van Hoof et al., 2017; Xenia et al., 2018). Understanding that fractions and decimals are an extension of whole numbers with a specific magnitude can support students in developing an understanding of the complexities of our number system (Petit et al., 2022). In the Delaware Longitudinal Study in the U.S.A, Jordan et al. (2017) examined the development of fraction concepts in students from age eight through to 12 years old. They found two issues within classroom fractions instruction. One is that a part-whole focus in classroom instruction of fractions can cause students to have an incomplete view of rational numbers. For example, students whose only experience with fractions is cutting a whole into equal-sized parts where the denominator matches the number of partitions in the whole will never see fractions as anything more than one whole. They conclude that another area for improvement is that the narrow focus on teaching and learning proper fractions in primary school may lead students to believe all fractions (a/b) are only numbers between zero and one. They propose a number-line approach to remedy some of the difficulties described above.

Number lines reinforce the concept that fractions are numbers that represent magnitude. Fractions can be ordered and positioned on a number line to develop an understanding of the density property of rational numbers, meaning that there are an infinite number of fractions between any two fractions (Winsløw, 2019). Siegler et al. (2011) conducted research with 9–12-year-old students assessing fraction magnitude knowledge involving number line estimation. Students demonstrated that they extended their magnitude understanding of whole numbers to include rational numbers and that number lines were a powerful representation that supported estimation. They also pointed out that previous research, referred to earlier in this review regarding the whole number bias as hindering rational number understanding, may instead be helpful. In their study, Van Hoof et al. (2017) concluded that teachers must decide when to emphasise the profound similarities between the whole and rational numbers systems and when to highlight the differences. One similarity is that whole and rational numbers measure quantities, have magnitudes and can be placed on number lines. In addition, research by Yu et al. (2022) found that number lines efficiently free the working memory for students. They state that the two proportional components of a fraction, the numerator and the denominator, are reduced to one point on a line, resulting in a single dimension to comprehend. Children can extend their mental number lines from whole familiar numbers to new rational numbers, building on what they know and applying this to their learning. In another study examining how young students build reasoning in fractions, Maher and Yankelewitz (2017) undertook a year-long intervention study with a class of 9-10 year old students that focussed on building student reasoning in fractions. Their research intended to co-ordinate students' understanding of fractions as operators with fractions as numbers. They discovered a clear theme from their studies. Children make a conceptual movement from thinking of fractions as parts of a whole to abstracting fractions and as positions on a number line. The finding suggests that a robust understanding of fractions as number are foundational for rational number development. Positioning fractions on a number line and reasoning about what fraction sits at a position indicated by an arrow are described as powerful learning experiences that support rational number understanding. Number lines are proven to be a vital representation in primary school mathematics.

2.3 Effective pedagogy

It can be challenging for teachers to support all students to learn mathematics with understanding. Teachers can influence how their students develop mathematical reasoning through the learning environments they develop, the learning opportunities they provide, how they orchestrate equitable opportunities to participate in rich discussions about the mathematics, and through their responses to student understanding. The following section will review literature regarding effective pedagogical approaches to teaching mathematics.

2.3.1 Developing a classroom community

Developing an effective classroom community is important because students need to feel safe and supported in order to engage in mathematical discourse. The classroom is a learning community where students actively construct knowledge while participating in and contributing to the mathematical discourse between students and with the teacher. A mathematics-talk inquiry community is a classroom where all participants engage in discourse to learn mathematics (Hufferd-Ackles et al., 2004). When students engage in collaborative discourse and argumentation in mathematics classrooms there is increased student learning and deep-level understanding (Franke et al., 2009; Nussbaum, 2008). A discourse-based mathematics communities approach to teaching mathematics is based on the teaching model, complex instruction, where students work together in small groups and act as resources for each other's learning (Boaler, 2006; Cohen et al., 1999). Students are grouped heterogeneously to collaboratively solve complex mathematical tasks. Anthony et al. (2015) refer to this discourse-based inquiry approach to teaching mathematics as "ambitious teaching" because it is complex and requires significant effort by teachers to facilitate discussions that focus on student reasoning. In their case study they focused on the teaching practice of one teacher as she facilitated a mathematics lesson with 30 students (aged 7-8 years). They found that ambitious teaching involved letting students guide the mathematical discussion and the teacher responded in the moment to the needs of the learners. A study by Averill (2012) researched caring teacher practices in mathematics classrooms. The study involved six teachers of students aged 14-15 years in three New Zealand secondary schools. The focus of the study was to determine what teacher pedagogies and behaviours' were effective for developing caring student-teacher relationships and supporting mathematical progress. Teacher behaviours were analysed in relation to a four sided model called whare tapa wha. The four sides related to four different dimensions including, taha hinengaro (expression of thoughts and feelings), taha whānau (relationships), taha wairua (spiritual elements), and taha tinana (health and well-being). Their study found that the lessons with the most teacher-care elements had the greatest student engagement and the most student-initiated interactions about mathematics. Teacher care and effective teacher-student relationships is essential for student learning in mathematics.

The importance of norms for participation in a community of learning are outlined in the next section.

2.3.2 Norms

The role of the teacher in setting the expectations for participation and collaboration in classroom discussions is paramount to the quality of discussion. Students must feel safe and respected in the community to participate in discussions and contribute mathematical reasoning. Conditions for respectful discourse include clear rules for what constitutes respect and disrespect so that everyone is listened to and all ideas are taken seriously (Chapin & O'Connor, 2007). Goos (2004) details that learning mathematics entails communication in social contexts and that the teacher must model a mathematical attitude as one of the community members. Establishing socio-mathematical norms where clear meta-discursive rules are co-constructed between teachers and students is an essential component of a classroom community where discussion is valued (Cobb et al., 2000; Hunter, 2010; Mueller et al., 2014; Sfard, 2001; Stovner et al., 2021; Yackel & Cobb, 1996). Mueller et al. (2014) outline norms as clear expectations for participation in respectful classroom discussions. They conjecture that this is a requirement for opening the space up for all learners to access the learning conversations within the community. Hunter and Hunter (2018) elaborate further to define three categories of norms, social norms, mathematical norms and cultural norms. Social norms are guidelines for the community to participate and communicate. Mathematical norms relate to mathematical behaviours required for learning and doing mathematics, such as explaining and justifying. Cultural norms refer to the ways of being in the classroom that reflect students' cultures, backgrounds and home lives within the community. For example, Hunter (2010) writes about a teacher making connections between listening to a speaker at a hui with pauses and thinking time, and the wait time required for working in a collaborative mathematics group. This analogy supported the students to see that their cultures are linked with their learning in school.

2.3.3 Mathematical practices

Engagement with mathematical practices is an important part of deepening mathematical understanding. Many researchers agree that growing students' conceptual understanding of mathematics requires teachers to set expectations for the use of mathematical practices (Anthony et al., 2015; Boaler, 2006; Cobb et al., 2000; Hunter, 2010; Hunter & Hunter, 2018; Moschkovich, 2015; Selling, 2016b). Boaler (2006) describes mathematical practices as mathematical ways of working. She explains that developing student proficiency in using mathematical practices will prepare students to be active citizens in the 21st century, where they will require the disposition to puzzle through unfamiliar problems. Mathematical practices are evident in classrooms when learners speak and act mathematically (Goos, 2004). Some of the practices include making mathematical explanations and justifications (Boaler, 2006; Hunter & Hunter, 2018; Staples et al., 2012; Yackel & Cobb, 1996), engaging in mathematical argumentation (Hunter & Hunter, 2018; Mueller et al., 2014; Yackel & Cobb, 1996), representing reasoning using tools, symbols and language (Cobb et al., 2000; Cordero-Siy & Ghousseini, 2022; Sfard, 2001), as well as looking for patterns and generalising (Moschkovich, 2015).

When students engage in mathematical practices teachers can gain insights into their conceptual understanding of mathematics. Research by Staples et al. (2012) investigated justification within mathematics classrooms of 12 teachers of students aged 12-14 years. The teachers were involved in a project that focused on developing students justification and argumentation. Analysis of classroom interactions and teacher interviews focused on the purpose of developing student justification skills. Findings concluded that justification is an important mathematical practice for students to develop because it builds connections between mathematical concepts and leads to the development of deeper mathematical understanding. In a New Zealand study, Hunter and Hunter (2018) researched student talk as a means of addressing equity issues of participation in mathematics classroom communities. The research project involved 400 teachers from 32 schools with students ages 5-13 years. They found that to get students engaging in mathematical practices of explanation, argumentation, and generalisation teachers opened the class to discussion and encouraged students to talk. Teachers also scaffolded students to engage in mathematical practices while participating in mathematics and explicitly named the practices students demonstrated. Engaging in mathematical practices supported students to engage with challenging mathematics tasks and mathematical concepts. In other research, Selling (2016b) studied the way in which teachers make mathematical practices explicit in classroom discourse. Her study analysed discussions from 3 mathematics classes of students aged 11 to 18 years old in the United States. Her aim was to understand how mathematical practices can be taught without becoming prescriptive because she wanted to shift away from explicit direct instruction as a dominant form of mathematics pedagogy. She found that teachers use three different moves to initiate, sustain, and reprise student engagement with mathematical practices. Her analyses led to the development of a framework of eight reprising moves that detail how teachers make the practices explicit after they have engaged in them. These are highlighting, naming, evaluating, explaining the goal or rationale, connecting student engagement, framing expansively, eliciting self-assessment, and making the teaching narrative explicit. The notion that being explicit is an essential teacher action is highlighted while also emphasising that this is not the same as direct teaching or the teacher action of telling students what to do. She concludes that making mathematical practices explicit supports students to develop an understanding of mathematical content as well as developing normative ways of working mathematically.

2.3.4 Mathematical tasks

The tasks students engage with in mathematics classrooms highly influence student outcomes. Stein et al. (1996) consider a high-level cognitively demanding task to encourage the use of multiple representations and solution strategies, as well as requiring explanation and justification of how the students arrived at their answers. Mathematical tasks have the potential to support students to extend and build on their informal knowledge of rational numbers. Martinie (2014) outlines a good problem as a broad task with a clear purpose that allows the teacher to formatively assess student understanding. When the contexts of the problems are set in familiar situations then students can draw on their informal knowledge to solve them. By providing common and familiar contexts students can draw on their experiences and reasoning to create representations. Teachers can then guide students to connect their representations to the corresponding symbolic representations (Mack, 2012). Research undertaken by Irwin (2001) involved 11-12 year old students in New Zealand. The focus of her study was to investigate if contextually familiar and relevant tasks influenced student understanding of decimals. This study found that students who worked on the contextualised problems improved their competence with decimals more than the group that did not work on contextualised problems. She concluded that solving tasks that required everyday knowledge of the students supported them to overcome misconceptions about decimals because the contexts were meaningful. Similarly, other research by Silver et al. (1993) investigated student solutions and sense making when solving division problems with remainders. The study involved 195 students (ages 10-13 years) in the United States. In their findings, the older students were more likely to give consideration to the interpretation of their numerical results with respect to the problem situation. Furthermore, the real life task context supported students to develop arguments to justify their solutions and interpretations of remainders. A variety of student responses was prompted by the task context.

2.3.5 Teacher actions to facilitate discussion

Communication within the classroom can influence learning by supporting students to make sense of their own ideas and the ideas of other. Thinking can be considered a form of communication. Chapin and O’Connor (2007) outline that academically productive talk supports the development of student reasoning and the ability to clearly express their thoughts. In the U.S.A, in their intervention work with Project Challenge, Chapin and O’Connor (2007), worked with 400 students and 18 teachers over a four year period. They outline five talk moves that were beneficial for facilitating academic productive discourse. These talk moves include asking students to repeat or to revoice what others have said, to add on and contribute more to an idea, to agree or disagree with the ideas shared and provide reasoning for their position, and to allow wait time for students to think about their reasoning. Through the facilitation of discussion amongst students in a classroom students learned to reflect on mathematical concepts, strategies, and language. They conclude that engaging in talk supported student learning. Similarly, Sfard and Kieran (2001) studied the effectiveness of communication between two students (aged 12 years) learning mathematics. The students were observed over 2 months while they interacted and discussed algebra concepts. They concluded that talking about mathematics is a powerful way to enhance mathematic learning, however the skills of communication must be taught for it to be most effective. Stein et al. (2008) suggest a series of five practices teachers can follow to prepare themselves to facilitate discussions when presenting challenging tasks to be solved by students in mathematics communities. These practices are, anticipating students' solutions before introducing the task, monitoring while students work together, selecting and sequencing student solutions to be explained and justified to the class and making connections to support understanding of mathematical concepts. The more comprehensively the teachers have anticipated student solutions and misconceptions the more prepared they will be when monitoring students during group work and discussion. The following section will elaborate specific teacher actions effective teachers of mathematics make while monitoring students solving mathematics tasks.

2.3.6 Responsive teaching actions

While students are working through their solutions to a mathematical task teachers make in the moment decisions to respond to what they notice. Student learning can be influenced by these responses. It is therefore important for teachers to take the most appropriate action to elicit, clarify and extend student reasoning. Questioning, probing, scaffolding and providing praise and feedback are all actions teachers may take in response to what they notice while listening to and monitoring student reasoning. Research by Jacobs and Empson (2016) studied responsive teaching actions in mathematics lessons. Their seven-teacher case study analysed the actions of highly skilled teachers. In this study the focus was on one teacher’s responses to student mathematical thinking that advanced student understanding of fractions (25 students aged 9-11 years). They created a framework with four categories of teaching moves. The categories included making sure that the child is making sense of the story problem, exploring details of the child’s strategy for solving the problem, encouraging the child to consider other strategies, and connecting student thinking with symbolic notation. They found that the second category highlighted the importance of teachers understanding students existing thinking before building on from it. This category involved the teacher asking questions to the student such as, “Can you tell me what you did?” to elicit students explanations of mathematical understanding and then pressing the student for a more in-depth explanation and justification.

Another teacher action within the second category was to link the students representation to the story in the problem. They found that this action supported students to keep track of the problem and the quantities involved in the solution. Research about other responsive teaching actions are outlined below.

2.3.6.2 Questioning

One of the key roles of the teacher is to ask questions to enhance students' understanding of mathematical concepts. Researchers have explored the different types of questions asked by teachers in a mathematics classroom (Franke et al., 2009; Hufferd-Ackles et al., 2004; Moschkovich, 2015). Sahin and Kulm (2008) investigated how frequently two teachers used probing, factual, and guiding questions in a mathematics class of 11–12-year-old students in the United States. They found that despite using a reform-based curriculum designed to encourage student discussion and reasoning, teachers predominantly asked factual questions. Hufferd-Ackles et al. (2004) describe this type of questioning as the lowest level in their framework for a math-talk community. Their year-long case study tracked the development of one teacher of students aged 7-8 years in the United States as they worked to build a discourse-focused learning community. The framework outlines four levels of questioning:

1. Level 0: Teacher asks short frequent questions.
2. Level 1: Teacher questions begin to focus on student thinking.
3. Level 2: Teacher asks open questions and more probing questions.
4. Level 3: Teacher expects students to ask one another questions

They found that questioning to reveal student thinking is a crucial skill for teachers, as it supports the development of students' mathematical understandings. Similarly, Franke et al. (2009) conducted a study examining the impact of teacher questioning on student outcomes. They analysed teacher questioning in three mathematics classes with 7–9-year-old students in the United States. The study found that probing questions help students clarify their thinking, develop stronger justifications, and enable other students to connect with the ideas being discussed. A sequence of probing questions was shown to effectively improve student explanations and learning outcomes. Specifically, the benefits include teachers making more informed decisions about instruction, the student being asked the probing questions gaining clarity in their own thinking, and other students in the class being able to connect their own thoughts to the discussion and correct any misconceptions. The study highlights the significant influence teachers have on student learning through their questioning techniques.

2.3.6.3 Feedback

Teacher reactions to student participation and their evolving mathematical understanding can positively influence student learning outcomes when these reactions are responsive and purposeful (Averill, 2012). Stovner et al. (2021) investigated the types of feedback 47 teachers in Norway provided students (13-14 years old). Their study involved analysis of 172 video recorded lessons and specifically focused on the feedback quality and the reasons for the feedback. One aim of the study was to identify how often teachers give feedback on procedures, concepts and mathematical practices. They concluded that providing conceptual feedback is essential for advancing students' conceptual understanding and engaging them in mathematical practices. They found that substantive feedback addresses students' mathematical challenges when solving complex tasks when greater clarity is necessary. In contrast, procedural feedback is aimed at guiding students through specific procedures, which reduces their cognitive struggle. Overall, teachers use of the scaffolding technique, such as giving feedback to students while they are working on mathematical tasks, plays an important role in shaping interactions among all members of the learning community and conceptual understanding.

2.4 Summary

This chapter examined the research literature on teaching and learning rational number concepts. The essential components of the learning trajectory were explored including the key mathematical big ideas of rational number and the prevalent misconceptions that student encounter. The significance of visual representations and effective pedagogy was highlighted as crucial elements for developing a robust understanding of rational numbers. Additionally, the establishment of a positive classroom community, along with the implementation of supportive norms and developing mathematical practices, is vital for enhancing student engagement and learning outcomes. Through thoughtful task design, meaningful discussion, and responsive teaching actions teachers can create an environment that nurtures mathematical reasoning and builds student confidence.

Chapter Three: Methodology

3.1 Introduction

The previous chapter discussed the literature related to the current study. This chapter outlines the research design and methods used in the study. Section 3.2 presents justification for the use of a qualitative, case study methodology. Section 3.3 outlines the data collection methods, including interviews, observations and classroom artefacts. Section 3.4 explores the research setting and participants. Section 3.5 discusses the role of the researcher. Section 3.6 details the research schedule. Section 3.7 describes the data analysis used in the study. Finally, Section 3.8 summarises how ethical standards were upheld throughout the study.

3.2 Justification for the methodology

A qualitative inquiry with a case study research design was chosen for this research project. The goal was to provide insight into how a teacher's pedagogical actions impact students' understanding of rational numbers. This aim influenced the choice of methodology, as the research was conducted in the natural classroom setting where daily mathematics teaching and learning occur.

3.2.1 Qualitative research

Qualitative research is an effective and powerful way to conduct educational research because it is about depth rather than breadth of a phenomenon (Neuman, 2014). Qualitative researchers collect data in a natural setting and undertake data analysis that is inductive and establishes patterns or themes (Creswell & Poth, 2016). Miller and Brewer (2003) outline the qualitative approach to research as one that stresses 'quality' over 'quantity' where social meanings are derived and interpreted. Qualitative researchers are interested in participant's actions and experiences within their environment. They want to understand the phenomena and make meaning by interpreting the lived realities of the participants in a natural social setting (Saracho, 2017).

In this study, the interactions between the teacher and students in the classoom setting allow for qualitative methods. A qualitative approach to data collection enables the researcher to delve into the nuanced pedagogical approaches and instructional strategies employed by the teacher providing a comprehensive understanding of how these factors shape student learning experiences.

3.2.2 Case study

A case study is an in-depth description and analysis of a bounded system within a real-life context (Gerring, 2006; Merriam, 2016). This research employs a case study approach because the phenomenon being researched is studied in its natural context, bounded by space and time (Hancock, 2017; Yin, 2018). This study is grounded in the mathematics classroom for the duration of the rational number unit of teaching and learning. The unit of analysis, not the topic of investigation, determines a case study. In this study, concentrating on a single teacher (the case), with one classroom (the bounded system) is the unit of analysis (Merriam, 2016)

The current study lends itself to in-depth analysis in a natural context using multiple sources of information to ensure researcher bias is limited (Oancea & Punch, 2014; Yin, 2018). This research design can be classified as an intrinsic case study because the case presents a unique situation (Creswell & Poth, 2016). In this case, the researcher is inquiring into how a teacher uses pedagogical approaches to support student understanding of rational number concepts. Intrinsic case study researchers do not seek to generalise their findings beyond the case but rather to understand the complexity of the case (Creswell & Poth, 2016; Hancock, 2017).

3.3 Data collection

An in-depth case study requires data from multiple sources (Bright et al., 1988). More than one method of data collection enhances the validity of the findings (Merriam, 2016).
The data collection tools used in this study include a semi-structured interview, video recorded lessons from within the classroom, and written documents containing student representations. All data collection methods are discussed in detail on the following sections.

3.3.1 Semi-structured interview

Semi-structured interviews are the most common type of interview in educational research (Merriam, 2016; Oancea & Punch, 2014). Longhurst (2003) describes a semi-structured interview as a verbal interchange where one person attempts to elicit information from another person by asking some questions that are predetermined and allowing the interview to unfold in a conversational manner. As the interview progresses the participants can explore issues that they feel are important. One semi-structured interview was conducted with the teacher participant before they began teaching the rational number unit at the centre of this case study. The semi-structured interview structure allowed the interviewer to adapt questions and respond to the participant's ideas as the interview progressed. The interview lasted approximately one hour and took place online using the communication platform, Zoom. It was conducted in a conversational manner allowing respectful interactions. The questions focused on the teachers beliefs and pedagogical choices in teaching mathematics allowing space for them to elaborate and provide further examples of teaching experiences (see Appendix A). The interview was video-recorded and wholly transcribed for coding and analysis.

3.3.2 Observational data

Observations are crucial in qualitative research as they allow the researcher to witness and hear participants' interactions and behaviours as they unfold in their natural setting (Oancea & Punch, 2014). In a classroom environment, the recording of lessons allows for the capture of subtle interactions, verbal cues, and non-verbal communication that influence student comprehension of the abstract concepts of rational numbers (Wang & Lien, 2013).

In the current study, seven lessons were video-recorded to capture a detailed view of classroom teaching and learning. The teacher was provided with an iPad and used an iOS application called Swivl. The iPad was mounted on a Swivl stand that followed and tracked the teacher as she moved around the classroom. Additionally, three sound recording devices were used: one was worn by the teacher on a lanyard to capture all her spoken words, while the other two devices were placed randomly at student tables to recoprd their conversations during the maths tasks. These recordings could be isolated to allow the researcher to focus on specific conversations during analysis. At the end of each week, the video recordings were downloaded and fully transcribed for coding and analysis.

3.3.3 Classroom artefacts

At the end of the study, the researcher collected the student group workbooks. These workbooks containnrecords of students' representations and responses to the rational number tasks they solved collaboratively during the lessons. For each lesson, student groups were assigned one of the class workbooks and recorded their names next to a paper copy of the task that was glued onto the page. This process helped the researcher to match student representations with the discussions captured on camera. The workbooks provided concrete evidence of the students' problem-solving methods and the types of representations they used to support their understanding.

3.4 Participants and research setting

The research was conducted with a single teacher and her mathematics class of 11-13 year olds, in an urban primary school in New Zealand. The students at this school represent a range of ethnicities and mainly come from middle socio-economic home environments. The project spanned a duration of five weeks during term three of the school year to align with the rational number unit of teaching and learning from their long term planning overview.

Purposive sampling was used in this study. The case study teacher was selected because she was involved with a mathematics professional development project called Developing Mathematical Inquiry Communities (DMIC). DMIC is a research based approach to teaching mathematics that requires teachers to become facilitators of rich mathematical discussion and to build on from student thinking to advance the understanding of all students in mathematics (Hunter et al., 2018). The case study teacher was using rich, culturally sustainable and contextually relevant mathematical tasks provided by the DMIC professional development team that were written to meet the achievement objectives at level 4 of the New Zealand curriculum.

The students involved in the study were assigned to the teachers' mathematics class from the wider student cohort as organised by the school. Students worked in collaborative groups of 4 with a maximum of 24 students in each lesson. Permission from students and their guardians was sought and 37 agreed to participate in the study. The students contribution to the research was agreeing to be visual and sound recorded while participating in their mathematics lessons for the duration of the study.

3.4.1 Classroom context

The mathematics lessons followed a consistent structure in all eight lessons of the case study. The teacher divided the class of 24 students into two groups for these lessons. Students attended a teacher-facilitated workshop twice a week and worked independently on follow-up tasks involving rational number on the other two days. Throughout the week, all students had the same learning opportunities and worked on the same challenging tasks. In workshops, the teacher assigned the students to work in small groups, either in pairs or groups of three. Each group shared a pen and workbook to visually represent their thinking as they worked together on a mathematical problem. In the workshop, students were seated at tables near the whiteboard, which both they and the teacher used to record various representations during the lesson. Students working independently were seated on the opposite side of the room to minimise distractions and maintain a quiet environment. The observations in this case study focused specifically on the teacher-facilitated workshops.

3.5 Researcher role

In this study, the researcher served as the sole collector of data. In qualitative case studies the researcher is the main instrument in the project (Merriam, 2016). Oancea and Punch (2014) describe qualitative research designs as frequently non-interventionist. In this project the role of the researcher was etic, meaning that the data collection was conducted through an outsider perspective. This enabled the researcher to be an objective viewer of teacher pedagogical actions as well as classroom interactions.

The relationship between the researcher and the teacher participant was professional and trust based. At the outset of the research project the researcher outlined the objectivity with the research to the teacher. It was explained that to support the study aims the research would not be a judgement or an evaluation of their teaching practice.

The previous mentor-mentee relationship between the teacher and the researcher could potentially have created some status issues in the research. There is the potential that the teachers views, opinions and actions were swayed by my role as a mathematics mentor and they may not reflect their true feelings. Self-concious behaviour of the teacher may have been reflected in more formalised behaviour than usual when they were being recorded. Responses to interview questions may have been what they think the researcher was wanting to hear rather than their true opinion. As a qualitative researcher, it was important to ensure the research practice was neutral and objective.

3.6 The research study schedule

The investigation was conducted over approximately five weeks (September – October, 2023) and consisted of two phases of data collection.

3.6.1 Preliminary phase

Before collecting data, the researcher held several meetings with the teacher participant. Initally, the researcher approached the teacher to explain the research focus, describe what participation would entail, and gauge interest in the project. The school principal, as the relevant gatekeeper, was also consulted and permission was obtained to conduct the research at the school. The teacher was provided information sheets detailing the students' involvement in the project and voluntary informed consent forms for the students and their guardians to review before agreeing to participate. The further three phases are outlined below (see Table 3).

Table 3

Phases of research

Phase One	Semi-structured interview with the teacher participant conducted over Zoom a week prior to the start of the rational number unit of mathematics.
Phase Two	Eight rational number lessons were video-recorded.
Phase Three	Retrospective analysis of the data.

3.7 Data analysis

Analysing data is a process of making meaning from the data (Merriam, 2016). The analysis of the data in this study involved interpreting the teacher's pedagogical strategies at different points in the mathematics lesson, particularly in response to student reasoning. Once the data from interview and video observation transcripts were collected, retrospective and formal analysis began.

3.7.1 Coding and developing themes

Reflexive thematic analysis was the method used in this study to develop, analyse and interpret patterns across the qualitatve dataset (Braun & Clarke, 2022). Data analysis in qualitative research follows a general process beginning with organising the data for analysis then reducing the data into themes through a process of coding and then further condensing the codes for discussion (Boaler, 2002; Oancea & Punch, 2014). Qualitative researchers build their patterns and themes by organising the data into increasingly more abstract categories over time. This inductive process involves constant review until a comprehensive set of themes is established (Creswell & Poth, 2016).

The systematic analysis of the data began with the process of coding. Initially the data were coded with respect to specific teacher actions, student responses and classroom norms. Codes were a combination of interpretative and descriptive. In the second phase the analysis shifted from codes to themes. Braun and Clarke (2022) describe the process of generating themes as clustering the codes into some groups connected by similarity of meaning. The next phase involved reviewing the emerging themes in relation to the coded data to ensure they addressed the research questions. As a result, the following themes were identified: attending to classroom community; supporting conceptual understanding; promoting mathematical reasoning; and fostering deep mathematical thinking.

3.7.2 Validity and reliability

All research studies need to be valid and reliable. Data collection methods are important in showing validity and reliability of the findings (Oancea & Punch, 2014; Yin, 2018). The collected data from the interview, lesson observations, transcripts and student workbooks were triangulated to ensure the validity of the project and to verify findings (Merriam, 2016). This provided a guarantee that the data was interpreted with clarity and rationality.

3.8 Ethical considerations

This research project was designed and conducted in accordance with the Massey University Code of Ethical Conduct for Research, Teaching and Evaluations Involving Human Participants (Massey University, 2017). The study prioritised participant well-being and ethical consideration through informed consent, confidentiality and data privacy. The potential vulnerability of the participants was considered and repected by the researcher. To guarantee the complete anonymity of participants, all video recordings were stored in the password protected Swivl online cloud to ensure their identities were not compromised.

Written consent was obtained from all participants, including the teacher, school principal, Board of Trustees, students, and their parents and guardians (see Appendices C1-C4). Consent was sought and obtained from parents of the students because they were under the age of fifteen years old. Alongside the consent form, a detailed information sheet was provided that included clear details of the research purpose, what was involved in the research process and what they were consenting to (see Appendices D1-D3). Efforts were made to ensure that no identifying information about the school or participants was included in any written reports. The teacher, the students and their guardians were informed that they had the right to withdraw from the research at any time. The risk of harm to the students was minimised as the study was conducted within their own learning environment during regular mathematics lessons.

3.9 Summary

This chapter has outlined the qualitative research design as the most suitable approach for this study, with a particular focus on conducting a single case study. Included is the rationale for selecting qualitative methods of data collection and analysis. A variety of methods of data collection were used, including semi-structured interview and classroom video recordings. Data were analysed using thematic analysis, which involved identifying codes and developing themes through an iterative process. The triangulation of data supported the integrity of the interpretations, and conduction the research in an ethical manner ensured the validity and reliability of the results. The findings and discussion of the study are presented in Chapter Four.

Chapter Four: Findings and Discussion

4.1 Introduction

The previous chapter outlined the design and methods used in this case study. This chapter presents the study's findings related to how teachers help students develop an understanding of rational numbers and how students communicate their mathematical understanding. Section 4.2 discusses the impact of contextual mathematics tasks on students' understanding of rational numbers. Section 4.3 highlights the visual representations used by teachers to enhance student understanding and reasoning about rational numbers. Section 4.4 explores the connections the teacher makes to the big mathematical ideas. Finally, Section 4.5 outlines specific actions by teachers to develop student mathematical reasoning.

4.2 Contextual mathematics tasks

Using mathematical language is crucial for demonstrating understanding of mathematical concepts. Specific terminology helps students communicate reasoning clearly and allows teachers to assess students' comprehension effectively. Abstract mathematical concepts can be made accessible by incorporating real-world contextual mathematical tasks into lessons and then using these contexts to establish mathematical terminology. The focus of this section is to present and analyse how mathematical langauge was used within the current study through the use of contextual tasks, and how this supported student reasoning with rational numbers.

At the beginning of the study, the teacher, in the interview, outlined her intention to use mathematics tasks that were contextually relevant for her students:

> *I try to use some real life situations that they can relate to. If it is not a situation that is familiar, or uses language that they've never heard of, it's really hard for them to get started. They have lots of questions like, What is that? What is this? But I think that it just makes sense if it's a situation that they can relate to, particularly if it's something around our school or in our community.*

In the excerpt, the teacher has stated the importance of designing tasks that are situated in familiar contexts for the students. By incorporating rational numbers in contexts that they already knew, she explained that students could more easily access the problems, as they could relate to their own experiences. This includes attention to language. She acknowledged the importance of using language the students are familiar with to avoid confusion. This is similar to research by Silver et al. (1993) that focused on division with whole numbers. These researchers concluded that the application of real-world knowledge was relevant to the study of rational numbers and that when students connect their computational results to the context of the problem they are more likely to achieve success. In the current study, the teacher intentionally presented mathematical tasks that were grounded in relatable contexts for the students. This choice reflected the teachers' belief that connecting mathematics to familiar school or community scenarios would facilitate deeper engagement and understanding. The teacher aimed to foster a more meaningful learning experience.

In the first lesson of the study, the teacher presented the following mathematical task involving the familiar context of a download bar on a computer (see Appendix B1). Before introducing the task, the teacher positioned the students on the floor around a long piece of tape she had stuck down. The tape represented a download bar. The teacher introduced the task and engaged the students in a discussion about the context of the download bar. This is illustrated in the following episode:

> *Teacher: Today we're going to pretend this is you are downloading a game, and this is the download bar. Do you know what I mean by download bar?*
>
> *Students: Yes.*
>
> *Teacher: So what do you notice? When you're like when you're downloading a game, what do you notice happens, is, or maybe you're downloading or your opening your game? Something's loading. So what do you notice happens?*
>
> *Georgie: It goes like it, the colour comes in the lines. There might be. Like colour and when it gets further, it changes.*
>
> *Teacher: It does. So what does that colour change represent?*
>
> *Jackson: How far away it is from done?*
>
> *Teacher: OK, cool. How far away it is from done or being fully downloaded or fully complete? Anything else to add?*
>
> *Emma: It usually goes up in percentages.*
>
> *Teacher: Okay, it usually goes up in percentages. What is the maximum percentage?*
>
> *Pippa: One hundred percent.*
>
> *Teacher: So, what does it. 100% represent. When you're downloading something?*

Lleyton: The whole.

This extract highlighted the teacher activating the students' prior knowledge about download bars. She used an open-ended question (what do you notice?) as a prompt. The teacher used the talk move of repeat to emphasise each student's response, before extending the prompt with another question. While the pattern of interaction appears to be student-teacher-student, this pattern funnelled the students towards further discussion.

Following the above exchange, the students were further prompted to interpret different percentages in terms of fractions. One student stated, "It's 33.333% - that's one third ", while another student stated, "I know 50% is equivalent to a half of the bar". These excerpts outlined how a task that was familiar to students supported them to develop an understanding of equivalency between fractions and percentages. The download bar was familiar to all students and this tangible concept from known experiences allowed them to engage in discussion to deepen their understanding of rational number equivalence. Earlier research by and Case (1999) used a 'number ribbon' to indicate progress of a file transfer on a Macintosh computer. Similar to the current study, the context and visual representation were familiar and contributed to building a strong connection between proportions and numbers for students to build secure rational number concepts.

As the lesson progressed and students were asked to explain and justify the proportion that had been downloaded at a specific marker on the line indicated by the teacher. The teacher provided wait time before asking the students to share their reasoning to the wider group. This is illustrated in the following episode where the teacher facilitated the large group discussion:

Teacher: What percentage of the game has been downloaded now?

James: Seventy-five.

Teacher: Seventy-five. What justifications can you make to prove its 75%?

James: Because it's not in the middle and it's not like really close to the end.

Teacher: Okay. You're saying there's a bit of space. Why is it important? Why did you say that it's not in the middle?

James: Because the middles about like here, half of the bar.

Teacher: Yeah, yeah. And what would it be if I put it in the middle?

Nick: 50%.

Teacher: It seems about 50%, OK, you're saying. Looks like a bit more, yeah. Georgie?

Georgie: I think. If you were to go like that with four of them and 25% taken away from the whole would be 75%. That’s the same as three quarters done.

Teacher: OK, so you notice that the smaller part here you think that's about 25%. Well done. So, at the end here we know it's 100%. You take away that 25%. We’re left with 75% here. Any other ideas to add to that?

In this episode, several teacher actions supported the students. Firstly, when the students said 75, the teacher repeated and then pressed for justification for why it would be 75 percent of the bar. Her actions highlighted the importance of using correct mathematical language and terminology when discussing percentages. Secondly, she pressed for justification, an important mathematical practice identified in findings from Staples et al. (2012). Pressing for justification publicises student thinking and positions student thinking as being valuable within the classroom learning community. This teacher action promoted students conceptual understanding of percentages and supported them to make sense of the proportion of a download being completed. The use of a download bar as a visual representation for making sense of rational numbers aligns with findings from Moss and Case (1999) who found that the download bar provided a unidimensional form of number representation for the students to access proportional reasoning using their whole number knowledge. Evident in the findings of this study is how the students naturally made references to the equivalent fractions which is further evidence of the download bar aiding their understanding of rational numbers.

After the discussion the teacher then recorded the students’ ideas on an equivalence chart. By beginning the unit of work focused on rational number with a task that focused on the concept of percentages on a download bar, the teacher offered students a powerful entry point to developing a deeper understanding of rational numbers. Using different representations and emphasising the relationships across the representations of the fractions, decimals and percentages supported students to develop understanding of equivalent rational number forms.

In this study, the teacher consistently corrected students' use of mathematical language, supporting them to replace it with the appropriate terminology. This is illustrated in the following episode:

> *Teacher: So, we have four tenths. Good explanation. What about our next one, four and what is this? Brogan?*
>
> *Brogan: Four point twenty-six.*
>
> *Teacher: Four point twenty-six?*
>
> *Brogan: Four point two six.*
>
> *Teacher: Or, as a fraction we have 26?*
>
> *James: Over 100.*
>
> *Teacher: Hundredths. Who can come and write this up on the place value chart?*

The teacher firstly questioned the terminology the student used when she said "*four point twenty-six*". Subsequently, the teacher prompted the student to clarify their response, guiding them towards the correct terminology, "four point two six". These actions encouraged precision in the language when reading decimal numbers. The teacher then introduced the correct terminology by referring to 'hundredths' when discussing the fraction (twenty six hundredths). This provided a clear model supporting students to learn the appropriate and specific vocabulary associated with decimal and fractional representations. After this interaction the teacher invited a student to write on the place value chart. This action reinforced the connection between the visual representation and the spoken language and further reinforced the students' understanding of the concepts of rational numbers. The students' incorrect interpretation and subsequent incorrect reading of the decimal number showed a misunderstanding about the density of decimal numbers and was evidence of a whole number bias. Developing student understanding of the density of rational numbers is a first step towards robust rational number knowledge. Supporting students understanding of this concept in these ways aligns with the development patterns for conceptual understanding of rational numbers by Van Hoof et al. (2018). The whole number bias that students hold was evident in this interaction and the teacher addressed this by highlighting the key understanding of decimal place-value. She highlighted this misconception by repeating the answer the student has given, "Four point twenty-six?" and then linked the way the student had read the number to the decimal form. The teacher connected "twenty-six" to the decimal value by directly asking what the denominator would be for the fraction with twenty-six as the numerator. When one of the student answered, “over one hundred”, the teacher revoived with the correct mathematical langauge of “hundredths”. In this way, the teacher was provided clear and explicit ways for students to conceptualize rational number concepts. The strong focus on oral language to promote the development of rational number conceptual understanding aligns with findings from Seethaler et al. (2011). The oral language illustrated in this interaction between the teacher and students faciliatated deeper conceptual understanding. The teacher was supporting the students to develop strong language skills by making explicit the different subconstructs of rational numbers such as the decimal form and the decimal fraction form. The spoken and written language were explored consecutively in the lesson which exposed students to a deeper conceptual understanding of rational numbers.

Group norms foster the conditions in a community of learners where understanding can occur through mathematical discourse and collaboration. Specifically, this teacher facilitated group norm discussions to create the conditions in which understandings about rational numbers could develop. In the interview at the beginning of the case study, the teacher explained that she creates a positive classroom environment to ensure students experience a strong sense of engagement and enjoyment in mathematics. This included taking time to establish important norms such as contributing to group discussions by sharing mathematical ideas, respecting the ideas of others and asking questions of each other. When asked why she believes fostering such an environment is important, she responded:

> *I make it a safe space for everyone so that they are comfortable sharing their ideas and their questions.*

The teacher has identified the importance of the classroom environment being a place where all students are able to freely share ideas and engage in mathematical discussion. This aligns with Anthony et al (2015) who describe ambitious forms of mathematics teaching as requiring all students to be positioned as competent and valued contributors within the learning community. Students feeling comfortable to share their ideas is vital to develop a classroom community with discussion about mathematics at the forefront. This means that the teacher can build on from student thinking and make connections to mathematical understandings from the reasoning that students share out loud. A safe space where all students feel comfortable to share their mathematical ideas is necessary for purposeful learning discussions within a mathematical community.

In each of the seven rational number lesson observations, the teacher consistently engaged students in a discussion about group norms before introducing the task. The teacher emphasised that the group norms were the expected ways for students to collaborate when working on rational number mathematical activity. This is illustrated by the following excerpt from lesson one where the teacher initiated a group norm discussion:

> *Teacher: OK, first of all, group norms. Hopefully you can remember the group norms or the expectations for when we are working in groups, Emily?*
>
> *James: Ask questions.*
>
> *Teacher: Ask questions, why is it important to ask questions? What do you think Maisy?*
>
> *Brogan: In case you're confused.*
>
> *Teacher: If you're confused about something, yeah, why else should we ask questions? Why do I ask you questions sometimes?*
>
> *Lilly: To see if you understand.*
>
> *Teacher: Yeah, it helps you understand things like Maisy said. Why else might I ask you questions? Hopefully the kinds of questions that I ask, make you think a little bit differently and to kind of extend your thinking or broaden you're thinking about something that we're doing.*

In the episode above, the students agreed that asking questions is an important group norm. The teacher pressed the students to justify the importance of this group norm. In addition, the teacher utilised talk moves, repeating what Maisy had stated, to emphasise this further. By asking the students to consider why she had asked questions, the teacher positioned herself as part of the group – in other words, any member of the group, including the teacher should ask questions if necessary. This aligns with Chapin and O'Connor (2007) who concluded that participating in academically productive talk provides greater opportunities for students to grasp mathematical concepts. The teacher embraced the students' suggestion that asking questions is an important group norm and explanded the discussion by prompting students to consider the purpose behind asking questions. She also encouraged them to explain why this practice is important and shared her own reasons for asking questions during mathematics lessons. Additionally, the teacher focused on establishing norms around active listening, leading the class to develop a shared understanding of this expectation. In this deliberate conversation, the teacher set clear guidelines for effective group work, acknowledging student input to clarify expectations.

The following episode is evidence of how the established group norms enabled students to explain and develop their understanding of decimals:

> *Georgie: What does that mean 0.53?*
>
> *Laura: It's just it's like a little bit over point five. Instead of 50%, its 53 percent, 53%.*
>
> *Georgie: How do you know its 53%?*
>
> *Laura: Because .53 is 53%.*
>
> *Teacher: Good use of language there. My question is can we have numbers between numbers?*
>
> *Students: Yes.*
>
> *Teacher: Why? How do you know that?*
>
> *Laura: There can be decimals.*
>
> *Georgie: You have a number, and it could be a little bit more.*

In this episode students participated in a mathematical discussion, during which one student asked questions to clarify the concepts of decimals and percentages. The teacher listened to the exchange between students and praised the mathematical language she heard. Praising the use of mathematical language corresponds with findings by Selling (2016b) specifically because it is naming the mathematical practice. In doing so publicly, the action emphasised to all students in the lesson that using mathematical language was a practice that is encouraged. The teacher then asked a probing question to move the discussion further. By asking students to consider if there are numbers between numbers, the teacher prompted students to clarify their own ideas about the number system. This teacher action aligns with the questioning framework developed by Hufferd-Ackles et al. (2004) where the teacher asks open ended questions and more probing questions. Further to this, when the teacher asked students to consider how they know, she encouraged students to think deeply about their understanding and to develop stronger explanations and justifications.

While nurturing the use of mathematical language was a deliberate teaching action to enhance student mathematical understanding, group norms and mathematical tasks supported this. The teacher also emphasised the importance of using appropriate mathematical representations to communicate their understanding of rational numbers as discussed in the next section.

4.3 Visual representations

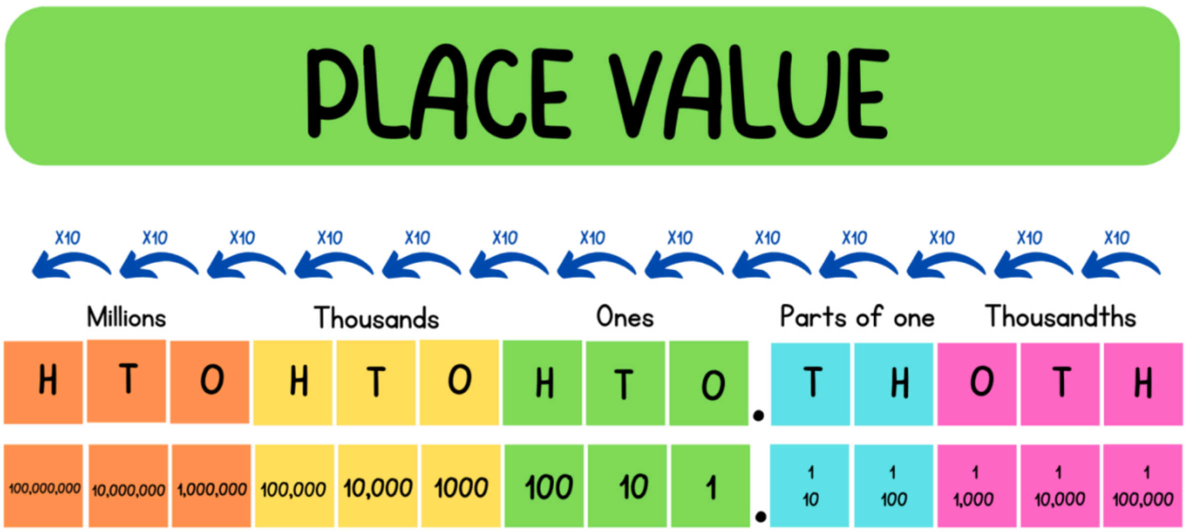
Using visual models and concrete representations is vital for mathematical understandings to develop. Connections can be made between the natural number system and the abstract nature of rational numbers through carefully selected visual representations. In this section, data will be presented to address how teacher actions utilising two kinds of representations supported student understanding and how the students demonstrated their understanding of rational numbers with the representations. Firstly, the use of visual representation wall charts and secondly, the use of an empty number line.

4.3.1 Wall charts

In the classroom, the teacher had two visual representation charts on display throughout the study. One was a place value chart printed on a large piece of paper attached to the whiteboard at the front of the room (see Figure 1).

Figure 1

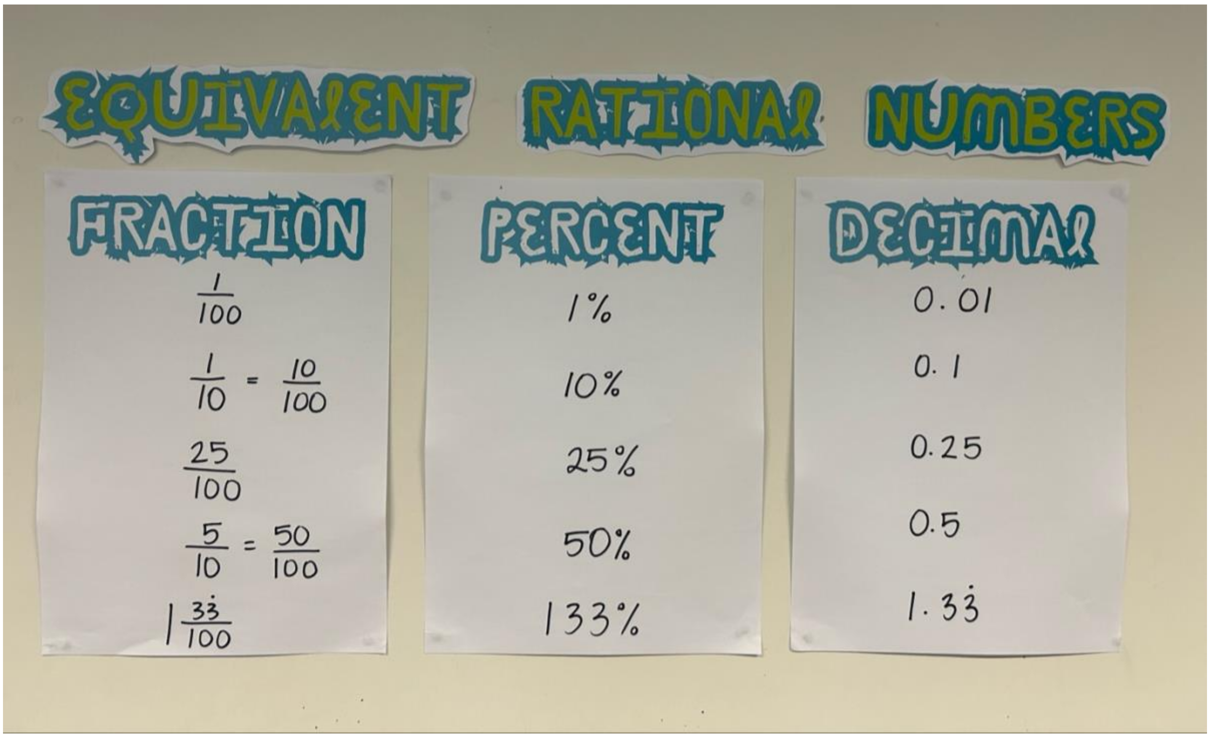
Place Value Chart



This visual representation was added to by the teacher during discussions about decimal numbers and sometimes by the students when they were asked to contribute to the chart by the teacher. It provided a reference point for the students as they engaged in discussion and solved rational number mathematical tasks. The other visual representation that was on display and added to throughout the study was a rational number equivalence chart made up of three columns with the headings, fraction, decimal and percentage (see Figure 2).

Figure 2

Equivalence Chart



The teacher often directed students to the chart when they were engaged in discussions about rational numbers as evident in the following interaction:

> *Teacher: How do you know 50% of 100 is equal to fifty?*
>
> *Lilly: Because fifty percent is the same as one half.*
>
> *Teacher: Fifty percent is equivalent to one half, good, which is on our equivalence chart here.*

The equivalence chart emphasised the decimal-fraction notation of rational numbers. Using this tool supports what Kieren (1976) describes as a critical aspect of learning related to rational numbers, generalisation of the symbols for decimal and fraction representation.

Benchmarks values were also predominantly recorded on the displayed equivalence chart. Displaying benchmark equivalence notations on a co-constructed chart for the duration of the study aligns with one of the effective teaching actions reported by Moss and Case (1999). The chart provided an anchor point for students as they worked on rational number tasks and reinforced the multiple forms that represent the same rational number. The different forms of representation permitted students to think flexibly when solving tasks.

4.3.2 Empty number lines

Empty number lines represent continuous measurement and the quantity of rational numbers. They provide a clear model for students to visualise the position of a rational number in relation to other numbers. Number lines can also support students to visualise the addition and subtraction of rational numbers, for example, moving to the right when adding and the left when subtracting. In the second and third lesson observations, the teacher regularly encouraged her students to use number lines when solving the mathematical task (see Appendix B2) involving addition of decimals and praised groups that represented their mathematical solutions on a number line.

In each lesson, the students were expected to explain and justify their group solution and mathematical reasoning to the class. In most lessons the teacher paused the groups approximately 10 minutes after they had begun to work on a task and asked students to explain how they had started to solve the task and their initial approach to solving it. This is demonstrated in the following excerpt:

> *Teacher: Alex's group. Explain to us what you have started doing first.*
>
> *Archie: We are putting everything on the number line.*
>
> *Teacher: Ok, why are you putting everything on a number line?*
>
> *Lilly: Because then you can see who came first, second and third at this point. And then you add the scores together.*
>
> *Teacher: So how does the number line help you see that?*
>
> *Lilly: I just find it easier.*

Teacher: You just find it easier. It's a good visual tool, isn't it? To lay it out. OK, so you said that you've started by adding the numbers together. Can you now explain how you started adding the numbers together?

In this interaction, the teacher began by asking a group to explain how they had started to solve the problem. The students explained they had used a number line. The teacher then pressed for justification for why they had selected a number line for their initial solution strategy. By pressing for justification, the teacher actions made the small group work public. By making the work public, other students in the classroom community were provided with the opportunities to think about explanation and justification for using a number line as a visual model to represent the numbers. When Lilly explained that she found it easier to use a number line, the teacher explicitly reinforced that a number line was a good visual tool. These actions highlighted to other students in the class that a number line was a valid means to visually represent their thinking too. In addition to drawing on the number line as a valid representation of mathematical reasoning, the teacher actions highlighted the use of socio-mathematical norms. Enacting socio-mathematical norms supported students to work towards reasoning with rational numbers through deepening explanation and justification. This aligns with the study by Yackel and Cobb (1996), which emphasised how teachers of 7- and 8-year-olds were supported to reason mathematically by explaining and justifying their thinking. Although the students in this case study are older, the vignette demonstrates the teacher's effectiveness in initiating and guiding the use of socio-mathematical norms within the classroom.

Later in the lesson, the teacher purposefully selected one group, who had added the decimals by breaking them down into place value and then added each place at a time, to share their solution strategy with the class. In this example, the students firstly added the ones, then the numbers in the places after the decimal point. To deepen student understanding, the teacher followed this explanation with an explicit demonstration of how to transfer their solution to a number line representation. She modelled this on the whiteboard next to the groups solution which had been written in various equations across the board:

Teacher: I'll show you how this can be transferred onto a number line, so we get like a visual representation. So, what has Ollie done? What numbers has he started adding first? Anna?

Anna: The smallest numbers.

Teacher: The smallest numbers? The first numbers which also known as the?

Anna: Ones.

Teacher: The ones. The ones good. So, they sit in our one's place which is before the decimal point. OK, so we've got 8.903, 7.0001 and 8.987. We know that we are going to add the whole numbers first. I'm just going to write those up here 8 + 7 + 8 = 23. OK, so we've added the whole numbers first and we've got 23 and then I'm going to look at the rest of my numbers like Ollie has. We are going to take the highest number beyond after the decimal and add that to the 23. So now we'll have 23.987. What do we have left to add on to 23.987? In our place value system, we've got ones we've got the tenths, the hundredths and the thousandths. And then after that, ten thousandths. So, what parts do we need to add on now?

Ted: The ten thousandths.

Teacher: OK, so if we add 0.0001 onto this number, what will that now become?

Ted: 23.9871.

In the above episode the teacher explicitly connected a student representation with a different visual model. Firstly, by repeating the steps that the students took, the teacher supported all students to make sense of the groups' mathematical reasoning. Students were carefully oriented to the ideas in the student generated solution strategy. In doing so, the teacher highlighted the mathematical concept of rational number place value. This aligns with the findings of Cordero-Siy and Ghousseini (2022) who found that effective teachers purposefully select student representations to address specific mathematical goals. The mathematical goal the teacher focused on in this lesson was to extend the thinking that students shared and extend this with an appropriate visual representation that clearly shows the addition of rational numbers. The teacher-led discussion prompted students to carefully consider the place names and the value of the digits in the numbers further reinforcing rational number understanding. Furthermore, the teacher consistently used correct mathematical language and verbalised the equations as she wrote them up on the board for all students to see a visual representation. The teacher used a student representation as a springboard and explicitly transferred the reasoning to an unfamiliar representation. This teacher action also aligns with findings by Selling (2016a) who found that student reasoning is reinforced with representations. The students explained and gave justifications for their own representation and the teacher then deliberately modelled how to represent the students' reasoning on a number line. The teacher highlighted multiple representations to further student understanding and support their sense making of rational numbers.

The teacher continued to lead the class in co-constructing the representation of the equations on the number line. As she facilitated the discussion, the teacher drew a small jump every time a part of the number was added on and recorded the number that was added on above the curved line of the jump to show the magnitude of the jump, and the total number underneath the line. By connecting student reasoning to a visual model, the teacher built proactively on students' contributions including students' understanding from the previous explanation and using a number line to clarify the concept. This teaching action corresponds with findings of Jacobs and Empson (2016) who found that connecting children's thinking to symbolic notation is important when supporting children to build mathematical meaning.

Additionally, the above excerpt is an example of responsive teaching. By providing a visual representation to support students in adding decimals, the teacher advanced their mathematical thinking. She listened to student reasoning and chose to connect their understanding of operating with decimal numbers to both concrete representations and symbolic notation. This decision reflects her thoughtful and considered approach to guiding their learning and effectively consolidating the lesson. This teaching episode also connects with Sfard (2001) who states that the number line is a tool that supports communication of mathematical understanding. The teacher was intentional with her modelling of the representation to support and extend her students' learning about rational numbers. Furthermore, making connections between student thinking and key mathematical ideas aligns with Stein et al. (2008). The teacher facilitated a discussion at the conclusion of the lesson to directly connect student thinking with the visual model of a number line and the big mathematical understandings of adding decimals.

While maintaining a focus on visual representations was a conscious teaching action to support student mathematical understandings, the teacher also made connections to the big mathematical ideas of rational numbers to deepen student understanding.

4.4 Big mathematical ideas

Understanding mathematics involves making sense of key mathematical concepts and overarching ideas that connect various mathematical principles. This needs to go beyond rote memorisation of formulas and procedures; it requires students to see relationships between different mathematical situations and to recognise the relevance of these ideas in real-world contexts. Deeper understanding supports students to communicate mathematical understanding effectively. The research questions about teacher actions and students' communication of rational understanding will be answered within this section of the findings and discussion.

In lesson four, the students completed a task (see Appendix B3) about exchange rates. In this task students were required to convert New Zealand dollars to other currencies when travelling abroad. The task was divided into two parts. In the first part, with the exchange rate set at 0.9301 Australian dollars for every $1 New Zealand dollar, students calculated how much they would receive if they exchanged $10, $100 and $550 New Zealand dollars. In the second part, the exchange rate changed to 1.5204 Samoan tala for each $1 New Zealand dollar. Students were required to calculate the equivalent amounts in Samoan tala for the same amounts of $10, $100, and $550 New Zealand dollars. This real-world context engaged students in practical applications of decimal numbers. When the students worked on the task in their small groups, they consistently referred to dollars. By using the familiar context of money, the students were supported to enhance their understanding. This familiarity allowed them to focus on the process of multiplication without becoming overwhelmed by abstract numbers. In student conversations, they actively questioned one another about the currency they were discussing. For example, one student asked, *"Is that number for the New Zealand or Australian dollars?"* This focus on money as a context supported the students to make sense of the decimal numbers, reinforcing their understanding when working with decimals.

By incorporating conceptual tasks, the teacher effectively facilitated deeper understanding of decimals through relatable scenarios. Using contextual tasks as a means to support students to improve competence with decimals correlates with Irwin's (2001) study. In her research, students of a similar age worked collaboratively to solve contextual tasks that incorporated their everyday knowledge of decimals. She found that all students who worked on contextualised problems improved their understanding of decimals more than those who worked on tasks without a context. The reference to everyday concepts can advance student understanding of decimals. Therefore, using real-world contexts not only engages students but also deepens their mathematical understanding of decimals.

4.4.1 Using misconceptions to develop understanding

Teacher noticing and addressing student misconceptions about rational numbers is necessary to develop student understanding. In the following excerpt, the teacher noticed and responded to a group discussing their solution to a rational number task involving multiplying decimals. She overheard a misconception verbalised by one of the students about the decimal point moving when a decimal number is multiplied by a multiple of ten. This resulted in the following discussion:

> *Alex: Move the decimal point to there and then add an extra 0 at the end.*
>
> *Teacher: Do we actually move the decimal point?*
>
> *Ted: No, you don't move the decimal point, you move the number.*
>
> *Teacher: Good. Why?*
>
> *Laura: The decimal point is set in concrete. Its set in concrete.*
>
> *Teacher: So, what's the part that's moving?*
>
> *Laura: The numbers.*
>
> *Teacher: Where are they moving?*
>
> *Laura: It shows that the number is getting larger and larger or smaller and smaller. If it's going that way, to the left, then it's growing 10 times larger. If it's going to the right, then it goes 10 times smaller.*
>
> *Teacher: Amazing. So, what is that called?*
>
> *Ted: Powers of ten.*
>
> *Teacher: Powers of ten, and that's exactly as you explained Sally.*

The teacher actively listened during the group discussion and identified a student misconception. In response, she asked a probing question that transformed into a teachable moment, helping to deepen student understanding of rational numbers. By asking one question, "do we actually move the decimal point?", other students in the group were offered the opportunity to share their correct understanding about the decimal point staying in place, therefore correcting the misconception shared by the first student. The teacher did not correct the student directly. Instead, she opened up the discussion for other students to address the misconception. The teacher action of asking probing questions aligns with the findings from Franke et al. (2009) because it clarified student thinking and corrected the misconception.

The resulting discussion highlighted the big mathematical idea that our number system is based on ten and that the decimal system is an extension of the whole number system (Charles & Carmel, 2005). By questioning the validity of the first student statement, this exchange helped to shift the focus from a surface-level understanding to a deeper comprehension of how numbers behave in relation to the decimal number system. Furthermore, the student comment about the decimal point being, "set in concrete" shows that the student has a vivid metaphor to show their understanding of the fixed position of the decimal point. The teacher continued to press for deeper conceptual understanding by asking for students to name the process of digits moving to the left or to the right in the place value setting. The students' response that it is called the, "powers of ten" shows comprehension of the base-10 number system. Teacher noticing and subsequent questioning initiated a significant moment in student learning. The episode highlights a shift from misconception to clarity in key rational number understanding. Specifically, this aligns with Barnett-Clarke's (2010) description of big ideas related to rational numbers, '*computation with rational numbers is an extension of computation with whole numbers but introduces some new ideas and processes'*.

4.5 Developing students mathematical reasoning

It is essential that teachers follow a trajectory of mathematical learning. Doing so builds on student understanding of the big ideas of mathematics. A trajectory of learning also provides opportunities to develop students' mathematical language for reasoning, representations of reasoning, and building connections between mathematics concepts. As highlighted by Stein et al. (2008) to facilitate these opportunities, the teacher selects and sequences student solutions for discussion. This provides affordances for students to share and discuss each other's mathematical reasoning, and in the case of the current study, to support the development of rational numbers. In all aspects of mathematics teaching and learning, it is essential that the teacher holds high expectations that all students are capable of learning mathematics successfully. The data presented in the following sections is drawn from the same task (see Appendix B4).

4.5.1 High expectations

When teachers set high expectations for their students, they create a classroom environment that encourages student thinking to emerge. From the start of the current study, in the interview, the teacher consistently stated her high expectations for all students in mathematics:

> *It's not necessarily me saying I have really high expectations for you. It's more just maintaining being really positive and saying things like I know you can do this, and we believe in you. It is not me saying that you must do this, you have to do it. It's more just like a strong belief that they are able to do it and to the best of their ability.*

The teacher stated her strong belief that all students were capable of learning mathematics. This enacted belief was important and aligns with the ambitious teaching outlined by Anthony et al. (2015). In addition to interview statements, the teacher's high expectations were evident in the classroom interactions with the students as they worked on a task involving time and distance (see Appendix B4).

As the students worked together on the task in their small groups, the teacher listened and observed closely. She paused one group to question their solution they had represented in their workbook.

> *Teacher: What's going on here?*
>
> *Olivia: One third of that so 123 divided by three would be one.*
>
> *Teacher: Olivia your thought process is amazing. Can you explain why you have decided to divide 123 km by 3? You are on the right track. I can see you have written 1.5 divided by three equals .5 here. What does that mean?*
>
> *Olivia: So, we are thinking that if its .5 per hour then we could do that average. So, we could also divide it by three on this one, so we have to one third it. Since you're doing it to that we have to do it for both. I just don't know how to do this, I'm not good at this.*
>
> *Teacher: Really good explaining and justification. I can see you've tried to use this number line with jumps as well. Could you continue with this?*

In this episode, the teacher commended the student for her mathematical thinking and encouraged her to elaborate further. Rather than acknowledging the students' claim of not knowing how to solve the problem, the teacher redirected the students to notice the explanation and justification that had been made. By emphasising the positive aspects of the students’ mathematical practices and encouraging her to continue to apply them, the teacher maintained high expectations. Additionally, the teacher focused student attention on the importance of using mathematical practices to support their understanding. These actions not only reinforce students’ mathematical strengths but also foster perseverance when engaging in challenging tasks. These actions are similar to what Averill (2012) found. Although the students in Averill’s study were in secondary school, the findings presented in this study illustrate that developing mathematical practice is possible with younger students too. Furthermore, setting high expectations aligns closely with Averill’s findings on effective teacher-student relationships and caring teacher practices. As in Averill’s (2012) study, there are clear parallels with the interactions between the teacher and students in the current study which align closely with the taha hinengaro dimension from the whare tapa wha model. The taha hinengaro dimension emphasises teacher practices that foster a safe learning environment and this includes setting high and attainable expectations. In the current study, the teacher also cultivated student motivation by upholding high expectations and encouraging them to persevere through challenges. The teacher’s care and respect for her students, aligns with Averill’s assertion that such practices contribute to a supportive learning atmosphere. Consequently, through consistently upholding high expectations throughout the current study, the teacher created a caring mathematics classroom characterised by safety, purpose and engagement. This provided a classroom environment where students could freely share their mathematical reasoning thus deepening their understanding of fractions and decimals.

4.5.2 Developing mathematical practices

When teachers make student mathematical practices explicit, students are supported to engage with and learn mathematics effectively. Making mathematical practices explicit while students are engaged in mathematical activity is a significant teacher action. Drawing from the extract in Section 4.5.1, when the teacher asked, “Can you explain why you have decided to divide 123 km by three?” she enacted what Selling (2016a) refers to as an initiating talk turn. In this instance, the teacher invited the student to use the mathematical practice of making an explanation. This was followed up with what Selling (2016a) refers to as a sustaining talk turn, “I can see you have written 1.5 divided by three equals point five here. What does that mean?” Here, the teacher elicited an explanation from the student. The purpose of this sustaining talk turn was to prompt the student to explain her written representation. The student responded by giving a mathematical explanation. She explained that dividing by three is the same as finding a third of the total distance. The teacher then replied, “Really good explaining and justification”. She explicitly praised the student for making an explanation and justifying her reasoning. This aligns closely with the study by Selling (2016a) who refers to these types of talk turns as a reprising talk turn. Additionally, this teacher, specifically named the mathematical practice the students used. This aligns with an important finding from Selling, who advocated that students learn to know and use these practices when the teacher names these practices explicitly.

4.5.3 Sequencing student solutions

Later in the same lesson, the teacher monitored the students’ solutions and selected two groups to share their reasoning with the wider group. She intentionally sequenced the order in which they presented their explanations. These actions are similar to what Stein et al. (2008) refer to as the five practices to support student reasoning. The solution pathways two groups took when solving the problem are presented below:

Group 1	**Group 2**
Lleyton: So, 1.5 is how long it took so we did jumps and went 0 to 0.5 and then went that again to one whole and then again to 1.5. We know that .5 is half of 100 so we did half of an hour which is 30 minutes so the 0.5 equals 30 minutes. And then if you do that again it’s another 30 minutes which is 60 minutes and another 30 minutes which is one and a half hours. *Teacher: Good explanation. Ok everyone, talk to the people around you and come up with one question please for this group. Have a look at what they have done and come up with one question.* *Jackson: Why is .5 30 minutes?* *Georgie: Because point 5 is half of 100 which is half of a whole and would be 50. But you do the same thing with 60 and half that and that’s 30.*	*Olivia: First of all, we found out how many 30 minutes were in an hour and a half which was 3 so then you go 123 divided by 3 which then we broke it down into 100 divided by 3 and it equals 33.33 but we rounded it to 33 and we removed the extra 2 from 23 which made it 21. Then we divided 21 by 3 and that’s 7. And then we added the 7 and the 33 together to make 40. And the 3 divided by 3 is 1 so we added that to the 40. 41km is how fast they went in half an hour, and we doubled that to get 82km per hour.* *Laura: That was rocket science.* *Teacher: Turn to your group members and come up with a question for these two.* *Anna: How did you know the numbers after the 33 was a lot of threes?* *Olivia: Because 30 times 3 is 90 and 3 times 3 is 9. 3 doesn’t go into 100 evenly so basically if you add it three times you get 99.99999. You*

Archie: Did you try any other method than a number line? *Lleyton: No all we did was number lines.* *Anna: What was the thought process with a number line?* *Georgie: Because it's easier for me to see it spaced out like that.*	*need the smallest number to make it to 100. Like a millionth or something.* *Teacher: Its infinite, isn't it?* *Ted: Yeah.*

In the episode featuring Group one, Lleyton explained how the group decomposed the time of 1.5 hours using a number line. The statement, "point five is half of 100 so we did half of an hour which is 30 minutes so the zero point five equals 30 minutes", demonstrated an understanding of fractions, decimals and their relationship to time. The group's method of jumps on the number line visually represented the groups understanding that fractions are parts of a whole. The teacher prompted students to analyse Lleyton's explanation and ask a question. This action pressed them to consider the mathematical representation and clarify their own understanding. These teacher actions align with the math-talk learning community framework outlined by Hufferd-Ackles et al. (2004) which encourages everyone to make sense of others' ideas and ask questions to clarify understanding. Anna's question about the thought process behind using a number line further enriched the discussion. Georgie's response, "Because it's easier for me to see it spaced out like that", emphasised the value of the number line as a visual aid for understanding abstract concepts The number line served as a representation of travel time, supporting the students' developing understanding of fractions and decimals in a real-world context.

The second groups solution strategy built on the first groups and was purposefully selected by the teacher for this reason. When Olivia explained, "you go 123 divided by three which then we broke it down into 100 divided by three', she demonstrated the group's understanding of the rational number system by their decomposition of the whole number into smaller parts and by dividing 100 by 3. Further, when Olivia stated, "it equals 33.33 but we rounded it to 33". This showed emerging student understanding that decimal numbers can result from dividing whole numbers emerged. Anna's question, "How did you know the numbers after the 33 was a lot of threes?" further explored this concept. The group offered a justification for why there was a decimal number resulting from the equation by stating, “Because 30 times three is 90 and three times three is 9. Three doesn’t go into 100 evenly so basically if you add it three times you get 99.99999. You need the smallest number to make it to 100. Like a millionth or something”. The acknowledgement that, “three doesn’t go into 100 evenly” showed that the group understood the limitations of whole number division and the necessity of decimals to represent the result. The following statement made by the group, “if you add it three times you get 99.99999,” illustrated the group understanding of repeating decimals and how rational numbers can approach a whole without reaching it exactly.

In the above episodes, the teacher intentionally selected and sequenced the groups to present their mathematical solutions publicly. This action allowed her to maximise opportunities for in-depth reasoning and support her student to develop mathematical understanding. The second group’s explanation was purposefully selected to build on the first groups reasoning and extended to include mathematical explanation and justification, important mathematical practices. The teachers’ deliberate selection and sequencing of student presentations aligns with the teaching practices described by Stein et al. (2008). By making rich decimal number concepts public and encouraging discussion, the teacher deepened student understanding.

Chapter Five: Conclusion

5.1 Introduction

The previous chapter presented the findings and discussion regarding the teacher's actions to support the development of students' conceptual understanding of rational numbers. This chapter concludes with a review of the main findings in relation to the research questions. Section 5.2 summarises the research questions while Section 5.3 summarises the research findings. Limitations to the study are discussed in Section 5.4 and Section 5.5 outlines suggested areas for future research.

5.2 Key findings related to the research questions

To focus on specific pedagogical actions by teachers that effectively enhance and extend students' understanding of rational numbers, two research questions were addressed within the study. This section summarises the key findings that emerged during the analysis in relation to the research questions. Recommendations for teachers are also presented.

5.2.1 What pedagogical actions do teachers enact to support students in developing rational number understanding?

In this study, the teacher created a supportive learning environment that encouraged group discussion and established high expectations for all students. Students were prompted to engage in collaborative discourse, share their mathematical explanations, and justifications. The teacher selected tasks were carefully designed to incorporate real-life contexts, enabling students to relate the use of rational numbers to their everyday experiences. By using strategic talk moves, the teacher facilitated whole class discussions that made student thinking visible, allowing all learners to consider their classmates' reasoning and connect it to their own understanding. The teacher emphasised mathematical practices and pressed students to justify their thinking and ask questions in response to their peers' explanations. Additionally, the teacher highlighted explicit connections between student visual representations and key mathematical concepts related to rational numbers, praised the use of mathematical language, and addressed misconceptions through guided discussions.

5.2.2 How do students demonstrate their understanding of rational numbers?

Students expressed their understanding of rational numbers in discussions about mathematical tasks. They provided explanations of their solution strategies and offered justifications to reinforce their reasoning. Many of these explanations included mathematical representations such as number lines, equations, and conversions between decimals, fractions, and percentages. The use of mathematical language throughout these practices provided the teacher clear evidence of student understanding. Over the five weeks of the study, it became clear that students' understanding deepened as their explanations grew more detailed, and they increasingly questioned one another to clarify their own comprehension.

Using the key findings of the study presented above, the following recommendations for teacher practice have been formulated:

5.3 Recommendations for teacher action

Mathematical Tasks

Contextual tasks that involve relevant situations significantly impact how students engage with mathematics. It is essential for teachers to choose appropriate tasks that resonate with students, allowing them to develop understanding in meaningful ways. When students can relate mathematical concepts to real-life scenarios, they are more likely to see the relevance and applicability of what they are learning. The thoughful selection of contextual tasks not only aids in comprehension but also helps students develop a positive attitude towards mathematics, empowering them to apply their knowledge in diverse situations.

Visual representations

Representations support student understanding by providing concrete ways to engage with abstract ideas. Teachers should encourage the intentional use of various visual representations because it is crucial for helping students to understand rational numbers. Wall charts, such as place value charts and equivalence charts, serve as constant reference points that reinforce relationships between fractions, decimals, and percentages. Empty number lines enable students to visualise the placement, density, and operations of rational numbers. It is important for teachers to connect student strategies with these visual models to enhance understanding and promote deeper reasoning.

Big mathematical ideas

Teachers should develop their own understanding of the big mathematical ideas related to rational numbers in order to guide and support student understanding. This foundational knowledge enables educators to identify and address student misconceptions that arise during problem-solving activities. By being attuned to students' thought processes, teachers can recognise when student understanding deviates from the intended concepts and intervene appropriately. A teacher's robust mathematical knowledge directly influences students' learning outcomes.

Developing student reasoning

Collaborative discourse promotes student reasoning and the development of student understanding. Teachers should faciliate student discussions about mathematical concepts and encourage students to engage with mathematical practices of explanation, justification, and argumentation. Additionally, teachers should select and sequence student solutions to be shared back to the wider group. The purposeful selection and sequencing of student reasoning builds on and advances student understanding of rational number concepts.

These recommendations contribute to the development of student understanding of rational numbers.

5.4 Limitations

The current study has limitations that should be considered when interpreting the results. This investigation was conducted in a single classroom with one teacher, which may limit the generalisability of the findings. The teacher focused on some big mathematics ideas of rational number, however not all were addressed. Additionally, there was no researcher intervention in the classroom teaching and learning process, therefore the impact of external factors or different teaching methods was not explored. This lack of intervention means that the findings may not fully capture the dynamics of how various instructional strategies could influence student understanding of rational numbers. Furthermore, due to the short time frame and the small number of participants, the interpretation of the results can only provide an emerging understanding of the actions teachers take to support a shift in students' rational number understanding.

5.5 Suggested areas for future research

The current study investigated the teacher actions in one classroom over a five week period. Given that the development of rational number understanding is complex and takes time, a longitudinal study investigating student understanding would be warranted. Since students have multiple teachers throughout primary school, including multiple teacher participants would help determine whether the findings are applicable in other contexts and locations. Additionally, the students in this study were aged 11-13, but in New Zealand primary schools, rational number learning outcomes are introduced at a much younger age.

A more in-depth, extended study could explore the conceptual change students undergo as they develop their understanding of the rational number system from an earlier age. Future research could also explore the effectiveness of various instructional strategies across different year levels and classroom environments, examining how different teaching approaches might support students with diverse needs. Furthermore, investigating the impact of collaborative learning and peer interactions on students' understanding of rational numbers could provide valuable insights. By considering these factors, future studies could contribute to a deeper understanding of how to effectively teach rational numbers and support students' mathematical development throughout their primary education.

5.6 Final thoughts

Findings presented in this study offer a contribution to the literature on how teachers support students in developing an understanding of rational numbers. In this study, the teacher employed a variety of pedagogical actions that supported student thinking, deepening student conceptual understanding and extending their reasoning about rational numbers. The classroom environment was intentionally nurtured by the teacher to foster a culture of learning that encouraged collaborative discourse and shared reasoning of mathematical concepts.

Through the careful selection of relevant mathematical tasks, students were presented with opportunities to learn about complex mathematical ideas. Connections were made between visual representations, student reasoning, and big mathematics ideas. Teachers play a crucial role in improving student learning outcomes. Their actions can lead to the exposure and exploration of students to complex mathematical ideas and ensure that students have opportunities to build a solid foundation of mathematical understanding. This study underscores the importance of ongoing development of teachers to enhance their pedagogical skills to further student learning and engagement in mathematics.

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Appendices

Appendix A: Semi-structured interview

Questions for teacher interview:

<u>Teacher actions</u>
What are your primary goals and objectives when teaching mathematics to your students?

What steps do you take to foster a positive classroom environment that promotes mathematical discourse among students?

How do you encourage active participation and collaboration among students during math lessons?

How do you provide opportunities for students to explore multiple solution paths and justify their reasoning?

What resources or materials do you use to engage students and make math learning more interactive and hands-on?

How do you provide timely and constructive feedback to students to support their learning and growth in mathematics?

What strategies or techniques do you use to help students develop their mathematical communication skills?

Can you share any specific examples or activities that you find particularly effective in promoting mathematical reasoning among students?

<u>Rational Numbers</u>
What are some effective strategies or techniques you use to introduce rational numbers to students?

How do you help students understand the concept of rational numbers?

What real-life examples or applications do you use to illustrate the relevance and importance of rational numbers?

How do you support students who may be struggling with grasping the concept of rational numbers?

Are there any specific challenges or common misconceptions you have encountered while teaching rational numbers, and how do you address them?

Appendix B1: Mathematical Task One

What percentage have you downloaded of that app? How much more would you need to download to complete it?

Record using a range of different representations including symbols and be ready to explain and justify how they are equivalent.

Appendix B2: Mathematical Task Two

There are 4 finalists in a skateboarding competition. Here are their scores for their 2 runs and trick stage. This competition scores each stage out of 10.

Contestant	1	2	3	4
Run 1	8.903	7.796	7.897	8.03
Run 2	7.0001	9.9911	8.98	8.004
Tricks	8.987	7.5	8.0	8.039

Who came first? Second? Third?

How many points would the second finalist have needed to come first?

How many points would the third finalist have needed to come first?

Appendix B3: Mathematical Task Three

Cameron is going to a family reunion in Tuvalu. He has some money saved up. In Tuvalu they use Australian dollars but their own coins. The exchange rate is $1 New Zealand for $.9301 Australian.

How much Australian money will he get in exchange for NZ$10? NZ$100? NZ$550?

In Samoa the exchange rate is $1 New Zealand for 1.5204 tala.
How much will she get in exchange for NZ$10? NZ$100? NZ$550?

Appendix B4: Mathematical Task Four

Lily and her family travelled 123km from her home in Christchurch to Hanmer Springs. It takes 1.5 hours to complete the journey.

What is the average rate in kilometres per hour?

Be ready to explain and justify your reasoning.

Appendix C1: Board of Trustees consent form

MASSEY UNIVERSITY
INSTITUTE OF EDUCATION
TE KURA O TE MATĀURANGA

Teacher actions that support primary school students' understanding of rational numbers

CONSENT FORM: BOARD OF TRUSTEES

THIS CONSENT FORM WILL BE HELD FOR A PERIOD OF FIVE (5) YEARS

We have read the Information Sheet and have had details of the study explained. Our questions have been answered to our satisfaction, and we understand that we may ask further questions at any time.

We agree to ____ participating in this study under the conditions set out in the Information Sheet.

Signature: ____________________________________ **Date:** ___________

Full Name – printed ________________________________

Appendix C2: Student participant consent form

MASSEY UNIVERSITY
INSTITUTE OF EDUCATION
TE KURA O TE MATĀURANGA

Teacher actions that support primary school students' understanding of rational numbers

CONSENT FORM: STUDENT PARTICIPANTS

THIS CONSENT FORM WILL BE HELD FOR A PERIOD OF FIVE (5) YEARS

I have read the Information Sheet and have had the details of the study explained to me. My questions have been answered to my satisfaction, and I understand that I may ask further questions at any time.

I agree/do not agree to being sound recorded.

I agree/do not agree to being image recorded.

I agree to participate in this study under the conditions set out in the Information Sheet.

Child's Signature: .... **Date:** ....

Full Name - printed ....

Appendix C3: Parents of student participants consent form

MASSEY UNIVERSITY
INSTITUTE OF EDUCATION
TE KURA O TE MATĀURANGA

Teacher actions that support primary school students' understanding of rational numbers

CONSENT FORM: PARENTS/CAREGIVERS OF STUDENT PARTICIPANTS

THIS CONSENT FORM WILL BE HELD FOR A PERIOD OF FIVE (5) YEARS

I have read the Information Sheet and have had the details of the study explained to me. My questions have been answered to my satisfaction, and I understand that I may ask further questions at any time.

I agree/do not agree to ________________________________ being sound recorded.

I agree/do not agree to ________________________________being image recorded.

I agree to __________________________________ participating in this study under the conditions set out in the Information Sheet.

Parent's Signature: .... **Date:** ....

Full Name - printed ....

Appendix C4: Teacher participant consent form

MASSEY UNIVERSITY
INSTITUTE OF EDUCATION
TE KURA O TE MATĀURANGA

Teacher actions that support primary school students' understanding of rational numbers

CONSENT FORM: TEACHER PARTICIPANT

THIS CONSENT FORM WILL BE HELD FOR A PERIOD OF FIVE (5) YEARS

I have read the Information Sheet and have had the details of the study explained to me. My questions have been answered to my satisfaction, and I understand that I may ask further questions at any time.

I agree/do not agree to the interview being sound recorded.

I agree/do not agree to the interview being image recorded.

I agree to participate in this study under the conditions set out in the Information Sheet.

Signature: .... **Date:** ....

Full Name - printed ....

Appendix D1: Board of Trustees research information sheet

MASSEY UNIVERSITY
INSTITUTE OF EDUCATION
TE KURA O TE MATĀURANGA

Teacher actions that support primary school students' understanding of rational numbers

INFORMATION SHEET

Dear Board of Trustees

My name is Kelly Sheppard and I am a Master's student at Massey University. My Master's study is titled *Teacher actions that support primary school students' understanding of rational numbers* and will be conducted within the context of a primary school mathematics classroom. The main purpose of this study is to explore how teachers help students to learn about rational numbers. The focus of the exploration will be on the specific actions a teacher takes to deepen student understanding of fractions, decimals, and percentages. The study will also specifically monitor how students respond to teacher actions and how they communicate their understanding of rational numbers.

The teacher of students in Year 7 has agreed to participate in this study as the mathematics teacher. The duration of this study will be over five weeks in Term 3 while the teacher is facilitating a mathematics unit with a rational numbers learning focus. During this project, ten lessons during the normal classroom schedule, will be filmed. The teacher, the students and their parents/caregivers will be given full information and consent will be requested. Specifically, permission to allow the students to be filmed. Work samples from each lesson may also be collected and photo-copied. The teacher will be interviewed prior to teaching the rational number teaching unit and will be approximately 30 minutes in duration. The interview with the teacher will be audio recorded.

The time involved in the complete study for the teacher will be no more than eleven hours over a period of five weeks. There is no expectation that the usual classroom programme will be disrupted in any way.

All project data collected during individual interviews and filming will be stored in a secure location, with no public access and used only for this research and any publication arising from this research. After completion of five years, all data pertaining to this study will be destroyed in a secure manner. All efforts will be taken to maximize confidentiality and anonymity for participants. Names of all participants and the school will not be used once information has been gathered and only pseudonyms and non-identifying information will be used in reporting.

Please note that the Board of Trustees is under no obligation to accept this invitation. If you decide to participate you have the right to:

- Decline to answer any particular question.
- Withdraw from the study at any time.
- Ask any questions about the study at any time during participation.
- Provide any information on the understanding that your name will not be used unless you give permission to the researcher
- Be given access to a summary of the project findings when it is concluded.

If you have any further questions about this project you are welcome to discuss them with me personally:

Kelly Sheppard. Phone: [redacted]. Email: k.sheppard@massey.ac.nz

Or contact my supervisors at Massey University

- Dr Generosa Leach. Phone: 021 709621. Email: G.Leach@massey.ac.nz
  Institute of Education, Private Bag 102 904, North Shore, Auckland 0745

- Dr Jodie Hunter (09) 4140800 ext.43518. Email. J.Hunter1@massey.ac.nz
  Institute of Education, Private Bag 102 904, North Shore, Auckland 0745

This project has been evaluated by peer review and judged to be low risk. Consequently, it has not been reviewed by one of the University's Human Ethics Committees. The researcher(s) named above are responsible for the ethical conduct of this research.

If you have any concerns about the conduct of this research that you wish to raise with someone other than the researcher(s), please contact Dr Gerald Harrison, Ethicist, telephone (06) 356 9099, email humanethicsouthb@massey.ac.nz

Appendix D2: Student participant research information sheet

MASSEY UNIVERSITY
INSTITUTE OF EDUCATION
TE KURA O TE MATĀURANGA

Teacher actions that support primary school students' understanding of rational numbers

INFORMATION SHEET

Dear Students and Parents/Caregivers,

My name is Kelly Sheppard and I am a Master's student at Massey University. My Master's study is titled *Teacher actions that support primary school students' understanding of rational numbers* and will be conducted within the context of a primary school mathematics classroom. The main purpose of this study is to explore how teachers help students to learn about rational numbers. The focus of the exploration will be on the specific actions a teacher takes to deepen student understanding of fractions, decimals, and percentages.

I would like to invite you with your parent's permission to be involved in this study. Your classroom teacher has also agreed to participate in this study. The school principal has also given her approval for me to invite you to participate, and for me to do this research in your classroom.

I will be observing you participating in 10 mathematics lessons in your classroom in Term 3. Your classroom teacher will be teaching you at this time and these lessons will be part of your normal mathematics programme, whether you agree to be in the study or not. These lessons will be filmed, and you may at any time ask that the camera be turned off and any comments you have made deleted. With your permission I might sometimes collect copies of written work to support your mathematical thinking. You have the right to refuse to allow the copies to be taken.

All the information gathered will be stored in a secure location and used only for this research. After completion of the research the information will be destroyed. All efforts will be taken to maximize your confidentiality and anonymity which means that your name will not be used in this study and only non-identifying information will be used in reporting.
I ask that you discuss all the information in this letter fully with your parents before you give your consent to participate.

Please note that you have the following rights:

- To say that you do not want to participate in the study
- To withdraw from the study at any time
- To ask for the audio or video recorder to be turned off at any time during the lessons and any comments you have made be deleted
- To refuse to allow copies of your written work to be taken
- To ask questions about the study at any time
- To participate knowing that you will not be identified at any time
- To be given a summary of what is found at the end of the study

If you have any further questions about this project you are welcome to discuss them with me personally:

Kelly Sheppard. Phone: [redacted]. Email: k.sheppard@massey.ac.nz

Or contact my supervisors at Massey University

- Dr Generosa Leach. Phone: 021 709621. Email: G.Leach@massey.ac.nz
  Institute of Education, Private Bag 102 904, North Shore, Auckland 0745

- Dr Jodie Hunter (09) 4140800 ext.43518. Email. J.Hunter1@massey.ac.nz
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Appendix D3: Teacher research information sheet

MASSEY UNIVERSITY
INSTITUTE OF EDUCATION
TE KURA O TE MATĀURANGA

Teacher actions that support primary school students' understanding of rational numbers

Dear Teacher

My name is Kelly Sheppard and I am a Master's student at Massey University. My Master's study is titled *Teacher actions that support primary school students' understanding of rational numbers* and will be conducted within the context of a primary school mathematics classroom. The main purpose of this study is to explore how teachers help students to learn about rational numbers. The focus of the exploration will be on the specific actions a teacher takes to deepen student understanding of fractions, decimals, and percentages. The study will also specifically monitor how students respond to teacher actions and how they communicate their understanding of rational numbers.

I am formally inviting you to be a part of this research as I examine the ways in which you as the teacher support students to develop an understanding of rational numbers.
Permission to participate in the study will be sought from both the parents/caregivers of the students in your class and the students themselves. The students and their parents/caregivers will be given full information, and consent will be requested in due course. Consent will be sought for permission to be filmed during mathematics lessons.

I will interview you at the beginning of the study. The time involved for the interview will be no more than 30 minutes. The interview with you will be audio recorded.
The duration of this project will be over five weeks in Term 3. During this project, ten mathematics lessons will be filmed. Work samples from each lesson may also be collected and photocopied. The observations will take place in the classroom and be part of your normal mathematics programme.
The time involved in the complete study for you will be no more than eleven hours over the period of five weeks.

All project data collected during individual interviews and filming will be stored in a secure location, with no public access and used only for this research and any publication arising from this research. After completion of five years, all data pertaining to this study will be destroyed in a secure manner. All efforts will be taken to maximize confidentiality and anonymity for participants. Names of all participants and the school will not be used once information has been gathered and only pseudonyms and non-identifying information will be used in reporting.

Please note that you are under no obligation to accept this invitation. If you decide to participate you have the right to:

- Decline to answer any particular question.
- Withdraw from the study at any time.
- Ask any questions about the study at any time during participation.
- Provide any information on the understanding that your name will not be used unless you give permission to the researcher
- To ask for the audio or video recorder to be turned off at any time during the interviews and any comments you have made be deleted.
- Be given access to a summary of the project findings when it is concluded.

If you have any further questions about this project you are welcome to discuss them with me personally:

Kelly Sheppard. Phone: [redacted]. Email: k.sheppard@massey.ac.nz

Or contact my supervisors at Massey University

- Dr Generosa Leach. Phone: 021 709621. Email: G.Leach@massey.ac.nz
  Institute of Education, Private Bag 102 904, North Shore, Auckland 0745

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Appendix E1: Thematic analysis table used to group codes into themes

Theme	**Contextual maths tasks**	**Visual representations**	**Big ideas of mathematics**	**Mathematical reasoning**
Codes	Real life contexts Highlighting connections Launching the task Teacher talk moves	Mathematical language Facilitating discussion Questioning for elaboration	Teacher re-voicing student explanations Building on from student reasoning Addressing misconceptions Generalisations	High expectations Selecting student solutions Sequencing student solutions