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Uniform Convergent Methods on Arbitrary Meshes for Singularly Perturbed Problems with Piecewise Smooth Coefficients

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Abstract
This paper deals with uniform convergent methods for solving singularly perturbed two-point boundary value problems with piecewise smooth coefficients. Construction of the numerical methods is based on locally exact schemes or on local Green's functions. Uniform convergent properties of the proposed methods on arbitrary meshes are proven. Numerical experiments are presented.

1 Introduction

We consider the following two linear singularly perturbed problems. The first is the selfadjoint problem

$$L_\mu u(x) = \mu^2 u^{''}(x) - \beta(x)u(x) = f(x),\ x \in \Omega = (0,1), \tag{1}$$

$$u(0) = u(1) = 0, \quad \beta(x) \geq \beta_*,\ \beta_* = \text{const} > 0,$$

where μ is a small positive parameter. The second problem is the non-selfadjoint problem

$$L_\epsilon u(x) = \epsilon u^{''}(x) + \alpha(x)u^{'}(x) - \beta(x)u(x) = f(x),\ x \in \Omega = (0,1), \tag{2}$$

$$u(0) = u(1) = 0, \quad \alpha(x) \geq \alpha_*,\ \alpha_* = \text{const} > 0,\ \beta(x) \geq 0,$$

where ϵ is a small positive parameter. Suppose that the coefficients in (1) and (2) are piecewise smooth functions, i.e.

$$\alpha(x), \beta(x), f(x) \in Q_p^n(\bar{\Omega}),\ n \geq 0.$$

We say that $v(x) \in Q_p^n(\bar{\Omega})$ if it is defined on $\bar{\Omega}$ and has derivatives up to order n, the function itself and its derivatives can only have discontinuity of the first kind at a finite set of points $p = \{p_1, \ldots, p_J\}, 0 < p_j < p_{j+1}, j = 1, \ldots, J-1$, i.e. $Q_p^n(\bar{\Omega}) = C^n(\bar{\Omega} \backslash p)$.

The solutions to (1) and (2) are functions with a continuous first derivative, which satisfies the boundary conditions and the equation everywhere, with the exception of the points in p. Problems (1) and (2) have an unique solution (see for details in [1])

$$u(x) \in C^1(\bar{\Omega}) \cap Q_p^{n+2}(\bar{\Omega}).$$

In general, the solution to (1) has boundary layers at the end-points $x = 0, 1$ and interior layers at the points of p (see, below, Lemma 1), and the solution to (2) possesses a boundary layer only at $x = 0$ (Lemma 2).

It is well-known that classical numerical methods for solving singularly perturbed problems are inefficient, since in order to resolve layers they require a fine mesh covering the whole domain. For constructing effective numerical algorithms to handle these problems, there are two general approaches: the first one is based on layer-adapted meshes and the second is based on exponential fitting or on locally exact schemes. The basic property of the effective numerical methods is uniform convergence with respect to the perturbation parameter. The three books [2, 3, 4] develop these approaches and give comprehensive applications to wide classes of singularly perturbed problems.

We note here the survey [5] concerning with the recent progress made on the layer-adapted mesh approach.

In this paper we concentrate on the locally exact schemes applied to problems (1), (2) with the piecewise smooth coefficients. The main theoretical results for convergent properties of numerical methods based on the locally exact scheme approach have been obtained on uniform computational meshes (see for details in [2, 4]), and this fact may play a negative role in the practical use of this approach. But as it is mentioned in [2], the locally exact scheme approach can be applied successfully on nonuniform meshes too.

We highlight here the attractive feature of the uniform convergent numerical methods on arbitrary meshes: only a locally fine mesh is required for resolving local details. For example, problem (1) with the piecewise smooth coefficients possesses in general the boundary layers near the endpoints and the interior layers at the points of discontinuity of the coefficients. If one needs to resolve accurately the exact solution only inside one layer, then in the case of the methods on layer-adapted meshes, the fine mesh is required near all the layers to get uniform convergence. But in the case of the methods on arbitrary meshes, for maintaining uniform convergence, it is enough to introduce the fine mesh only inside the layer of interest and the uniform mesh outside this layer (strictly speaking, the points of discontinuity have to be included in this uniform mesh).

The aim of this work is to construct uniform in the small parameter convergent methods on arbitrary meshes. We apply the locally exact scheme approach based on local Green's functions. To illustrate this approach, consider a linear two-point boundary value problem

$$Lu(x) = f(x),\ x \in (a,b), \quad u(a), u(b) \text{ given},$$

where L is the differential operator.

Introduce an arbitrary partition

$$\bar{\Omega}^h = \{x_i, i = 0, 1, \ldots, N, x_0 = a, x_N = b\},\ \omega_i = (x_i, x_{i+1}),\ h_i = x_{i+1} - x_i, h = \max h_i.$$

We single out the principal part $\tilde{L}$ of L such that this operator can be inverted explicitly on each interval $\bar{\omega}_i, i = 0, \ldots, N-1$, and it satisfies the following estimate

$$\int_a^b |(L - \tilde{L})u(x)|dx \le Ch^k,\ k > 0,$$

where C is a constant independent of h.

We now use local Green's functions to construct an integral-difference scheme satisfied by the exact solution. For $i = 0, \ldots, N-1$, consider the following linear two-point boundary value problems on $\bar{\omega}_i$

$$\tilde{L}\phi_{mi} = 0,\ x \in (x_i, x_{i+1}),\ m = 1, 2,$$

$$\phi_{1i}(x_i) = \phi_{2i}(x_{i+1}) = 1,\ \phi_{1i}(x_{i+1}) = \phi_{2i}(x_i) = 0.$$

Denoting by $G_i(x,s)$ the local Green's function for the operator $\tilde{L}$ on $\bar{\omega}_i$, we represent the exact solution on each interval $\bar{\omega}_i$ in the form

$$u(x) = u_i\phi_{1i}(x) + u_{i+1}\phi_{2i}(x) + \int_{x_i}^{x_{i+1}} G_i(x,s)\psi(s)ds,\ u_i = u(x_i), \tag{3}$$

$$\psi(x) = (\tilde{L} - L)u(x) + f(x).$$

Assume that $\tilde{L}$ on $\bar{\omega}_i$ is the operator with sufficiently smooth coefficients and thus $\tilde{L}$ is the piecewise smooth operator on the whole domain $[a,b]$. From this, $u(x) \in C^1[a,b]$ and we must have

$$\frac{du(x_i - 0)}{dx} = \frac{du(x_i + 0)}{dx},\ i = 1, \ldots N-1.$$

Equating these derivatives calculated from (3), we obtain the required integral-difference scheme

$$a_i u_{i-1} - c_i u_i + b_i u_{i+1} = \Psi_i,\ i = 1, \dots, N-1,\ u_0 = u(a),\ u_N = u(b), \tag{4}$$

$$a_i = \phi'_{1,i-1}(x_i),\ b_i = -\phi'_{2i}(x_i),\ c_i = \phi'_{1i}(x_i) - \phi'_{2,i-1}(x_i),$$

$$\Psi_i = -\int_{x_{i-1}}^{x_i} [G_{i-1}(x,s)]'_{x=x_i}\psi(s)ds + \int_{x_i}^{x_{i+1}} [G_i(x,s)]'_{x=x_i}\psi(s)ds,$$

where the prime denotes differentiation. In general, the integrals cannot be evaluated exactly, but we can overcome this difficulty by appropriate approximations to ψ. Some implementations of this approach can be found in [6, 7, 8].

In [2], the difference-integral scheme (4) is constructed by representing the exact solution in the form (3) on each interval $[x_{i-1}, x_{i+1}]$ and putting $x = x_i$ to get the three-point scheme. In [4], for constructing the locally exact scheme for (2), the same property as in the present work that $u'(x) \in C^1(\bar{\Omega})$ is exploited, but instead of the local Green's functions, the so-called L-splines (the adapted-spline functions) are used to express the exact solution on $[x_i, x_{i+1}]$.

Some preliminary results concerned with a priori estimates and continuous dependence of the solutions to problems (1) and (2) with piecewise smooth coefficients are given in section 2. In section 3, the uniform difference schemes on arbitrary meshes are constructed and uniform convergent properties are proved. Results of numerical experiments are presented in section 4.

2 Some preliminary results

In this section we formulate and prove some preliminary results necessary below.

2.1 A priori estimates

The following lemmas contain estimates of the solutions to problems (1) and (2).

Lemma 1 *If the functions $\beta(x), f(x) \in Q^n_p(\bar{\Omega}), n \geq 0$, then an unique solution exists and $u(x) \in C^1(\bar{\Omega}) \cap Q^{n+2}_p(\bar{\Omega})$. The solution $u(x)$ to (1) satisfies the following estimates*

$$\max_{0 \leq x \leq 1} |u(x)| \leq \beta_*^{-1} \|f\|;$$

$$\left| \frac{d^n u(x)}{dx^n} \right| \leq K\|f\|[1 + \mu^{-n}\Pi(x)],\ x \in \bar{\Omega},\ n = 1, 2,$$

$$\Pi(x) = \exp(-m_s x/\mu) + \sum_{j=1}^{J} \exp(-m_s \mid x - p_j \mid /\mu) + \exp(-m_s(1-x)/\mu),$$

$$m_s = \beta_*^{1/2},\ \|f\| = \sup\{f(x), x \in \bar{\Omega}\}, \quad u''(p_j) = (u''(p_j - 0), x = p_j - 0; u''(p_j + 0), x = p_j + 0),$$

here K denotes a generic positive constant independent of μ and $f(x)$.

Proof. Note that [9] contains the proof of the lemma but it is an inconvenient source. The result that problem (1) with the piecewise smooth coefficients has an unique solution can be found in [1].

Prove that the maximum principle for (1) holds true: if a function $w(x) \in C^1(\bar{\Omega}) \cap Q^{n+2}_p(\bar{\Omega})$ and satisfies $L_\mu w(x) \leq 0, x \in (0,1),\ w(0), w(1) \geq 0$ then $w(x) \geq 0,\ x \in [0,1]$. Suppose to the contrary that there is a point x_* where $w(x_*) < 0$. If $x_* \notin p$, where p is the set of the points of discontinuity then from $w'(x_*) = 0$ and $w''(x_*) \geq 0$, it follows that $L_\mu w(x_*) > 0$, so we get a contradiction with our assumption. Now suppose that $x_* \in p$. Since $w'(x)$ is a continuous function then $w'(x_*) = 0$ and $w'(x) \geq 0$ in some small vicinity $[x_*, x_* + \delta], \delta > 0$. In general, $w''(x)$ has a jump point at x_*, but on the interval $(x_*, x_*+\delta]$, it is a continuous function. Now, if δ is small enough, then $\beta(x),\ f(x)$ are continuous functions and $w^{''}(x)$ does not change a sign in this interval. Representing $w^{'}(x)$ in the form $\int_{x_*+0}^{x} w^{''}(s)ds$, we conclude that $w''(x)\geq 0, x\in(x_*, x_*+\delta]$. Hence, $L_\mu w(x) > 0, x\in(x_*, x_*+\delta]$, that contradicts our assumption. The bound on $u(x)$ is derived by applying the maximum principle to the functions $-\|f\|/\beta_* \pm u(x)$.

Now prove the estimate on $u^{'}(x)$ and $u^{''}(x)$. For simplicity, suppose that the set p contains only one point p_1. Denoting by $u_r(x)$ the solution of the reduced problem with $\mu = 0$, on $[0, p_1]$ introduce the function $z(x) = u(x) - u_r(x)$ which satisfies the problem

$$L_\mu z(x) = -\mu^2 u_r^{''}(x),\ x\in(0,p_1),\ z(0)=z_0,\ z(p_1)=z_1.$$

On $[0, p_1]$, consider the function $w(x)$:

$$w(x) = -\mu^2(\sup_{x\in[0,p_1-0]} |u_r^{''}(x)|/\beta_*) - |z_0|\exp(-m_s x/\mu) - |z_1|\exp[-m_s(p_1-x)/\mu].$$

Since $L_\mu(w\pm z)\geq 0, x\in(0,p_1),\ w\pm z\leq 0, \mathrm{at}, x=0,p_1$, from the maximum principle it follows that $|z(x)|\leq -w(x)$ and

$$|u(x)|\leq K[1+\Pi(x)],\ x\in[0,p_1].$$

From (1) and the definition of z, we have $u^{''}(x) = \mu^{-2}\beta(x)z(x)$, and, hence the estimate for $u^{''}$ holds true.

To estimate $u^{'}(x)$, we integrate with respect to t the equality

$$u^{'}(x) - u^{'}(t) = \int_t^x u^{''}(s)ds,$$

over the interval $[t_1, t_2]$ and obtain

$$u^{'}(x) = (t_2-t_1)^{-1}\Big[u(t_2)-u(t_1)+\int_{t_1}^{t_2}(x-t)u^{''}[t+\theta(t)(x-t)]dt\Big],\ 0<\theta<1.$$

From here, we conclude that

$$|u^{'}(x)|\leq (t_2-t_1)^{-1}|u(t_2)-u(t_1)| + |x-(t_1+t_2)/2|\max_{t\in[t_1,t_2]}|u^{''}(t)|.$$

Choosing $t_1 = x, t_2 = x+\mu$ if $x\in[0,p_1/2]$ or $t_1 = x-\mu, t_2 = x$ if $x\in[p_1/2, p_1]$, we get the estimate for $u^{'}(x)$. In the case $x\in[p_1,1]$, the estimates can be proved analogously.

From this lemma it follows that the exact solution of (1) has the boundary layers near the end-points and the interior layers near the points of discontinuity of the set p.

Lemma 2 *If the functions $\alpha(x), \beta(x), f(x)\in Q_p^n(\bar\Omega), n\geq 0$, then unique solution to (2) exists and $u(x)\in C^1(\bar\Omega)\cap Q_p^{n+2}(\bar\Omega)$. The following estimates of the solution hold*

$$\max_{0\leq x\leq 1}|u(x)|\leq K\|f\|;$$

$$|u^{'}(x)|\leq K\|f\|[1+\epsilon^{-1}\exp(-\alpha_* x/\epsilon)],\ x\in\bar\Omega,$$

where a generic positive constant K is independent of ϵ and $f(x)$.

Proof. It is known (see [1] for details) that if the coefficients of the equation from (2) satisfy the conditions $\inf\alpha(x)\geq\alpha_*>0,\ \inf\beta(x)\geq 0$, then unique solution to (2) exists.

Firstly, we estimate the solution to (2). The transformation $u(x) = \exp(-\sigma x)v(x)$, for a positive constant σ, yields the equation and the boundary conditions

$$L_\epsilon^\sigma v = \epsilon v^{''} + \alpha_\sigma v^{'} - \beta_\sigma v = f_\sigma,\ v(0)=v(1)=0,$$

$$\alpha_\sigma = \alpha - 2\epsilon\sigma,\ \beta_\sigma = \beta + \alpha\sigma - \epsilon\sigma^2,\ f_\sigma = \exp(\sigma x)f.$$

If we choose $\sigma = \alpha_*/4$, then in the equation for v, both the coefficients of $v^{'}(x)$ and that of $-v(x)$ are positive, i.e. $\alpha_\sigma \geq \alpha_*/2,\ \beta_\sigma \geq \beta_* = 3\alpha_*^2/16$, (where we suppose that $\epsilon \leq 1$).

The maximum principle for the differential operator L_ϵ^σ with the piecewise smooth coefficients holds true: if a function $w(x) \in C^1(\bar\Omega) \cap Q_p^{n+2}(\bar\Omega)$ and satisfies $L_\epsilon^\sigma w(x) \leq 0, x \in (0,1),\ w(0), w(1) \geq 0$ then $w(x) \geq 0,\ x \in [0,1]$. This result is proved as in Lemma 1.

The bound on $v(x)$, $\|v\| \leq \beta_*^{-1}\|f_\sigma\|$ is an immediate consequence of the maximum principle and is derived by applying the maximum principle to the functions $-\|f_\sigma\|/\beta_* \pm v(x)$. Thus, the estimate for $u(x)$ follows.

Now, we prove the estimate for $u^{'}(x)$. Integrating (2), we get

$$u^{'}(x) = A\exp(-q(x)) + \epsilon^{-1}\int_0^x [\beta(s)u(s) + f(s)]\exp(q(s) - q(x))ds,$$

$$A = u^{'}(0), \quad q(x) = \epsilon^{-1}\int_0^x \alpha(s)ds.$$

From here and using the estimate for $u(x)$, it follows that

$$|u^{'}(x)| \leq |A|\exp(-q(x)) + K_1\|f\|\epsilon^{-1}\int_0^x \exp(\alpha_*(s-x)/\epsilon)ds.$$

We only need to check that

$$|u^{'}(0)| \leq K_2\|f\|/\epsilon.$$

Writing the equation from (2) in the form

$$\epsilon(u^{'}(x)\exp(q(x)))^{'} = [\beta(x)u(x) + f(x)]\exp(q(x)),$$

and integrating it, we obtain

$$u^{'}(x) = \exp(-q(x))\Big[\epsilon^{-1}\int_0^x (\beta u(s) + f)\exp(q(s))ds - R(1)\Big(\int_0^1 \exp(-q(s))ds\Big)\Big],$$

$$R(x) = \epsilon^{-1}\int_0^x\int_0^s (\beta u(\tau) + f)\exp(q(\tau) - q(x))d\tau ds.$$

From here, the estimate for $u^{'}(0)$ follows.

Remark 1 *We note that using a more complicated approach, it is possible to prove the estimate for $u(x)$ with $K = \alpha_*^{-1}$.*

2.2 Continuous dependence of the solutions

Similarly to (1), introduce the following selfadjoint problem

$$\mu^2\bar{u}^{''}(x) - \bar\beta(x)\bar{u}(x) = \bar{f}(x),\ x \in \Omega = (0,1), \tag{5}$$

$$\bar{u}(0) = \bar{u}(1) = 0, \quad \bar\beta(x) \geq \beta_*,\ \beta_* = \text{const} > 0,$$

Lemma 3 *If the coefficients in (1), (5) $\beta(x), \bar\beta(x), f(x), \bar{f}(x) \in Q_p^n(\bar\Omega), n \geq 0$, then for $z(x) = u(x) - \bar{u}(x)$, where $u(x), \bar{u}(x)$ are the solutions to (1) and (5), respectively, the following estimate holds*

$$\max_{x\in\bar\Omega}|z(x)| \leq C(\|\beta - \bar\beta\| + \|f - \bar{f}\|),$$

where a constant C is independent of μ.

Proof. Subtracting (5) from (1), we obtain

$$L_\mu z(x) = (\beta - \bar{\beta})\bar{u}(x) + (f - \bar{f}),\ x \in \Omega,\ z(0) = z(1) = 0.$$

Applying Lemma 1, it follows that

$$\max_{0\le x\le} |z(x)| \le \beta_*^{-1}((\max_{x\in\bar{\Omega}} |\bar{u}(x)|)\|\beta - \bar{\beta}\| + \|f - \bar{f}\|).$$

Estimating $\bar{u}(x)$ by Lemma 1, we prove the required estimate with $C = \max(\|\bar{f}\|/\beta_*^2, \beta_*^{-1})$.

Consider the non-selfadjoint problem

$$\epsilon\bar{u}^{''}(x) + \bar{\alpha}(x)\bar{u}^{'}(x) - \bar{\beta}(x)\bar{u}(x) = \bar{f}(x),\ x \in \Omega = (0,1), \tag{6}$$

$$\bar{u}(0) = \bar{u}(1) = 0, \quad \bar{\alpha}(x) \ge \alpha_*,\ \alpha_* = \text{const} > 0,\ \bar{\beta}(x) \ge 0,$$

Lemma 4 *If the coefficients in (2), (6) $\alpha(x), \bar{\alpha}(x), \beta(x), \bar{\beta}(x), f(x), \bar{f}(x) \in Q_p^n(\bar{\Omega}), n \ge 0$, then for $z(x) = u(x) - \bar{u}(x)$, where $u(x), \bar{u}(x)$ are the solutions to (2) and (6), respectively, the following estimate holds*

$$\max_{x\in\bar{\Omega}} |z(x)| \le C(\|\alpha - \bar{\alpha}\| + \|\beta - \bar{\beta}\| + \|f - \bar{f}\|),$$

where a constant C is independent of ϵ.

Proof. Introduce Green's function $G(x,s)$ for the differential operator $L_\epsilon^\alpha = \epsilon d^2/dx^2 - \alpha d/dx$:

$$G(x,s) = \frac{1}{\epsilon W(s)} \begin{cases} \phi_2(x)\phi_1(s), & 0 \le x \le s \le 1, \\ \phi_2(s)\phi_1(x), & 0 \le s \le x \le 1, \end{cases}$$

$$\phi_1(x) = (Q(1) - Q(x))/Q(1),\ \phi_2(x) = Q(x)/Q(1),\ Q(x) = \int_0^x \Phi(s)ds,$$

$$\Phi(s) = \exp\Big(-\epsilon^{-1}\int_0^s \alpha(\tau)d\tau\Big),\ W(s) = -\Phi(s)/Q(1).$$

The functions $\phi_1(x), \phi_2(x)$ are the solutions of the following problems

$$L_\epsilon^\alpha \phi_{1,2} = 0,\ x \in \Omega,\ \phi_1(0) = \phi_2(1) = 1,\ \phi_1(1) = \phi_2(0) = 0.$$

Now we prove the estimate uniform in the small parameter

$$\max_{x\in\bar{\Omega}} \int_0^1 |G(x,s)|ds \le K. \tag{7}$$

Using the explicit formula for $G(x,s)$, we get

$$\int_0^1 G(x,s)ds = -\epsilon^{-1}\int_0^x \frac{(Q(x) - Q(1))Q(s)}{Q(1)\Phi(s)}ds - \epsilon^{-1}\int_x^1 \frac{Q(x)(Q(s) - Q(1))}{Q(1)\Phi(s)}ds.$$

From here, it follows that

$$\int_0^1 |G(x,s)|ds \le \frac{2}{\epsilon}\int_0^1 \frac{Q(1) - Q(s)}{\Phi(s)}ds = \frac{2}{\epsilon}\int_0^1 P(s)ds,$$

$$P(s) = \Big[\exp\Big(\epsilon^{-1}\int_0^s \alpha(\tau)d\tau\Big)\Big]\int_s^1 \exp\Big(-\epsilon^{-1}\int_0^\eta \alpha(\tau)d\tau\Big)d\eta.$$

The function $P(s)$ is the solution of the initial value problem

$$P^{'}(s) = (\alpha(s)/\epsilon)P(s) - 1,\ P(1) = 0.$$

From the maximum principle for the initial value problem, we obtain the estimate

$$\max_{s\in\bar{\Omega}} |P(s)| \leq \epsilon/\alpha_*.$$

From here, we conclude (7) with $K = 2/\alpha_*$.

From (2), (6), it follows that $z(x) = u(x) - \bar{u}(x)$ is the solution of the following problem

$$L_\epsilon z(x) = \tilde{f}(x),\ x \in \Omega,\ z(0) = z(1) = 0,$$

$$\tilde{f}(x) = -(\alpha - \bar{\alpha})u^{'}(x) + (\beta - \bar{\beta})u(x) + (f - \bar{f}).$$

Denote by $Z(x)$ the solution of the problem

$$L_\epsilon Z(x) = -|\tilde{f}(x)|,\ x \in \Omega,\ Z(0) = Z(1) = 0.$$

From the maximum principle, the following inequality holds

$$|z(x)| \leq Z(x),\ x \in \bar{\Omega}.$$

Now using Green's function $G(x, s)$ of the differential operator L^α_ϵ, we write down $Z(x)$ in the form

$$Z(x) = \int_0^1 G(x, s)\beta(s)Z(s)ds + \int_0^1 G(x, s)(-|\tilde{f}(s)|)ds.$$

Since $G(x, s) \leq 0, Z(x) \geq 0, \beta(x) \geq 0$, it follows that

$$Z(x) \leq \int_0^1 |G(x, s)\tilde{f}(s)|ds.$$

From here, (7), and Lemma 2, we prove Lemma 4.

3 Uniform difference schemes

Applying the integral-difference method (3), (4), we construct uniform convergent methods on arbitrary meshes for solving (1) and (2). On $\bar{\Omega}$, introduce an arbitrary mesh $\bar{\Omega}^h$:

$$x_i,\ i = 0, \ldots, N,\ x_0 = 0,\ x_N = 1,\ \max_i(x_{i+1} - x_i) = h.$$

We suppose that the points of discontinuity of the functions $\alpha(x), \beta(x), f(x)$ in (1), (2) belong to the mesh Ω^h, i.e. $p \subset \Omega^h$.

3.1 Difference scheme for the selfadjoint problem (1)

On $\bar{\Omega}$, introduce the piecewise-constant functions

$$\bar{\beta}(x) = \beta(x_i + 0),\ \bar{f}(x) = f(x_i + 0),\ x_i + 0 \leq x \leq x_{i+1} - 0,$$

$$x_i \in \bar{\Omega}^h,\ i = 0, \ldots, N - 1, \quad (f(x_i \pm 0) = \lim_{x\to x_i\pm 0} f(x)).$$

Applying the integral-difference method to the problem

$$\bar{L}_\mu \bar{u}(x) = \mu^2 \bar{u}^{''}(x) - \bar{\beta}(x)\bar{u}(x) = \bar{f}(x),\ x \in \Omega,\ \bar{u}(0) = \bar{u}(1) = 0,$$

we write down (4) in the explicit form

$$a_i\bar{u}_{i-1} - c_i\bar{u}_i + b_i\bar{u}_{i+1} = F_i,\ i = 1, \ldots, N - 1, \tag{8}$$

$$\bar{u}_0 = \bar{u}_N = 0,\ a_i = (\mu^2\kappa_{i-1})/\sinh(\kappa_{i-1}h_{i-1}),\ b_i = a_{i+1},\ \kappa_i = \beta_i^{1/2}/\mu,$$

$$c_i = \mu^2[\kappa_{i-1}\coth(\kappa_{i-1}h_{i-1}) + \kappa_i\coth(\kappa_i h_i)],$$

$$F_i = f_{i-1}\kappa_{i-1}^{-1}\tanh(\kappa_{i-1}h_{i-1}/2) + f_i\kappa_i^{-1}\tanh(\kappa_i h_i/2)],$$

$$\bar{\phi}_{1i}(x) = \sinh(\kappa_i(x_{i+1}-x))[\sinh(\kappa_i h_i)]^{-1},\ \bar{\phi}_{2i}(x) = \sinh(\kappa_i(x-x_i))[\sinh(\kappa_i h_i)]^{-1},$$

where $\beta_i = \beta(x_i+0), f_i = f(x_i+0)$. Since the difference scheme (8) is the exact scheme for the solution $\bar{u}(x)$, then we can determine $\bar{u}(x)$ by the explicit formula

$$\bar{u}(x) = \bar{u}_i\bar{\phi}_{1i}(x) + \bar{u}_{i+1}\bar{\phi}_{2i}(x) + \bar{\phi}_{3i}(x), x \in [x_i, x_{i+1}],\ i = 0,\dots,N-1, \tag{9}$$

$$\bar{\phi}_{3i}(x) = -(f_i/\beta_i)[1 - \bar{\phi}_{1i}(x) - \bar{\phi}_{2i}(x)].$$

Theorem 1 *Let $\beta(x), f(x) \in Q_p^1(\bar{\Omega})$ and $\bar{\Omega}^h$ $(p \in \Omega^h)$ be an arbitrary mesh.*

i) The difference scheme (8) converges to the exact solution $u(x)$ of (1) uniformly with the first order in h

$$\max_{0\le i\le N} |u(x_i) - \bar{u}_i| \le Ch.$$

ii) On the whole interval $\bar{\Omega}$, the formula (9) represents the continuous approximate solution with the first order of accuracy.

iii) The flux $\mu\bar{u}'(x)$ approximates $\mu u'(x)$ uniformly with the first order in h, i.e.

$$\max_{x\in\bar{\Omega}} |\mu u'(x) - \mu\bar{u}'(x)| \le Ch.$$

Proof. Since $\|\beta - \bar{\beta}\|, \|f - \bar{f}\| \le Ch$, then properties i) and ii) are immediate consequences of Lemma 3. To prove iii), we note that $z(x) = u(x) - \bar{u}(x)$ can be represented in the form

$$z(x) = z(x_i)\bar{\phi}_{1i}(x) + z(x_{i+1})\bar{\phi}_{2i}(x) + \int_{x_i}^{x_{i+1}} \bar{G}_i(x,s)[(f(s) - \bar{f}(s)) + (\beta(s) - \bar{\beta}(s))u(s)]ds,$$

where $\bar{G}_i(x,s)$ is the Green's function of the differential operator $\bar{L}_\mu$ on $[x_i, x_{i+1}]$. Now differentiating this formula and using i) and Lemma 1, we prove iii).

Remark 2 *The difference scheme (8) has the following nonuniform estimate*

$$\max_{0\le i\le N} |u(x_i) - \bar{u}_i| \le C\min(h, h^2/\mu).$$

See for details in [10].

Remark 3 *If we apply the integral-difference method (3), (4) to the differential operator $\mu^2 d^2/dx^2$, we obtain the classical difference scheme on the uniform mesh*

$$a_{i+1}(\bar{u}_{i+1} - \bar{u}_i) - a_i(\bar{u}_i - \bar{u}_{i-1}) - \beta_i\hbar_i\bar{u}_i = f_i\hbar_i,\ i = 1,\dots,N-1,$$

$$\bar{u}_0 = \bar{u}_N = 0,\ a_i = \mu^2/h_{i-1},\ \hbar_i = (h_{i-1} + h_i)/2,$$

$$\bar{\phi}_{1i}(x) = (x_{i+1} - x)/h_i,\ \bar{\phi}_{2i}(x) = (x - x_i)/h_i.$$

3.2 Difference scheme for the non-selfadjoint problem (2)

As before, introduce the piecewise-constant functions

$$\bar{\alpha}(x) = \alpha(x_i + 0),\ \bar{\beta}(x) = \beta(x_i + 0),\ \bar{f}(x) = f(x_i + 0),\ x_i + 0 \le x \le x_{i+1} - 0.$$

For simplicity, we consider two cases of problem (2). The first one is (2) with $\beta(x) \ge \beta_*, \beta_* = \mathrm{const} > 0$, and the second $\beta(x) = 0,\ x \in \bar{\Omega}$.

If $\beta(x)$ is strictly positive, then applying the integral-difference method (3), (4) to the problem

$$\bar{L}_\epsilon \bar{u}(x) = \epsilon \bar{u}^{''}(x) + \bar{\alpha}(x)\bar{u}^{'}(x) - \bar{\beta}(x)\bar{u}(x) = \bar{f}(x),\ x \in \Omega,\ \bar{u}(0) = \bar{u}(1) = 0,$$

we obtain the following difference scheme

$$a_i \bar{u}_{i-1} - c_i \bar{u}_i + b_i \bar{u}_{i+1} = F_i,\ i = 1, \ldots, N-1,\ \bar{u}_0 = \bar{u}_N = 0, \tag{10a}$$

$$a_i = \kappa_{i-1} \exp(-\lambda_{i-1} h_{i-1})[\sinh(\kappa_{i-1} h_{i-1})]^{-1},\ b_i = \kappa_i \exp(\lambda_i h_i)[\sinh(\kappa_i h_i)]^{-1},$$

$$c_i = c_i^{(1)} + c_i^{(2)},\ c_i^{(1)} = -\lambda_{i-1} + \kappa_{i-1}\coth(\kappa_{i-1}h_{i-1}),\ c_i^{(2)} = \lambda_i + \kappa_i \coth(\kappa_i h_i),$$

$$F_i = (f_{i-1}/\beta_{i-1})(c_i^{(1)} - a_i) + (f_i/\beta_i)(c_i^{(2)} - b_i),\ \lambda_i = \alpha_i/(2\epsilon),\ \kappa_i = (\lambda_i^2 + \beta_i/\epsilon)^{1/2},$$

$$\bar{\phi}_{1i}(x) = [\exp(-\nu_{1i}(x_{i+1} - x)) - \exp(-\nu_{2i}(x_{i+1} - x))][\exp(-\nu_{1i}h_i) - \exp(-\nu_{2i}h_i)]^{-1},$$

$$\bar{\phi}_{2i}(x) = [\exp(\nu_{1i}(x - x_i)) - \exp(\nu_{2i}(x - x_i))][\exp(\nu_{1i}h_i) - \exp(\nu_{2i}h_i)]^{-1},$$

$$\nu_{1i} = -\lambda_i - \kappa_i,\ \nu_{2i} = -\lambda_i + \kappa_i,$$

where $\alpha_i = \alpha(x_i + 0), \beta_i = \beta(x_i + 0),\ f_i = f(x_i + 0)$.

If $\beta(x) = 0, x \in \bar{\Omega}$, the difference scheme has the form

$$\alpha_i r_i(\bar{u}_{i+1} - \bar{u}_i) - \alpha_{i-1} r_{i-1}(\bar{u}_i - \bar{u}_{i-1}) + \alpha_i(\bar{u}_{i+1} - \bar{u}_i) = F_i,\ i = 1, \ldots, N-1, \tag{10b}$$

$$\bar{u}_0 = \bar{u}_N = 0,\ r_i = \exp(-\alpha_i h_i/\epsilon)[1 - \exp(-\alpha_i h_i/\epsilon)]^{-1},$$

$$F_i = f_i h_i + f_i(d_{i-1} - d_i),\ d_i = \epsilon/\alpha_i - r_i h_i,$$

$$\bar{\phi}_{1i}(x) = r_i[\exp(\alpha_i(x_{i+1} - x)) - 1],\ \bar{\phi}_{2i}(x) = 1 - \bar{\phi}_{1i}(x),$$

As before for the selfadjoint problem, the difference schemes (10a,b) are the exact scheme for the solution $\bar{u}(x)$. Hence, we can determine $\bar{u}(x)$ explicitly. For the difference scheme (10a), $\bar{u}(x)$ is determined by (9) with the functions $\bar{\phi}_{1i}(x), \bar{\phi}_{2i}(x)$ from (10a). In the case of (10b), we have

$$\bar{u}(x) = \bar{u}_i \bar{\phi}_{1i}(x) + \bar{u}_{i+1}\bar{\phi}_{2i}(x) + \bar{\phi}_{3i}(x),\ x \in [x_i, x_{i+1}],\ i = 0, \ldots, N-1, \tag{11}$$

$$\bar{\phi}_{3i}(x) = (f_i/\alpha_i)[(x - x_i)\bar{\phi}_{1i}(x) - (x_{i+1} - x)\bar{\phi}_{2i}(x)].$$

Theorem 2 *Let $\alpha(x), \beta(x), f(x) \in Q_p^1(\bar{\Omega})$ and $\bar{\Omega}^h$ $(p \in \Omega^h)$ be an arbitrary mesh.*

i) The difference schemes (10a,b) converge to the exact solution $u(x)$ of (2) uniformly with the first order in h

$$\max_{0 \le i \le N} |u(x_i) - \bar{u}_i| \le Ch.$$

ii) On the whole interval $\bar{\Omega}$, the formulas (9) and (11) represent the continuous approximate solution with the first order of accuracy.

iii) The flux $\epsilon\bar{u}^{'}(x)$ approximates $\epsilon u^{'}(x)$ uniformly with the first order in h, i.e.

$$\max_{x \in \bar{\Omega}} |\epsilon u^{'}(x) - \epsilon \bar{u}^{'}(x)| \le Ch.$$

Proof. Since $\|\alpha - \bar{\alpha}\|, \|\beta - \bar{\beta}\|, \|f - \bar{f}\| \le Ch$, then properties i) and ii) are immediate consequences of Lemma 4. To prove iii), we note that $z(x) = u(x) - \bar{u}(x)$ on $[x_i, x_{i+1}]$ can be represented in the form

$$z(x) = z(x_i)\bar{\phi}_{1i}(x) + z(x_{i+1})\bar{\phi}_{2i}(x) + \int_{x_i}^{x_{i+1}} \bar{G}_i(x,s)\tilde{f}(s)ds,$$

$$\tilde{f}(s) = [f(s) - \bar{f}(s)] - (\alpha - \bar{\alpha})u^{'}(s) + (\beta - \bar{\beta})u(s),$$

where $\bar{G}_i(x,s)$ is Green's function of the differential operator $\bar{L}_\epsilon$ on $[x_i, x_{i+1}]$. Now differentiating this formula and using i) and Lemma 2, we prove iii).

Remark 4 *In [11], for the semilinear problem*

$$\epsilon u^{''}(x) + \alpha(x)u(x) = g(x,u),\ x \in \Omega,\ u(0) = u(1) = 0,$$

$$\alpha(x) \ge \alpha_*,\ \partial g/\partial u \ge 0,$$

with sufficiently smooth functions $\alpha(x), g(x,u)$, the nonlinear scheme (10b) with $f_i = g(x_i, \bar{u}_i)$ was introduced. The construction of this scheme is based on the integral-difference method (3), (4) applied to the differential operator $\bar{L}_\epsilon^\alpha = d^2/dx^2 + \bar{\alpha}(x)d/dx$. Using the method of the discrete Green function, it was proved that on an arbitrary mesh, the difference scheme (10b) converges to the exact solution of the semilinear problem uniformly with the first order in h.

If $g(x,u)$ is a linear function $g(x,u) = f(x) + \beta(x)u(x)$, then we get the problem (2). It means that for sufficiently smooth functions $\alpha(x), \beta(x), f(x)$, the difference scheme (10b) with the right-hand side

$$F_i = (f_i + \beta_i\bar{u}_i)(h_i + d_{i-1} - d_i)$$

possesses uniform convergence of the first order in h.

Note here that the left-hand side in (10b) consists of the selfadjoint part and the so-called upwind part, thus the integral-difference method "automatically" gives us the upwind discretization of the convection term.

4 Numerical results

We consider problems (1), (2) with smooth coefficients and introduce two layer-adapted meshes.

The first type of meshes is a modification of Bakhvalov's mesh (B-mesh) from [6] with logarithmic mesh generating functions inside the boundary layers and uniform mesh generating functions outside the boundary layers. For problem (1), we choose the two transition points $\sigma, 1-\sigma$, such that

$$x_{N_\mu} = \sigma,\ x_{2N_\mu} = 1 - \sigma,\ \sigma = -\mu\ln(\mu)/\sqrt{\beta}_*.$$

Inside the boundary layers $[0,\sigma], [1-\sigma, 1]$, we logarithmically grade the mesh

$$x_i = \rho(i, N_\mu),\ x_{3N_\mu - i} = 1 - \rho(i, N_\mu),\ i = 0, \ldots, N_\mu,\ 3N_\mu = N,$$

$$\rho(i, N_\mu) = -(\mu/\sqrt{\beta}_*)\ln[1 - (1-\mu)N_\mu^{-1}i],$$

and outside the layers, the mesh is uniform

$$x_i = \sigma + (1 - 2\sigma)N_\mu^{-1}i,\ i = N_\mu + 1, \ldots, 2N_\mu - 1.$$

For problem (2), we choose the transition point σ such that

$$x_{N_\epsilon} = \sigma,\ \sigma = -(\epsilon/\alpha_*)\ln(\epsilon),\ x_i = -(\epsilon/\alpha_*)\ln[1 - (1-\epsilon)N_\epsilon^{-1}i],\ i = 0, \ldots, N_\epsilon,$$

$$x_i = \sigma + (1-\sigma)N_\epsilon^{-1} i,\ i = N_\epsilon + 1, \dots, 2N_\epsilon,\ 2N_\epsilon = N.$$

The second type of meshes is the piecewise equidistant meshes (S-mesh) from [3], where the uniform meshes are constructed inside the layers and outside them as well. For problem (1) the two transition points $\sigma, 1-\sigma$ are defined by

$$x_{N_\mu} = \sigma,\ x_{2N_\mu} = 1-\sigma,\ \sigma = \mu \ln(N)/\beta_*, N = 3N_\mu.$$

Inside the layers $[0,\sigma], [1-\sigma, 1]$, we choose the uniform mesh with the step spacing σ/N_μ and outside the transition points, the uniform mesh with the step spacing $(1-2\sigma)/N_\mu$. Similarly, for problem (2) the transition point $\sigma = \epsilon \ln(N)/\alpha_*$, and in $[0,\sigma]$, $[\sigma, 1]$ the step spacing is defined by σ/N_ϵ and $(1-\sigma)/N_\epsilon$, respectively, where $N = 2N_\epsilon$.

For problem (1), introduce the classical difference scheme on the nonuniform mesh

$$\frac{\mu^2}{\hbar_i}\Big(\frac{U_{i+1}-U_i}{h_i} - \frac{U_i - U_{i-1}}{h_{i-1}}\Big) - \beta_i U_i = f_i,\ i = 1, \dots, N-1,\ U_0 = U_N = 0,$$

and for problem (2), the simple upwind scheme on the nonuniform mesh

$$\frac{\epsilon}{\hbar_i}\Big(\frac{U_{i+1}-U_i}{h_i} - \frac{U_i - U_{i-1}}{h_{i-1}}\Big) + \alpha_i\Big(\frac{U_{i+1}-U_i}{h_i}\Big) - \beta_i U_i = f_i,$$

$$i = 1, \dots, N-1,\ U_0 = U_N = 0.$$

4.1 Numerical results for problem (1)

As a test problem, consider (1) with $\beta(x) = 1 + x, \beta_* = 1$ and the exact solution

$$u(x) = \exp(-x/\mu) + \exp(-(1-x)/\mu) - (1 + \exp(-1/\mu)),\ u(0) = u(1) = 0.$$

In this case we have

$$f(x) = -x[\exp(-x/\mu) + \exp(-(1-x)/\mu)] + (1+x)[1 + \exp(-1/\mu)].$$

N_μ	δ_B; δ_B^{sp}			
16	1.15-2; 1.39-2	3.30-2; 1.50-2	4.29-2; 1.50-2	4.79-2; 1.50-2
64	1.30-3; 3.62-3	7.02-3; 3.81-3	1.02-2; 3.81-2	1.17-2; 3.81-2
256	8.58-5; 9.14-4	1.45-3; 9.68-4	2.41-3; 9.69-4	2.80-3; 9.69-4
1024	5.40-6; 2.26-4	2.26-4; 2.43-4	5.38-4; 2.43-4	6.81-4; 2.43-4
μ	10^{-2}	10^{-4}	10^{-6}	10^{-8}

Table 1: Errors on B-mesh

In Tables 1 and 2, for various values of μ and N_μ, we give the maximum errors of the approximate solutions obtained by the classical difference scheme and by the special difference scheme (8) on B-mesh and S-mesh, respectively, where

$$\delta = \max_{0\le i\le N} |u(x_i) - U_i|,\ \delta^{sp} = \max_{0\le i\le N} |u(x_i) - \bar{u}_i|.$$

The numerical results are clear illustrations of the convergent estimate from Theorem 1.

In Tables 3 and 4, we represent the numerical results on modified B_*-mesh and S_*-mesh, respectively. The modified meshes are constructed in such a way that the fine mesh is introduced only inside the boundary layer at $x = 0$, and the uniform mesh is chosen outside this layer including

N_μ	δ_S; δ_S^{sp}			
16	1.01-2; 1.41-2	2.03-2; 1.41-2	2.04-2; 1.41-2	2.04-2; 1.41-2
64	4.92-4; 5.03-3	5.01-3; 5.12-3	5.24-3; 5.12-3	5.24-3; 5.12-3
256	1.02-5; 1.60-3	1.22-3; 1.64-3	1.33-3; 1.64-3	1.33-3; 1.64-3
1024	9.36-7; 4.93-4	2.16-4; 4.96-4	3.24-4; 4.96-4	3.25-4; 4.96-4
μ	10^{-2}	10^{-4}	10^{-6}	10^{-8}

Table 2: Errors on S-mesh

N_μ	δ_{B_*}; $\delta_{B_*}^{sp}$			
16	2.32-2; 2.27-2	3.30-2; 8.59-6	4.29-2; 8.66-8	4.79-2; 8.66-10
64	5.50-3; 3.67-2	7.01-3; 3.17-6	1.02-2; 3.25-8	1.17-2; 3.25-10
256	3.64-4; 1.10-2	1.43-3; 1.01-6	2.40-3; 1.08-8	2.81-3; 1.08-10
1024	2.30-5; 2.91-3	1.62-4; 4.71-3	5.38-4; 3.37-9	6.81-4; 3.38-11
μ	10^{-2}	10^{-4}	10^{-6}	10^{-8}

Table 3: Errors on B_*-mesh

the boundary layer near $x = 1$. These numerical results for the special scheme (8) show a significant improvement in the accuracy for sufficiently small values of the perturbation parameter μ. This fact can be explained if we indicate that for sufficiently small values of μ, the maximum errors $\delta_{B,S}^{sp}$ in Tables 1, 2 occur at mesh points inside the boundary layer at $x = 1$. Now for the modified meshes, where in the region outside the boundary layer at $x = 0$ the coarse uniform mesh is in use, the reduced problem for $\mu = 0$ is actually calculated. In the case of the classical scheme, the maximum errors $\delta_{B,S}$ occur at mesh points inside the boundary layer at $x = 0$. It explains the fact that for sufficiently small μ, the accuracy of the classical scheme on the modified B_*, S_*-meshes is approximately the same as in Tables 1, 2.

4.2 Numerical results for problem (2)

Consider (2) with $\alpha(x) = 1 + x, \alpha_* = 1, \beta(x) = 0$ and the exact solution

$$u(x) = [1 - \exp(-x/\epsilon)][\exp(-1/\epsilon) - 1]^{-1} + x,\ u(0) = u(1) = 0.$$

The right hand side in (2) is defined by

$$f(x) = (x/\epsilon)\exp(-x/\epsilon)[\exp(-1/\epsilon) - 1]^{-1} + x + 1.$$

N_μ	δ_{S_*}; $\delta_{S_*}^{sp}$			
16	2.32-2; 2.23-2	2.01-2; 3.94-6	2.04-2; 3.94-8	2.04-2; 3.94-10
64	5.40-3; 3.65-2	4.93-3; 1.36-6	5.20-3; 1.36-8	5.20-3; 1.36-10
256	3.51-4; 1.08-2	1.36-3; 4.32-7	1.31-3; 4.32-9	1.31-3; 4.32-11
1024	2.14-5; 2.80-3	1.62-2; 4.70-3	3.23-4; 1.31-9	3.25-4; 1.31-11
μ	10^{-2}	10^{-4}	10^{-6}	10^{-8}

Table 4: Errors on S_*-mesh

N_ϵ	δ_B; δ_B^{sp}			
16	5.93-2; 3.03-2	6.47-2; 4.21-2	6.55-2; 4.24-2	6.59-2; 4.24-2
64	1.52-2; 5.81-3	1.61-2; 1.07-2	1.61-2; 1.08-2	1.62-2; 1.08-2
256	3.82-3; 1.25-3	3.84-3; 2.62-3	4.07-3; 2.71-3	4.08-3; 2.71-3
1024	9.52-4; 2.79-4	9.79-4; 6.24-4	9.81-4; 6.76-4	9.81-4; 6.77-4
ϵ	10^{-2}	10^{-4}	10^{-6}	10^{-8}

Table 5: Errors on B-mesh

N_ϵ	δ_S; δ_S^{sp}			
16	6.85-2; 2.94-2	6.97-2; 4.25-2	6.97-2; 4.26-2	6.97-2; 4.26-2
64	2.62-2; 5.01-3	2.64-2; 1.07-2	2.64-2; 1.08-2	2.64-2; 1.08-2
256	8.82-3; 9.09-4	8.84-3; 2.66-3	8.84-3; 2.73-3	8.84-3; 2.73-3
1024	2.73-3; 1.89-4	2.75-3; 6.23-4	2.78-3; 6.76-4	2.78-3; 6.76-4
ϵ	10^{-2}	10^{-4}	10^{-6}	10^{-8}

Table 6: Errors on S-mesh

In Tables 5 and 6, for various values of ϵ and N_ϵ, we represent the maximum errors for the simple upwind scheme and the special difference scheme (10b) on B-mesh and S-mesh, respectively. The numerical experiments confirm our theoretical results from Theorem 2 concerning the uniform convergence of (10). They also indicate that the theoretical results are fairly sharp.

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