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# Nonsmooth folds as tipping points

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## ABSTRACT

A nonsmooth fold occurs when an equilibrium or limit cycle of a nonsmooth dynamical system hits a switching manifold and collides and annihilates with another solution of the same type. We show that beyond the bifurcation, the leading-order truncation to the system, in general, has no bounded invariant set. This is proved for boundary equilibrium bifurcations of Filippov systems, hybrid systems, and continuous piecewise-smooth ordinary differential equations, and grazing-type events for which the truncated form is a continuous piecewise-linear map. The omitted higher-order terms are expected to be incapable of altering the local dynamics qualitatively, implying the system has no local invariant set on one side of a nonsmooth fold, and we demonstrate this with an example. Thus, if the equilibrium or limit cycle is attracting, the bifurcation causes the local attractor of the system to tip to a new state. The results also help explain global aspects of bifurcation structures of the truncated systems.

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**Tipping points are thresholds where the state of a dynamical system changes drastically. In abstract mathematical settings, they can be realized by passing slowly through a bifurcation that brings about the loss of a local attractor, the simplest example being a saddle-node bifurcation. Nonsmooth folds are piecewise-smooth counterparts to saddle-node bifurcations, although they may involve additional invariant sets including chaos. We argue that nonsmooth folds universally have no local invariant sets on one side of bifurcation and hence when they involve an attractor can be interpreted as tipping points, as has been observed in nonsmooth climate models.**

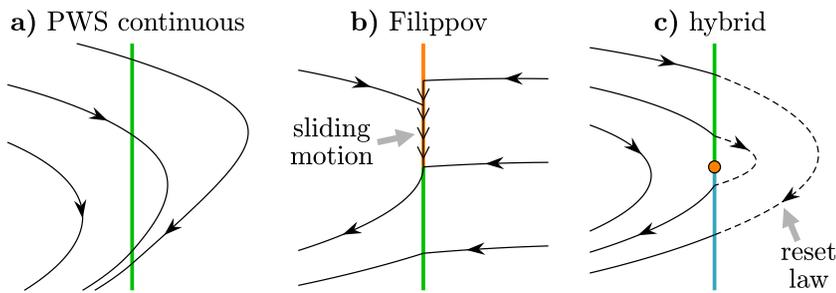
## I. INTRODUCTION

Bifurcations are critical parameter values at which the dynamical behavior of a system changes in a fundamental way. For example, oscillations in chemical kinetics are often onset by Hopf bifurcations,<sup>1</sup> turbulent fluid flow is usually generated through period-doubling cascades,<sup>2</sup> and Type I excitability in neurons is caused by saddle-node on invariant circle (SNIC) bifurcations.<sup>3</sup> These examples require local smoothness in the equations of motion. If the equations are nonsmooth, additional bifurcations are possible.<sup>4</sup>

Nonsmooth models commonly belong to one of three classes: piecewise-smooth continuous ordinary differential equations (ODEs), Filippov systems, and hybrid systems. The phase space of these systems contain switching manifolds where the ODEs are non-differentiable, discontinuous, or a map is applied (Fig. 1). In each case, a *boundary equilibrium bifurcation* (BEB) occurs when an equilibrium of a smooth component of the ODEs hits a switching manifold as parameters are varied. Functionally, the equilibrium is well-defined on both sides of the bifurcation, but only on one side of the bifurcation is an equilibrium of the system where it is said to be *admissible*; on the other side of the bifurcation, it is *virtual*.

Generic BEBs involve two equilibria, where for Filippov and hybrid systems, the second equilibrium is a pseudo-equilibrium (an equilibrium of the sliding or sticking motion).<sup>8</sup> Both equilibria are admissible on exactly one side of the bifurcation, and this immediately presents us with two cases: Either the equilibria are admissible on different sides of the bifurcation, as in Fig. 2, or the equilibria are admissible on the same side of the bifurcation, as in Fig. 3. The first case is termed *persistence*. The second case is termed a *nonsmooth fold* because if we consider only admissible solutions, then at the bifurcation, two equilibria collide and annihilate, analogous to a saddle-node bifurcation or fold.

BEBs readily generate other invariant sets growing out of the bifurcation point, such as limit cycles<sup>9,10</sup> and chaotic sets,<sup>8,11,12</sup> and the various transitions that BEBs bring about have been heavily



**FIG. 1.** Schematic phase portraits of two-dimensional nonsmooth systems with one switching manifold (vertical line). In (a), the ODEs are continuous but non-differentiable on the switching manifold. In (b), the ODEs are discontinuous on the switching manifold and Filippov's convention is used to specify sliding motion on the switching manifold. In (c), the system involves ODEs and a map (reset law) that instantaneously transports the system state from the top half of the switching manifold to the bottom half of the switching manifold. Such hybrid systems are commonly used to model mechanical systems with hard impacts.<sup>5–7</sup>

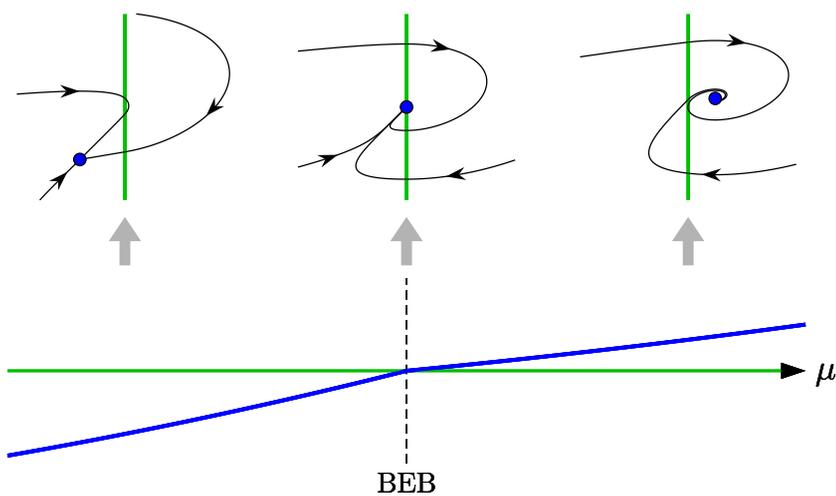
studied. It is well known that the local dynamics is usually captured by the leading-order terms of a Taylor expansion of each smooth component of the system centered at the bifurcation point. This leads to a truncated form that is more amenable to an exact analysis. Indeed, when analyzing families of BEBs in mathematical models, it is usually convenient to work with the truncated form in order to accurately identify codimension-two points.

In this paper, we describe a simple property of BEBs that appears to have been overlooked. We show that on the side of a nonsmooth fold where both equilibria are virtual, the system has no local invariant sets, at least for the truncated form. This result applies to piecewise-smooth continuous ODEs, Filippov systems, and hybrid systems, and with no restriction on the dimension of the system. Also, if a limit cycle has a degenerate interaction with a switching manifold, the same result applies when the associated Poincaré map is continuous and asymptotically piecewise-linear. This occurs for grazing-sliding bifurcations,<sup>13</sup> corner collisions,<sup>14</sup> and event collisions for piecewise-smooth ODEs with time-delay in the switching condition.<sup>15</sup> To prove the result, different calculations are required for each setting, but in each case, there is a certain direction in which solutions evolve monotonically and hence eventually diverge. This is a consequence of piecewise-linearity, the absence of admissible equilibria, and a continuity constraint on the vector field or map (for Filippov systems, the vector field can be viewed as continuous through tangency points).

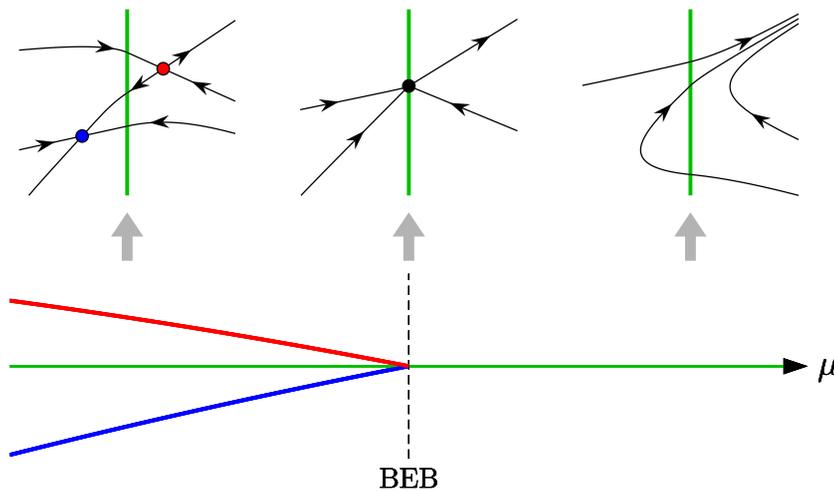
We expect that in all settings if the truncated form has no local invariant set, then neither does the full system because the terms that have been omitted are in higher order. In Sec. III E, we verify this for an example and outline the technical difficulties that need to be overcome in order to prove this in general.

Our results were inspired by a 2002 paper of Carmona *et al.*<sup>16</sup> Their work provided canonical forms for piecewise-linear ODEs arising in control circuitry. As a side note (Proposition 19), they proved that if their two-piece canonical form has no admissible equilibria, then it has no periodic solutions. We generalize their result beyond the canonical form, from periodic solutions to arbitrary bounded invariant sets; place the result in the context of BEBs; and extend the result to Filippov systems, hybrid systems, and maps.

The remainder of the paper is organized as follows. In Sec. II, we show how a nonsmooth fold in Stommel's ocean circulation model acts as a tipping point. We also describe two ways by which global features of a dynamical system can create a limit cycle in the presence of two virtual equilibria. In Sec. III, we prove the main result in the case of maps. We explore the consequences of the result to bifurcation structures of families of piecewise-linear maps and study the influence of higher order terms in an example. In Sec. IV, we prove the main result for the various ODE settings. To handle discontinuities, we use Filippov's convention to define a codimension-one sliding vector field that specifies motion on switching manifolds.<sup>4,17,18</sup> In the case of hybrid systems, we assume



**FIG. 2.** A bifurcation diagram and sample phase portraits of a piecewise-smooth continuous ODE system that experiences a persistence-type BEB as a parameter  $\mu$  is varied. The blue curves indicate the location of the admissible equilibrium in relation to the switching manifold.



**FIG. 3.** A bifurcation diagram and sample phase portraits of a piecewise-smooth continuous ODE system that experiences a nonsmooth fold BEB as the parameter  $\mu$  is varied. To the left of the bifurcation, the system has two admissible equilibria; to the right of the bifurcation, it has no admissible equilibria. Throughout this paper, stable solutions are colored blue, and unstable solutions are colored red.

reset laws model impact events in mechanical systems because this introduces a constraint that ensures nonsmooth folds occur in a local fashion.<sup>8</sup> To describe sustained contact with the impact surface, a codimension-two sticking vector field is needed. Concluding remarks are provided in Sec. V.

## II. TIPPING POINTS, GLOBAL BIFURCATIONS, AND CLIMATE

Tipping points refer to critical events that bring about a sudden and drastic change to the state of a dynamical system. This behavior can be simulated by considering a bifurcation at which an attractor vanishes and allowing the parameter that controls the bifurcation to vary slowly in time.<sup>19,20</sup>

Figure 4 illustrates this for a nonsmooth fold. The blue and red curves are stable and unstable branches of equilibria of

$$\begin{aligned} \dot{T} &= 1 - T - k(T, S)T, \\ \dot{S} &= \beta(\mu - S) - k(T, S)S, \end{aligned} \tag{1}$$

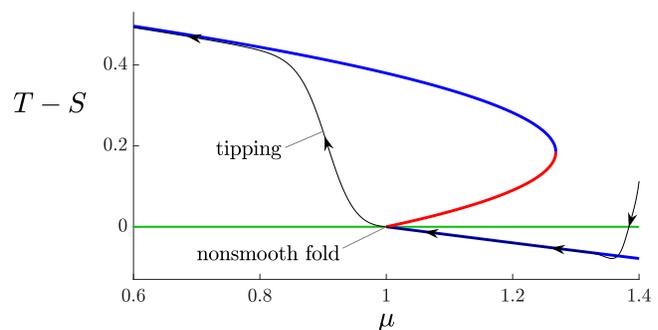
where  $k(T, S) = \alpha\beta|T - S|$ . This is a version of Stommel's two-box model<sup>21</sup> for ocean circulation ( $T$  and  $S$  represent temperature and salinity differences). We treat  $\mu$  as the bifurcation parameter. A nonsmooth fold occurs at  $\mu = 1$ .

By now, allowing  $\mu$  to vary according to  $\dot{\mu} = -0.01$ , the curve in black shows a typical trajectory. The trajectory is drawn to the lower stable equilibrium branch, then tracks this branch until the branch is destroyed at the nonsmooth fold. Now, tipping occurs: the trajectory is forced to a different area of phase space, in this case to the upper stable equilibrium branch. For Stommel's model, this has been studied further by Roberts and co-workers.<sup>22,23</sup> More recently, Budd *et al.*<sup>24</sup> report that, unlike tipping in smooth systems, since the stability of the stable equilibrium does not weaken as the bifurcation is approached, statistical tests find it difficult to generate warning that tipping is about to occur.

In Stommel's model, we know there is no local invariant set of the other side of the nonsmooth fold because (1) is a two-dimensional system and the local invariant sets associated with BEBs in two-dimensional systems have been completely classified.<sup>10,25</sup> This paper is a first step toward showing that nonsmooth fold BEBs for systems with any number of dimensions always bring about the loss of a local invariant set, in which case when one of the equilibria involved in the bifurcation is attracting, slow passage through the nonsmooth fold will induce tipping.

It should be stressed that in general the absence of an attractor only occurs locally. For example, Fig. 5 illustrates the dynamics of

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{cases} \begin{bmatrix} -2x_1 + x_2 \\ \mu - \frac{1}{2}x_1 \end{bmatrix}, & x_1 \leq 0, \\ \begin{bmatrix} \frac{1}{2}x_1 + x_2 \\ \mu + x_1 - x_1^2 \end{bmatrix}, & x_1 > 0, \end{cases} \tag{2}$$



**FIG. 4.** A bifurcation diagram of Stommel's ocean circulation model with  $\alpha = 5$  and  $\beta = 0.2$ . The blue and red curves are branches of stable and unstable equilibria. The black solution uses  $\dot{\mu} = -0.01$  and experiences a tipping point by passing the nonsmooth fold at  $\mu = 1$ .

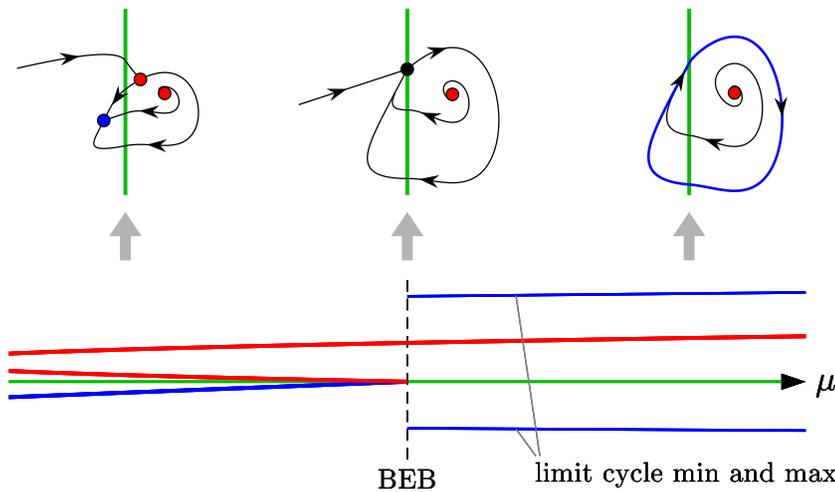


FIG. 5. A bifurcation diagram and sample phase portraits of the piecewise-smooth continuous system (2) which experiences a nonsmooth fold BEB as a parameter  $\mu$  is varied. The bifurcation creates a large amplitude limit cycle due to global features of the dynamics.

which has a nonsmooth fold BEB at  $\mu = 0$ . The nonsmooth fold creates a large amplitude limit cycle because just before the bifurcation occurs there is a global connection from the saddle equilibrium to the stable equilibrium. This is analogous to a SNIC bifurcation in a smooth system.<sup>26</sup>

Furthermore, the occurrence of two virtual stable equilibria can create stable oscillations.<sup>27-29</sup> For example, Fig. 6 uses again (1), but now with  $k(T, S) = 1$  if  $-\alpha T + S > \varepsilon$ , and  $k(T, S) = 0$  otherwise (this is Welander’s ocean circulation model<sup>30</sup> in the discontinuous limit<sup>31</sup>). Orbits move toward one equilibrium until crossing the switching manifold, then move toward the other equilibrium until crossing the switching manifold, and so on, leading to a stable limit cycle.

### III. MAPS

In this section, we first state the main result in the context of border-collision bifurcations of maps. We then explore bifurcation

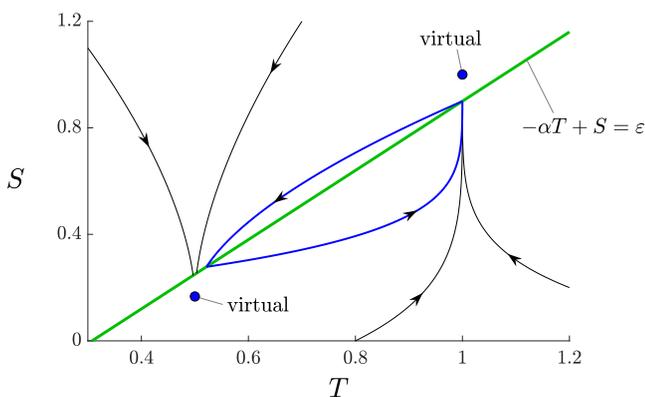


FIG. 6. A phase portrait of Welander’s ocean circulation model with  $\alpha = 1.3$ ,  $\beta = 0.2$ ,  $\varepsilon = -0.4$ , and  $\mu = 1$ . There are two virtual equilibria and a stable limit cycle (blue).

structures of piecewise-linear families, prove the result, and discuss the effect of higher order terms.

#### A. Border-collision bifurcations and a statement of the main result

Let  $f$  be a piecewise- $C^1$  continuous map with vector variable  $x = (x_1, \dots, x_n) \in \mathbb{R}^n$  and scalar parameter  $\mu \in \mathbb{R}$ . Suppose  $f$  has one switching manifold and coordinates are chosen so that this manifold is  $x_1 = 0$ . Then,  $f$  has the form

$$f(x; \mu) = \begin{cases} f^L(x; \mu), & x_1 \leq 0, \\ f^R(x; \mu), & x_1 \geq 0, \end{cases} \quad (3)$$

where  $f^L$  and  $f^R$  are  $C^1$  and satisfy  $f^L(x; \mu) = f^R(x; \mu)$  for all  $(x; \mu) \in \mathbb{R}^n \times \mathbb{R}$  with  $x_1 = 0$ .

As  $\mu$  is varied a border-collision bifurcation occurs when a fixed point of  $f^L$  or  $f^R$  hits the switching manifold. If  $f$  is a Poincaré map of an piecewise-smooth ODE system, its fixed points correspond to periodic orbits and the border-collision bifurcation corresponds to a periodic orbit that interacts degenerately with a switching manifold of the ODEs.<sup>4</sup> Examples of this include grazing-sliding bifurcations,<sup>13</sup> corner collisions,<sup>14</sup> and event collisions.<sup>15</sup>

Suppose  $f$  has a border-collision bifurcation at  $x = 0$  when  $\mu = 0$ . Then,  $f^L(0; 0) = f^R(0; 0) = 0$ , so we can write

$$f(x; \mu) = \begin{cases} A_L x + b\mu + E^L(x; \mu), & x_1 \leq 0, \\ A_R x + b\mu + E^R(x; \mu), & x_1 \geq 0, \end{cases} \quad (4)$$

where  $A_L, A_R \in \mathbb{R}^{n \times n}$  differ only in their first columns (by continuity),  $b \in \mathbb{R}^n$ , and  $E^L(x; \mu)$  and  $E^R(x; \mu)$  contain only higher order terms. By dropping  $E^L$  and  $E^R$ , we obtain the truncated form

$$g(x; \mu) = \begin{cases} A_L x + b\mu, & x_1 \leq 0, \\ A_R x + b\mu, & x_1 \geq 0, \end{cases} \quad (5)$$

which approximates  $f$  in a neighborhood of  $(x; \mu) = (0; 0)$ . We now focus our attention on the truncated form  $g$ ; the general map  $f$  is revisited in Sec. III E.

If  $\det(I - A_L) \neq 0$  and  $\det(I - A_R) \neq 0$ , as is usually the case, the pieces of  $g$  have the unique fixed points

$$\begin{aligned} x^L(\mu) &= (I - A_L)^{-1}b\mu, \\ x^R(\mu) &= (I - A_R)^{-1}b\mu. \end{aligned} \tag{6}$$

The point  $x^L(\mu)$  is admissible if  $x_1^L(\mu) < 0$  and virtual if  $x_1^L(\mu) > 0$ . Similarly,  $x^R(\mu)$  is admissible if  $x_1^R(\mu) > 0$  and virtual if  $x_1^R(\mu) < 0$ . We can now state the main result for maps.

**Theorem 1:** *If  $\det(I - A_L) \neq 0$ ,  $\det(I - A_R) \neq 0$ , and both fixed points of (5) are virtual, then every forward orbit of (5) diverges.*

In Sec. III D, we prove Theorem 1 by showing that for any suitable value of  $\mu$  there exists,  $s > 0$  and  $p \in \mathbb{R}^n$  such that

$$p^T g(x; \mu) \geq p^T x + s, \tag{7}$$

for all  $x \in \mathbb{R}^n$ . This shows that the value of  $p^T x$  increases without bound along forward orbits of  $g$ , hence all forward orbits diverge.

### B. A typical divergent orbit

As an example for which this can be seen graphically, consider (5) with

$$\begin{aligned} A_L &= \begin{bmatrix} \delta_L + 1 - \alpha & 1 \\ -\delta_L & 0 \end{bmatrix}, \\ A_R &= \begin{bmatrix} \delta_R + 1 + \alpha & 1 \\ -\delta_R & 0 \end{bmatrix}, \\ b &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \end{aligned} \tag{8}$$

where  $\delta_L, \delta_R \in \mathbb{R}$  and  $\alpha > 0$  are additional parameters. The fixed points are

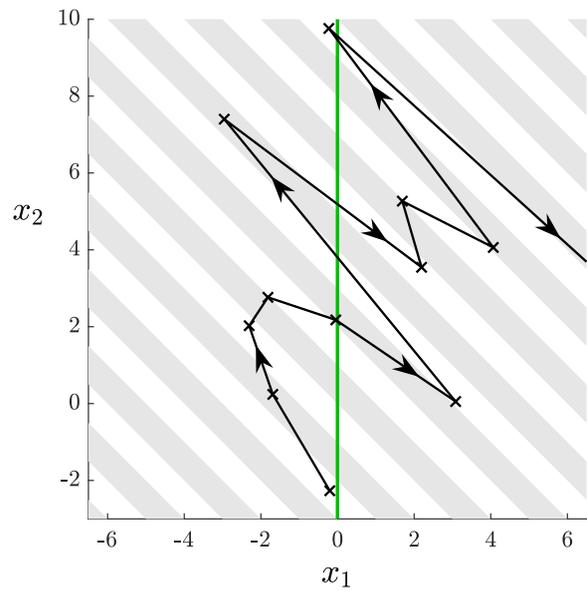
$$\begin{aligned} x^L(\mu) &= \left( \frac{\mu}{\alpha}, \frac{-\delta_L \mu}{\alpha} \right), \\ x^R(\mu) &= \left( \frac{-\mu}{\alpha}, \frac{\delta_R \mu}{\alpha} \right), \end{aligned}$$

so are both virtual when  $\mu > 0$ . Inequality (7) holds for all  $x \in \mathbb{R}^2$  using  $s = \mu$  and  $p^T = [1 \ 1]$  (obtained by applying formulas in Sec. III D to this example). Figure 7 uses  $\mu = 1$  and shades regions between integer values of  $p^T x$ . Inequality (7) implies that no two points of an orbit of  $g$  belong to the same region, and this is evident for the sample orbit shown.

### C. Bifurcation structures of piecewise-linear families

The ‘‘robust chaos’’ paper of Banerjee *et al.*<sup>32</sup> considered a four-parameter family of two-dimensional piecewise-linear maps of the form (5) with  $p^T b \mu = 1$  fixed. Their main result was identification of a parameter region where the fixed points  $x^L(\mu)$  and  $x^R(\mu)$  are admissible saddles and the map has a chaotic attractor. One boundary of this region is  $\det(I - A_L) = 0$ , where  $x^L(\mu)$  changes from admissible to virtual by (13). However, the attractor is unrelated to  $x^L(\mu)$  so, in fact, it persists beyond this boundary.<sup>33</sup>

Similarly, crossing  $\det(I - A_R) = 0$  rarely affects invariant sets other than  $x^R(\mu)$ . However, once  $\det(I - A_L) = 0$  and



**FIG. 7.** A forward orbit of the piecewise-linear map (5) with (8) and  $\delta_L = 1.2$ ,  $\delta_R = -2.4$ ,  $\alpha = 0.1$ , and  $\mu = 1$ . The shaded regions are bounded by integer values of  $p^T x = x_1 + x_2$ .

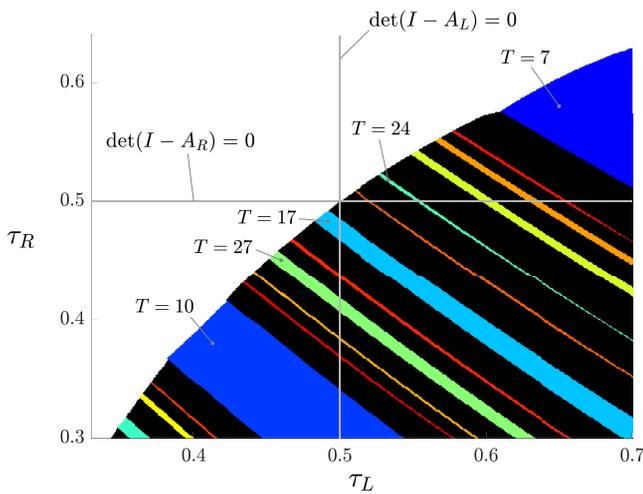
$\det(I - A_R) = 0$  have both been crossed, Theorem 1 shows that the map can no longer have an attractor or, in fact, any bounded invariant set. In this way, the boundaries  $\det(I - A_L) = 0$  and  $\det(I - A_R) = 0$  together have a global effect on the dynamics.

As an example, Fig. 8 shows how the attractor of the three-dimensional border-collision normal form varies with respect to two of its six parameters (white represents the absence of an attractor). This form consists of (5) with

$$\begin{aligned} A_L &= \begin{bmatrix} \tau_L & 1 & 0 \\ -\sigma_L & 0 & 1 \\ \delta_L & 0 & 0 \end{bmatrix}, \\ A_R &= \begin{bmatrix} \tau_R & 1 & 0 \\ -\sigma_R & 0 & 1 \\ \delta_R & 0 & 0 \end{bmatrix}, \\ b &= \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}. \end{aligned} \tag{9}$$

Notice an attractor exists for parameter values arbitrarily close to the corner of the top-left quadrant where no attractor exists by Theorem 1. Assuming Fig. 8 is a typical two-dimensional slice of six-dimensional parameter space, as appears to be the case from additional numerical explorations, we can make the following conclusion: Individually the boundaries  $\det(I - A_L) = 0$  and  $\det(I - A_R) = 0$  do not affect the attractor, but together they form a codimension-two surface where the attractor is destroyed.

A general theory for this phenomenon remains for future work. There does not always exist an attractor in a neighborhood such



**FIG. 8.** A two-parameter bifurcation diagram of (5) with (9) and  $\sigma_L = 0, \delta_L = 0.5, \sigma_R = 1,$  and  $\delta_R = 1.5$ . This diagram was produced by computing  $10^6$  iterates of the forward orbit of  $x = 0$  over a  $500 \times 500$  equi-spaced grid of  $(\tau_L, \tau_R)$  values and using the last few iterates to predict the existence of an attractor and its period. The numerics identified a stable period- $T$  solution with  $T \leq 50$  in the colored regions, a higher period or aperiodic solution in the black regions, and no attractor in the white regions.

a codimension-two point; this depends on other features of the dynamics. In scenarios for which an attractor does exist, its existence could be demonstrated formally by constructing a trapping region.<sup>34</sup> Presumably, the size (diameter) of the attractor necessarily tends to infinity at the codimension-two point in which case, it may be helpful to compactify phase space.<sup>35</sup>

**D. Proof of Theorem 1**

To prove Theorem 1, we use matrix adjugates.<sup>36,37</sup> The adjugate  $\text{adj}(A)$  of a matrix  $A \in \mathbb{R}^{n \times n}$  is defined entry-wise by  $\text{adj}(A)_{ij} = (-1)^{i+j} m_{ji}$ , where  $m_{ji}$  is the determinant of the  $(n - 1) \times (n - 1)$  matrix obtained by removing the  $j$ th row and  $i$ th column from  $A$ . Every  $A \in \mathbb{R}^{n \times n}$  satisfies

$$A \text{adj}(A) = \text{adj}(A)A = \det(A)I, \tag{10}$$

so, if  $A$  is invertible,

$$A^{-1} = \frac{\text{adj}(A)}{\det(A)}. \tag{11}$$

The first row of  $\text{adj}(A)$  is independent of the first column of  $A$ , so if two  $n \times n$  matrices differ in only their first columns (such as  $A_L$  and  $A_R$  above), the first rows of their adjugates will be the same. Algebraically, this means

$$e_1^T \text{adj}(A) = e_1^T \text{adj}(A + ce_1^T), \tag{12}$$

where  $c \in \mathbb{R}^n$  and  $e_1$  denotes the first standard basis vector of  $\mathbb{R}^n$ .

*Proof of Theorem 1.* By (11) applied to (6),

$$x_1^L(\mu) = \frac{p^T b \mu}{\det(I - A_L)}, \tag{13}$$

where we let

$$p^T = e_1^T \text{adj}(I - A_L)$$

denote the first row of  $\text{adj}(I - A_L)$ . By (12), this is also the first row of  $\text{adj}(I - A_R)$ , so

$$x_1^R(\mu) = \frac{p^T b \mu}{\det(I - A_R)}. \tag{14}$$

Let

$$s = p^T b \mu.$$

Notice  $s \neq 0$ , for otherwise, the fixed points are not virtual. Without loss of generality, suppose  $s > 0$ . Then,  $\det(I - A_L) > 0$  and  $\det(I - A_R) < 0$  because the fixed points are virtual.

For any  $x \in \mathbb{R}^n$  with  $x_1 \leq 0$ ,

$$\begin{aligned} p^T g(x; \mu) &= p^T (A_L x + b \mu) \\ &= p^T (-x + A_L x + x + b \mu) \\ &= -p^T (I - A_L)x + p^T x + s. \end{aligned}$$

By (10), we have  $p^T (I - A_L)x = \det(I - A_L)x_1$ . Importantly,  $\det(I - A_L) > 0$  and  $x_1 \leq 0$ , hence

$$p^T g(x; \mu) \geq p^T x + s. \tag{15}$$

Similarly, for any  $x \in \mathbb{R}^n$  with  $x_1 \geq 0$ ,

$$p^T g(x; \mu) = -p^T (I - A_R)x + p^T x + s,$$

where  $p^T (I - A_R)x = \det(I - A_R)x_1 \leq 0$  because  $\det(I - A_R) < 0$ , so again we have (15). Thus, for any forward orbit of  $g$ , the value of  $p^T x$  increases by at least  $s > 0$  every iteration, so the orbit diverges.  $\square$

**E. The effect of higher order terms**

The quantities  $E^L$  and  $E^R$  in (4) that were omitted to produce the truncated form (5) are higher order. To be precise,  $\frac{\|E^L(x; \mu)\|}{\|x\| + |\mu|} \rightarrow 0$  and  $\frac{\|E^R(x; \mu)\|}{\|x\| + |\mu|} \rightarrow 0$  as  $(x; \mu) \rightarrow (0; 0)$ , where  $\|\cdot\|$  denotes the Euclidean norm on  $\mathbb{R}^n$ . For this reason, if (5) satisfies the conditions of Theorem 1, we expect (4) has no local invariant set on one side of the nonsmooth fold. The following conjecture formalizes this claim. Without loss of generality, we suppose  $p^T b > 0$  and consider  $\mu > 0$  in which case  $\det(I - A_L) > 0$  and  $\det(I - A_R) < 0$  are needed for both fixed points to be virtual by (13) and (14). We write  $B_\eta = \{x \in \mathbb{R}^n \mid \|x\| \leq \eta\}$  for the closed ball of radius  $\eta > 0$  centered at the origin.

**Conjecture 2:** *Let  $f$  be a map of the form (4) with  $\det(I - A_L) > 0, \det(I - A_R) < 0,$  and  $p^T b > 0$ . Then, there exists  $\mu_0 > 0$  and  $\eta > 0$  such that for all  $\mu \in (0, \mu_0)$  and all  $x \in B_\eta$ , there exists  $m \geq 1$  such that  $\|f^m(x; \mu)\| > \eta$ .*

That is,  $f$  has no invariant set in  $B_\eta$  for all  $\mu \in (0, \mu_0)$ . To illustrate Conjecture 2, we add the quadratic  $E^L(x; \mu) = E^R(x; \mu) = \begin{bmatrix} 0 \\ -x_2^2 \end{bmatrix}$  to our previous example of (5) with (8) to obtain the map

$$f(x; \mu) = \begin{cases} \begin{bmatrix} (\delta_L + 1 - \alpha)x_1 + x_2 + \mu \\ -\delta_L x_1 - x_2^2 \end{bmatrix}, & x_1 \leq 0, \\ \begin{bmatrix} (\delta_R + 1 + \alpha)x_1 + x_2 + \mu \\ -\delta_R x_1 - x_2^2 \end{bmatrix}, & x_1 \geq 0. \end{cases} \quad (16)$$

Conjecture 2 is true for this map with the parameter values of Fig. 7, but to prove this, it is insufficient to use the linear function  $p^T x = x_1 + x_2$ . This is because

$$p^T f(x; \mu) - p^T x = \begin{cases} \mu - \alpha x_1 - x_2^2, & x_1 \leq 0, \\ \mu + \alpha x_1 - x_2^2, & x_1 \geq 0, \end{cases}$$

so for any  $\eta > 0$  and  $\mu \in (0, \eta^2)$ , at the point  $x = (0, \eta) \in B_\eta$  the difference  $p^T f(x; \mu) - p^T x = \mu - \eta^2$  is negative. To prove Conjecture 2 for (16), we can instead use  $\Phi(x) = x_1 + x_2 - 2x_2^2$ , as shown in the Appendix. It remains for future work to show that such a function  $\Phi$  can always be constructed, prove Conjecture 2 in different way, or find a counterexample.

#### IV. ODEs

In this section, we consider BEBs of piecewise-smooth continuous ODEs, Filippov systems, and hybrid systems, in that order. In each setting, we construct a truncated form by discarding higher order terms and prove the main result for this form. For piecewise-smooth continuous ODEs, the proof follows closely to that given above for maps. For Filippov systems, we need to perform some calculations of the sliding vector field, while for hybrid systems, we need to use the sticking vector field. The nuances of the result described above for maps (i.e., bifurcation structures and the effect of higher order terms) are expected to occur analogously for the three ODE settings.

##### A. Piecewise-smooth continuous ODEs

We consider  $n$ -dimensional ODE systems of the form

$$\dot{x} = \begin{cases} f^L(x; \mu), & x_1 \leq 0, \\ f^R(x; \mu), & x_1 \geq 0, \end{cases} \quad (17)$$

where  $f^L$  and  $f^R$  are  $C^1$  and  $f^L(x; \mu) = f^R(x; \mu)$  at all points on the switching manifold  $x_1 = 0$ . Suppose (17) has a BEB at  $x = 0$  when  $\mu = 0$ . Then, by replacing  $f^L$  and  $f^R$  with their linearizations, we obtain

$$\dot{x} = \begin{cases} A_L x + b\mu, & x_1 \leq 0, \\ A_R x + b\mu, & x_1 \geq 0, \end{cases} \quad (18)$$

where  $A_L, A_R \in \mathbb{R}^{n \times n}$  differ in only their first columns (by continuity) and  $b \in \mathbb{R}^n$ . Solutions to (18) are well-defined for all  $x \in \mathbb{R}^n$  and  $t \in \mathbb{R}$  because (18) is Lipschitz and each piece is linear.

If  $\det(A_L) \neq 0$  and  $\det(A_R) \neq 0$ , the pieces of (18) have the unique equilibria

$$\begin{aligned} x^L(\mu) &= -A_L^{-1} b\mu, \\ x^R(\mu) &= -A_R^{-1} b\mu. \end{aligned} \quad (19)$$

The equilibrium  $x^L(\mu)$  is admissible if  $x_1^L(\mu) < 0$  and virtual if  $x_1^L(\mu) > 0$ , while  $x^R(\mu)$  is admissible if  $x_1^R(\mu) > 0$  and virtual if  $x_1^R(\mu) < 0$ .

**Theorem 3:** *If  $\det(A_L) \neq 0$ ,  $\det(A_R) \neq 0$ , and both equilibria of (18) are virtual, then every solution to (18) diverges as  $t \rightarrow \pm\infty$ .*

*Proof.* By (11) applied to (19),

$$x_1^L(\mu) = -\frac{q^T b\mu}{\det(A_L)}, \quad (20)$$

where we let

$$q^T = e_1^T \text{adj}(A_L).$$

By (12), this is also the first row of  $\text{adj}(A_R)$ , so

$$x_1^R(\mu) = -\frac{q^T b\mu}{\det(A_R)}. \quad (21)$$

The value

$$s = q^T b\mu$$

cannot be zero because the equilibria are virtual. Without loss of generality suppose  $s > 0$ , then  $\det(A_L) < 0$  and  $\det(A_R) > 0$ .

For any  $x \in \mathbb{R}^n$  with  $x_1 \leq 0$ ,

$$q^T f^L(x; \mu) = q^T (A_L x + b\mu) = q^T A_L x + s.$$

Notice  $q^T A_L x = \det(A_L) x_1 \geq 0$ , thus

$$q^T \dot{x} \geq s. \quad (22)$$

Similarly, for any  $x \in \mathbb{R}^n$  with  $x_1 \geq 0$ ,

$$q^T f^R(x; \mu) = q^T (A_R x + b\mu) = q^T A_R x + s,$$

where  $q^T A_R x = \det(A_R) x_1 \geq 0$ , so again we have (22). Thus, any solution  $\phi(t)$  to (18) has  $\frac{d}{dt} q^T \phi(t) \geq s > 0$  for all  $t \in \mathbb{R}$  and so  $\|\phi(t)\| \rightarrow \infty$  as  $t \rightarrow \pm\infty$ .  $\square$

##### B. Filippov systems

Here, we treat systems

$$\dot{x} = \begin{cases} f^L(x; \mu), & x_1 < 0, \\ f^R(x; \mu), & x_1 > 0, \end{cases} \quad (23)$$

where again  $f^L$  and  $f^R$  are  $C^1$ , but now (23) is not necessarily continuous on the switching manifold  $\Sigma = \{x \in \mathbb{R}^n \mid x_1 = 0\}$ . To describe the dynamics of (23), it is useful to partition  $\Sigma$  into regions throughout which  $f^L$  and  $f^R$  each either direct solutions into  $\Sigma$  or away from  $\Sigma$ .

**Definition 1:** Consider a system (23) with fixed  $\mu \in \mathbb{R}$ . A subset  $S \subset \Sigma$  is

- (1) a *crossing region* if  $f_1^L(x; \mu) f_1^R(x; \mu) > 0$  for all  $x \in S$  and
- (2) a *sliding region* if  $f_1^L(x; \mu) f_1^R(x; \mu) < 0$  for all  $x \in S$ .

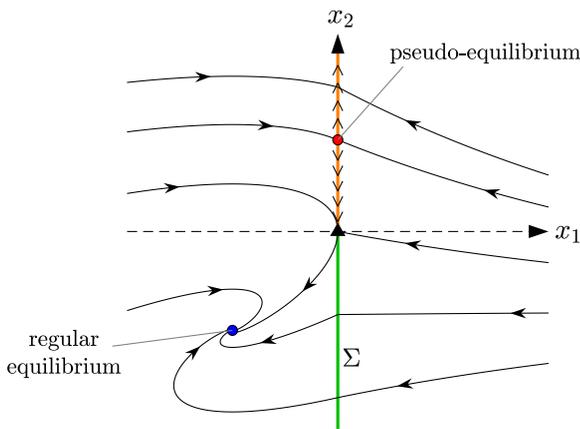


FIG. 9. A schematic phase portrait of a two-dimensional Filippov system of the form (23).

A sliding region is *attracting* if  $f_1^L(x; \mu) > 0$  and  $f_1^R(x; \mu) < 0$  and *repelling* if  $f_1^L(x; \mu) < 0$  and  $f_1^R(x; \mu) > 0$ .

For example, in Fig. 9, the top half of  $\Sigma$  is an attracting sliding region, while the bottom half of  $\Sigma$  is a crossing region.

To define solutions on sliding regions, we use Filippov’s convention.<sup>17,38</sup> That is, solutions on sliding regions evolve according to  $\dot{x} = f^S(x; \mu)$ , where  $f^S$  is the *sliding vector field*

$$f^S(x; \mu) = \frac{f_1^L(x; \mu)f^R(x; \mu) - f_1^R(x; \mu)f^L(x; \mu)}{f_1^L(x; \mu) - f_1^R(x; \mu)}, \quad (24)$$

defined as the unique convex combination of  $f^L$  and  $f^R$  that is tangent to  $\Sigma$ . With this convention (23) is termed a *Filippov system*. Solutions are termed *Filippov solutions* and are concatenations of smooth segments of motion in  $x_1 < 0$  under  $f^L$ , in  $x_1 > 0$  under  $f^R$ , and on sliding regions under  $f^S$ .

Equilibria of  $f^L$  and  $f^R$  are admissible or virtual as in Sec. IV A. They are sometimes called regular equilibria to distinguish them from equilibria of  $f^S$ :

**Definition 2:** A point  $x \in \Sigma$  is a *pseudo-equilibrium* of (23) if  $f^S(x; \mu) = 0$ . It is admissible if it belongs to a sliding region and virtual if it belongs to a crossing region.

Now, suppose (23) has a BEB caused by an equilibrium of  $f^L$  hitting  $\Sigma$  at  $x = 0$  when  $\mu = 0$ . Then,  $f^L(0; 0) = 0$ , but unlike in Sec. IV A, typically  $f^R(0; 0) \neq 0$ . By replacing  $f^L(x; \mu)$  with its linearization about  $(x; \mu) = (0; 0)$  and  $f^R(x; \mu)$  with its value at  $(x; \mu) = (0; 0)$ , we obtain the truncated form

$$\dot{x} = \begin{cases} Ax + b\mu, & x_1 < 0, \\ c, & x_1 > 0, \end{cases} \quad (25)$$

where  $A \in \mathbb{R}^{n \times n}$  and  $b, c \in \mathbb{R}^n$ . If  $\det(A) \neq 0$ , then  $f^L$  has the unique equilibrium

$$x^L(\mu) = -A^{-1}b\mu, \quad (26)$$

which is admissible if  $x_1^L < 0$  and virtual if  $x_1^L > 0$ . If  $c_1 \neq 0$  and  $q^T c \neq 0$ , where again

$$q^T = e_1^T \text{adj}(A),$$

then (25) has a unique pseudo-equilibrium  $x^S(\mu)$  with  $x^S(0) = 0$ .<sup>8,12</sup>

**Theorem 4:** If  $\det(A) \neq 0$ ,  $c_1 \neq 0$ ,  $q^T c \neq 0$ , and  $x^L(\mu)$  and  $x^S(\mu)$  are virtual, then every Filippov solution to (25) diverges as  $t \rightarrow \infty$ .

Let us first give some intuition behind Theorem 4. As in the previous proof, below we obtain  $q^T f^L(x; \mu) > 0$  for points with  $x_1 \leq 0$ ; but instead of  $f^R$ , we work with  $f^S$  because the second equilibrium is a zero of  $f^S$  not  $f^R$ . Filippov solutions switch from following  $f^S$  to  $f^L$  by passing through points  $x \in \Sigma$  for which  $f_1^L(x; \mu) = 0$  and notice  $f^S(x; \mu) = f^L(x; \mu)$  at such points by (24). This is a type of continuity constraint that together with the assumption that  $x^S(\mu)$  is virtual leads to the inequality  $q^T f^S(x; \mu) > 0$  throughout sliding regions and the conclusion that all Filippov solutions diverge.

*Proof.* By (11) applied to (26),

$$x_1^L(\mu) = -\frac{q^T b \mu}{\det(A)}. \quad (27)$$

Thus, the admissibility of  $x^L(\mu)$  is determined by the signs of  $\det(A)$  and

$$s = q^T b \mu.$$

Notice  $s \neq 0$  because  $x^L(\mu)$  is assumed to be virtual.

We now derive an analogous characterization for the admissibility of  $x^S(\mu)$ . Since  $x^S(\mu)$  is zero of  $f^S$ ,

$$f_1^L(x^S(\mu); \mu)f^R(x^S(\mu); \mu) - f_1^R(x^S(\mu); \mu)f^L(x^S(\mu); \mu) = 0.$$

By multiplying both sides of this equation by  $q^T$ , then solving for  $f_1^L$ , we obtain

$$f_1^L(x^S(\mu); \mu) = \frac{f_1^R(x^S(\mu); \mu)q^T f^L(x^S(\mu); \mu)}{q^T f^R(x^S(\mu); \mu)}. \quad (28)$$

Notice  $f^R(x^S(\mu); \mu) = c$  and

$$\begin{aligned} q^T f^L(x^S(\mu); \mu) &= q^T (Ax^S(\mu) + b\mu) \\ &= \det(A)x_1^S(\mu) + s \\ &= s, \end{aligned}$$

where  $x_1^S(\mu) = 0$  because  $x^S(\mu) \in \Sigma$ . Inserting these into (28) produces

$$f_1^L(x^S(\mu); \mu) = \frac{sc_1}{q^T c}.$$

Thus, the admissibility of  $x^S(\mu)$  is determined by the sign of

$$f_1^L(x^S(\mu); \mu)f_1^R(x^S(\mu); \mu) = \frac{sc_1^2}{q^T c}. \quad (29)$$

Without loss of generality, suppose  $s > 0$ . Then,  $\det(A) < 0$  because  $x^L(\mu)$  is virtual, and  $q^T c > 0$  because  $x^S(\mu)$  is virtual. For any  $x \in \mathbb{R}^n$  with  $x_1 \leq 0$ ,

$$q^T f^L(x; \mu) \geq s,$$

as in Sec. IV A. For any  $x \in \Sigma$  for which the denominator of (24) is

nonzero,

$$q^T f^s(x; \mu) = \frac{f_1^L(x; \mu) q^T c - c_1 q^T f^L(x; \mu)}{f_1^L(x; \mu) - c_1} = \frac{s + r(x; \mu) q^T c}{1 + r(x; \mu)},$$

where  $q^T f^L(x; \mu) = s$  because  $x_1 = 0$ , and we have let  $r(x; \mu) = -\frac{f_1^L(x; \mu)}{c_1}$ . If  $f_1^L(x; \mu) c_1 \leq 0$ , i.e.,  $x$  does not belong to a crossing region, then  $r(x; \mu) \geq 0$ . Notice  $q^T f^s(x; \mu)$  ranges monotonically from  $s > 0$  at  $r = 0$  to  $q^T c > 0$  as  $r \rightarrow \infty$ . Thus, at all  $x \in \Sigma$  not on a crossing region the denominator of (24) is nonzero and

$$q^T f^s(x; \mu) \geq \min(s, q^T c) > 0.$$

Finally, consider the behavior of an arbitrary Filippov solution  $\phi(t)$  to (25) as  $t \rightarrow \infty$ . If  $c_1 < 0$  then  $\phi(t)$  eventually evolves exclusively under  $f^L$  and  $f^s$  and diverges, while if  $c_1 > 0$ , then  $\phi(t)$  either evolves exclusively under  $f^L$  and  $f^s$  and diverges or eventually evolves exclusively under  $f^R$  and diverges.  $\square$

### C. Impacting hybrid systems

Finally, we consider hybrid systems of the form

$$\begin{aligned} \dot{x} &= f(x; \mu), \quad \text{for } x_1 < 0, \\ x &\mapsto g(x; \mu), \quad \text{whenever } x_1 = 0, \end{aligned} \tag{30}$$

where  $f$  and  $g$  are  $C^1$ . The following assumptions are motivated by the view that  $x_1(t) < 0$  represents the displacement of a rigid object relative to a wall located at  $x_1 = 0$ . The map  $g$  is a reset law that models impacts as instantaneous events with velocity reversal and possibly energy loss.

Let  $\Sigma = \{x \in \mathbb{R}^n \mid x_1 = 0\}$  denote the switching manifold, and

$$v(x; \mu) = f_1(x; \mu) \tag{31}$$

denote the velocity of the object relative to the wall. We partition  $\Sigma$  into the *incoming set*  $\Sigma_{in} = \{x \in \Sigma \mid v(x; \mu) > 0\}$ , the *outgoing set*  $\Sigma_{out} = \{x \in \Sigma \mid v(x; \mu) < 0\}$ , and the *grazing set*  $\Gamma = \{x \in \Sigma \mid v(x; \mu) = 0\}$  (Fig. 10). We assume  $g$  maps  $\Sigma_{in}$  to  $\Sigma_{out}$  and is the identity map on  $\Gamma$ , thus

$$g(x; \mu) = x + v(x; \mu)h(x; \mu), \tag{32}$$

for some  $C^1$  function  $h$ . The following definition uses

$$a(x; \mu) = \nabla v(x; \mu)^T f(x; \mu), \tag{33}$$

which represents the acceleration of the object relative to the switching manifold.

**Definition 3:** Consider a system (30) with fixed  $\mu \in \mathbb{R}$ . A subset  $S \subset \Gamma$  is

- (1) a *sticking region* if  $a(x; \mu) > 0$  for all  $x \in S$  and
- (2) a *detaching region* if  $a(x; \mu) < 0$  for all  $x \in S$ .

Sticking regions and detaching regions are analogous to sliding regions and crossing regions of Filippov systems. On sticking

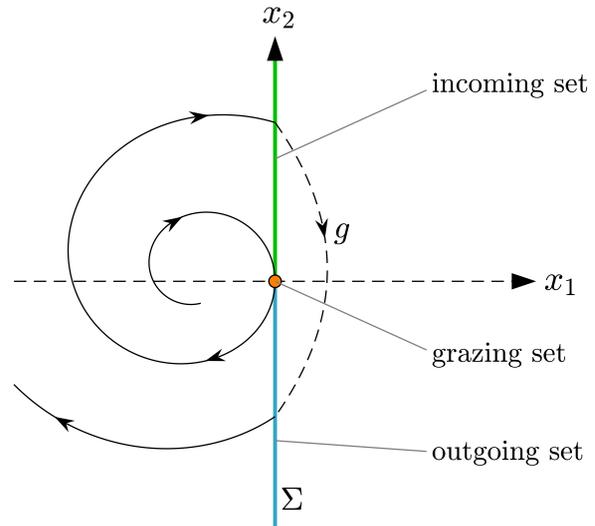


FIG. 10. A schematic phase portrait of a two-dimensional hybrid system of the form (30).

regions, it is natural to define the sticking vector field

$$f^{st}(x; \mu) = \left( I - \frac{h(x; \mu) \nabla v(x; \mu)^T}{\nabla v(x; \mu)^T h(x; \mu)} \right) f(x; \mu), \tag{34}$$

given by projecting  $f$  onto  $\Gamma$  by translating in the direction  $h$  (see Sec. 2.2.4 of di Bernardo *et al.*<sup>4</sup>). Solutions to (30) are concatenations of smooth segments of motion in  $x_1 < 0$  under  $f$ , instantaneous jumps defined by  $g$ , and smooth segments of motion on sticking regions under  $f^{st}$ . Regular equilibria of (30) are zeros of  $f$ ; pseudo-equilibria of (30) are zeros of  $f^{st}$ :

**Definition 4:** A point  $x \in \Gamma$  is a *pseudo-equilibrium* of (30) if  $f^{st}(x; \mu) = 0$ . It is admissible if it belongs to a sticking region, and virtual if it belongs to a detaching region.

Now, suppose (30) has a BEB caused by a regular equilibrium hitting  $\Sigma$  at  $x = 0$  when  $\mu = 0$ . By replacing  $f(x; \mu)$  with its linearization about  $(x; \mu) = (0; 0)$ , and  $h(x; \mu)$  with its value at  $(x; \mu) = (0; 0)$ , we obtain the truncated form

$$\begin{aligned} \dot{x} &= Ax + b\mu, \quad \text{for } x_1 < 0, \\ x &\mapsto x + v(x; \mu)c, \quad \text{whenever } x_1 = 0, \end{aligned} \tag{35}$$

where  $A \in \mathbb{R}^{n \times n}$  and  $b, c \in \mathbb{R}^n$  with  $c_1 = 0$ ; also

$$v(x; \mu) = e_1^T (Ax + b\mu). \tag{36}$$

The regular equilibrium  $x^L(\mu)$  is given by (26), assuming  $\det(A) \neq 0$ . Note that some authors prefer an equivalent truncated form for which the regular equilibrium is fixed at the origin.<sup>39</sup>

Let  $x' = x + v(x; \mu)c$  denote the image of the reset law. Then,

$$v(x'; \mu) = (1 + e_1^T A c) v(x; \mu),$$

and hence the condition

$$e_1^T A c < -1 \tag{37}$$

ensures the reset law maps the incoming set to the outgoing set. If  $q^T c \neq 0$ , then (35) has a unique pseudo-equilibrium  $x^{St}(\mu)$  with  $x^{St}(0) = 0$ .<sup>3</sup>

**Theorem 5:** *If  $\det(A) \neq 0$ ,  $e_1^T A c < -1$ ,  $q^T c \neq 0$ , and  $x^L(\mu)$  and  $x^{St}(\mu)$  are virtual, then every solution to (35) diverges as  $t \rightarrow \infty$ .*

Notice  $f^{St}(x; \mu) = f(x; \mu) - \frac{a(x; \mu)h(x; \mu)}{\nabla v(x; \mu)^T h(x; \mu)}$ , so at any point  $x \in \Gamma$  for which  $a(x; \mu) = 0$ , we have  $f^{St}(x; \mu) = f(x; \mu)$ . This continuity underpins the following proof. The proof is similar to that of Sec. IV B except we also need to consider the action of the reset law relative to the direction  $q$  [see (43)].

*Proof.* We first characterize the admissibility of  $x^{St}(\mu)$ . Observe  $h(x; \mu) = c$  and  $\nabla v(x; \mu)^T = e_1^T A$ , so the sticking vector field (34) can be written as

$$f^{St}(x; \mu) = \frac{1}{e_1^T A c} (e_1^T A c f(x; \mu) - a(x; \mu) c). \tag{38}$$

Since  $x^{St}(\mu)$  is a zero of  $f^{St}$ ,

$$0 = e_1^T A c f(x^{St}(\mu); \mu) - a(x^{St}(\mu); \mu) c. \tag{39}$$

By multiplying both sides of this equation by  $q^T = e_1^T \text{adj}(A)$ , then solving for  $a$ , we obtain

$$a(x^{St}(\mu); \mu) = \frac{e_1^T A c q^T f(x^{St}(\mu); \mu)}{q^T c}.$$

But  $x_1^{St}(\mu) = 0$ , so

$$\begin{aligned} q^T f(x^{St}(\mu); \mu) &= q^T (A x^{St}(\mu) + b \mu) \\ &= \det(A) x_1^{St}(\mu) + q^T b \mu \\ &= s, \end{aligned}$$

where we again let  $s = q^T b \mu$ , hence

$$a(x^{St}(\mu); \mu) = \frac{e_1^T A c s}{q^T c}. \tag{40}$$

The first component of  $x^L(\mu)$  is given by (27), so we require  $s \neq 0$  because  $x^L(\mu)$  is virtual. Without loss of generality, suppose  $s > 0$ , then  $\det(A) < 0$ . Also,  $x^{St}(\mu)$  is virtual, meaning  $a(x^{St}(\mu); \mu) < 0$ , thus by (40) we have  $q^T c > 0$  (notice  $e_1^T A c < 0$  by the theorem statement).

For any  $x \in \mathbb{R}^n$  with  $x_1 \leq 0$ ,

$$q^T f(x; \mu) \geq s, \tag{41}$$

as in the proof of Theorem 3. By (38), for any  $x \in \Gamma$ ,

$$q^T f^{St}(x; \mu) = q^T f(x; \mu) - \frac{a(x; \mu) q^T c}{e_1^T A c}.$$

So since  $q^T c > 0$  and  $e_1^T A c < 0$ , for any  $x \in \Gamma$  with  $a(x; \mu) \geq 0$ , i.e., not in a detaching region, we have

$$q^T f^{St}(x; \mu) \geq s, \tag{42}$$

using also (41). Finally, for any  $x \in \Sigma_{in}$ , let  $x' = x + v(x; \mu) c$  and observe

$$q^T (x' - x) = v(x; \mu) q^T c > 0, \tag{43}$$

because  $v(x; \mu) > 0$ .

By (41)–(43), any solution  $\phi(t)$  to (35) satisfies  $\frac{d}{dt} q^T \phi(t) \geq s > 0$  while in  $x_1 < 0$  and on sticking regions, plus whenever the reset law is applied the value of  $q^T \phi(t)$  increases, hence  $q^T \phi(t) \rightarrow \infty$  as  $t \rightarrow \infty$ , so the solution diverges.  $\square$

## V. DISCUSSION

We have considered previously established truncated forms for BEBs and border-collision bifurcations and proved that if two equilibria or fixed points are virtual then all orbits diverge.

The terms that were discarded to reach the truncated forms are higher order. Consequently, we expect these terms to be incapable of generating a local invariant set on the side of the bifurcation where both equilibria or fixed points are virtual, and this was formulated as Conjecture 2 in the case of the maps. If this is true then for general piecewise-smooth systems, nonsmooth folds involving a stable equilibrium or fixed point universally act as tipping points when the bifurcation parameter is varied slowly in time. To prove Conjecture 2, it would be sufficient to construct a function  $\Phi(x)$  with the property that  $\Phi(f(x; \mu)) - \Phi(x) \geq \mu$  for all sufficiently small  $x \in \mathbb{R}^n$  and  $\mu > 0$ . The problem of finding  $\Phi$  bears similarities to the problem of identifying a Lyapunov function to verify the stability of an equilibrium or fixed point. For systems with state-dependent switching, this is an on-going area of research.<sup>40–44</sup>

Also, we showed numerically that for piecewise-linear families of maps the condition for both fixed points to be virtual can be a codimension-two boundary for the existence of an attractor. This phenomenon should occur analogously in the ODE settings. We expect this observation to be useful when performing bifurcation analyses of the truncated forms.

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## AUTHOR DECLARATIONS

### Conflict of Interest

The authors have no conflicts to disclose.

### Author Contributions

**D. J. W. Simpson:** Conceptualization (equal); Formal analysis (equal); Funding acquisition (equal); Investigation (equal); Methodology (equal); Project administration (equal); Software (equal); Validation (equal); Visualization (equal); Writing – original draft (equal); Writing – review & editing (equal).

## DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

## APPENDIX: DETAILS OF THE HIGHER ORDER TERMS EXAMPLE

Here, we prove Conjecture 2 for the example map (16). Specifically, we show the following.

**Proposition 6:** Consider the map (16) with  $\delta_L > 0$ ,  $\delta_R < 0$ , and  $\alpha > 0$ . Then, there exists  $\eta > 0$  such that for any  $\mu > 0$  and  $x \in B_\eta$ , there exists positive  $m \leq \frac{6\eta}{\mu}$  such that  $\|f^m(x; \mu)\| > \eta$ .

*Proof.* Let  $\eta = \min\left(\frac{\alpha}{2\delta_L^2}, \frac{\alpha}{2\delta_R^2}, \frac{1}{\sqrt{2}}\right)$  and  $\Phi(x) = x_1 + x_2 - 2x_2^2$ . Notice  $-4\eta \leq \Phi(x) \leq 2\eta$  for any  $x \in B_\eta$ , so over  $B_\eta$ , the value of  $\Phi(x)$  ranges by at most  $6\eta$ . Using (16), direct calculations give

$$\begin{aligned} \Phi(f(x; \mu)) - \Phi(x) &= \begin{cases} \mu - (\alpha + 2\delta_L^2 x_1) x_1 - 4\delta_L x_1 x_2^2 + (1 - 2x_2^2) x_2^2, & x_1 \leq 0, \\ \mu + (\alpha - 2\delta_R^2 x_1) x_1 - 4\delta_R x_1 x_2^2 + (1 - 2x_2^2) x_2^2, & x_1 \geq 0. \end{cases} \end{aligned} \quad (\text{A1})$$

Notice  $\Phi(f(x; \mu)) - \Phi(x) \geq \mu$  for any  $\mu > 0$  and  $x \in B_\eta$ , using  $\delta_L > 0$ ,  $\delta_R < 0$ , and  $\alpha > 0$ . So, in  $B_\eta$ , the value of  $\Phi$  increases by at least  $\mu$  under every iteration of  $f$ , thus forward orbits of  $f$  escape  $B_\eta$  within at most  $\frac{6\eta}{\mu}$  iterations.  $\square$

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