Intuitive Transformation Geometry and Frieze Patterns

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Abstract

This study on intuitive frieze pattern construction and description was set up as an attempt to answer part of a general question: "Do students bring intuitive transformation geometry concepts with them into the classroom and, if so, what is the character of those concepts?" The motivation to explore this topic arose, in part, from the particular relevance that transformation geometry has to New Zealand: *kowhaiwhai* (Maori rafter patterns) are examples of frieze patterns and are suggested by recent mathematics curriculum documents as a way for Form 3 and 4 students to explore transformations.

When very few restrictions were put on the subjects, frieze patterns made by Standard 3 and 4 students displayed evidence of the use of transformations such as translation, vertical reflection, and half-turn. Transformations, such as horizontal reflection and glide reflection, were very rarely used by themselves. However, from the frieze group analysis alone, no strong conclusions could be drawn about the frieze patterns featuring a combination of two or more different symmetry types (besides translation). The Form 4 class surveyed showed similar results, with an increase in the proportion of students using half-turn by itself. Another contrast between the two age groups was the production of disjoint and connected patterns: the Primary students' patterns were mostly disjoint, whereas the Secondary students made almost equal numbers of disjoint and connected designs.

In a restricted frieze construction activity, which required the subjects to use asymmetric objects (right-angled scalene triangles), the use of non-translation transformations reduced considerably from the first exercise, although vertical reflection was still popular amongst 70% of the Primary students. However, the results of a small survey of 10 children suggested that if the strips to be filled in are aligned vertically, the rarer symmetries such as glide reflection may be used more easily than in the horizontal case.

The style analysis revealed that the Primary (pre-formal) and Tertiary (post-formal) groups were quite similar in the patterns they drew under the restricted conditions, and therefore in the probable construction methods used to produce them. The Form 4's patterns differed in several ways, especially by their extensive use of half turn and tilings. It seems that the Fourth Form students were affected by the formal transformation geometry framework to which they had been recently exposed.

Interviews of 10 Primary students provided information about the intentions and methods used to construct the frieze patterns under both restricted and unrestricted conditions. The case studies revealed that several standard approaches to frieze pattern construction were employed, none of which corresponded with the mathematical structure of a symmetry group. It was also found that a number of methods could be used to make the same pattern. The qualitative analysis highlighted some shortfalls of the quantitative approach. For example, some students used transformations not detected by the frieze group analysis, and some symmetries present in the children's patterns were incidental (a spin-off of another motivation) or accidental. Ambiguities in pattern classification also arose.

The Primary children's descriptions of the seven different frieze groups (which were discrete examples) displayed several characteristic features. For instance, they often used a form of simile or metaphor, comparing a pattern part to a real world object with the same set of symmetries. In addition, many children considered a pattern's translation unit to be 'the pattern'. In this case, the interviews suggested that the repetition (translation) was obvious to the students. Also interesting was the tendency of these subjects to write down orientation or direction judgements, omitting the relationships between adjacent congruent figures within a pattern. However, the Primary children did use more explicit transformation terminology when able to describe the patterns orally. A peculiar feature of these explanations was that the symmetry described was often not differentiable from another symmetry. For example, to a child, the phrase "turn upside down" can mean a half-turn or a horizontal reflection or both; the result *is* identical in many cases.

Secondary and Tertiary students tended not to use implicit phrases in their pattern descriptions but were more explicit and precise, using a wider range of criteria in their descriptions. The results from this activity also indicated that the Primary and older students alike did not perceive the patterns to extend infinitely beyond the confines of the the page, highlighting another difference between the mathematical structure of a symmetry group and the intuitive cognitive processes of the students.

An additional matching activity was conducted in the interviews, requiring the subjects to match various pairs of frieze patterns and discuss the similarities they saw. It appeared that transformation criteria were not verbalized predominantly over other criteria such as orientation or direction judgements, although many matches were made between patterns with the same underlying frieze group.

Finally, educational implications for mathematics were indicated and areas for further research were suggested.

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1 *Introduction*

1.1 An Explanation of the Topic

Aims

This thesis addresses the question of how students of various ages perceive, or make, frieze patterns. The purpose of this research is to decide whether *intuitive* transformation geometry concepts form a component of either of these processes and, if so, to what extent. Consequently, the main objective of this mathematics education study is to identify and describe the character of the conceptualization and utilization of transformation geometry in students' description or construction of frieze designs. In particular, we consider this conceptualization and utilization as an element of 'geometrical intuition', which is a perceptual function by which a person apprehends spatial relationships *independent of a formal geometry framework*. To outline the nature of this topic, there are three key phrases which probably deserve further explanation: *transformation geometry*, *frieze patterns*, and *intuition*. These terms are discussed in the three subsections which follow.

1.1.1 Transformation Geometry

Background

Transformation, or 'motion', geometry was secured on firm mathematical ground in the 1870's when Felix Klein and Sophus Lie produced their version of it. In one sense, it can be considered as a refashioning of Euclidean (Sinha, 1986) and other geometries. In hindsight, this progression seems to have been quite natural. For instance, David Hilbert praised Euclid for his foresight in perceiving that 'motion' is a prerequisite for establishing the congruence of two figures (Sinha, 1986).

Rosenfeld (1988) reported that, in 1872, Felix Klein presented a lecture outlining the *Erlangen Program*, entitled (in English) *Comparative Overview of Recent Geometric Investigations*. The types of motions which he considered varied from rigid, affine, and projective transformations to inversive, circular and conformal transformations. Klein noticed that such transformations form groups[1] under composition. His emphasis was therefore on groups of transformations of space or manifolds, and the geometric properties of spatial figures. In this present study, however, the consideration of transformations is generally restricted to the 'rigid' or congruence transformations of the plane, as well as a variety of associated groups. This begs the question: what is a rigid transformation?

Rigid Transformations

Loosely, a rigid transformation is a motion of the plane which doesn't change the size or shape of figures within that plane. However, from a mathematical point of view, a transformation is a *mapping* which describes the relationship of points and their images; the idea of a motion is informal. Martin (1982) showed that there are only four types of rigid 'motions': a *reflection* about a mirror line, a *rotation* about a point, a *translation* in the direction and length of a vector, and a *glide reflection* about a line. The transformation most likely to be unfamiliar to the reader is the glide reflection. This 'motion' can be understood as the composition of a reflection and a translation, although it is a transformation in its own right. If this seems somewhat unexpected or contrived, it may be helpful to remember that a rotation (or a translation) can both be thought of as a product of two reflections. For more formal definitions of these transformations, see section 2.4.

If a transformation maps a set of points onto itself (so that it appears unchanged), it is called a *symmetry* of that set of points. As a consequence, there are four types of symmetry associated with the four types of transformations, which is contrary to a popular view that symmetry is synonymous with reflection symmetry.

1.1.2 Frieze Patterns

Until their own work was published, Grünbaum and Shephard (1987) explained that the term *pattern* had not been defined, even by mathematicians, in a lucid and useful way. For the purposes of this thesis, the word pattern is employed very broadly in its popular use, that is, as some sort of 'regular design'. (A *design* is taken to mean any set of points in the plane). Unlike Grünbaum and Shephard's (1987) definition, tilings (partitions of the plane into regions) are considered to be a special kind of pattern. A *frieze pattern* can be understood, informally , as a set of points in the plane which has translation symmetry in only one direction. An example is shown below.

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[1] The algebraic properties of a group are assumed to be known to the reader.

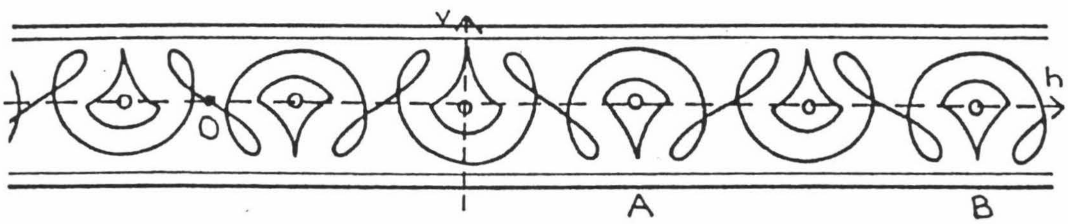


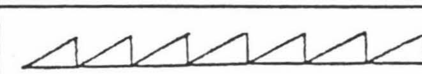
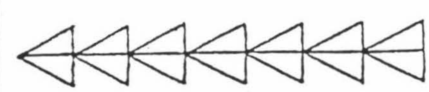
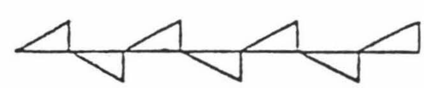
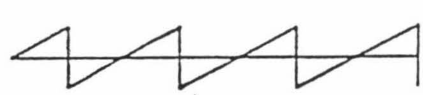
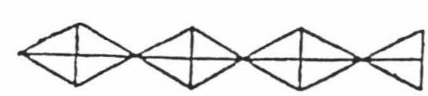
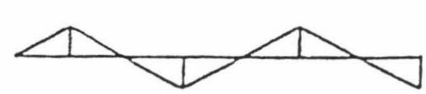
Figure 1.1

Reproduced from Shubnikov and Koptsik (1974), p 90

If we imagine the infinite extension of the pattern shown above (fig. 1.1) and translate it a distance AB (along its 'length'), then it will map onto itself. Of course, this particular example has other types of symmetries as well, such as a half-turn symmetry about the point O, a reflection symmetry about the line v, and a glide reflection symmetry about the line h (with the translation component in the direction of h).

In general, every frieze pattern has an underlying set of symmetries forming a *frieze group*. It was probably first proven by Federov a hundred years ago (Washburn and Crowe, 1988), and was shown again in detail by Martin (1982), that there are only seven different classes of frieze groups. Examples and the corresponding nomenclatures (Coxeter, 1987) are given in the following table:

Table 1

Belov's Crystallographic Notation	*Senechal's Abbreviated Notation*	*Martin's (1982) Notation*	*Examples*
p111	11	F_1	
p1m1	1m	F_1^1	
pm11	m1	F_1^2	
p1a1	1g	F_1^3	
p112	12	F_2	
pmm2	mm	F_2^1	
pma2	mg	F_2^2	

1.1.3 Intuition

Perhaps the most elusive ingredient of this thesis' title is that of *intuition*. Before proposing a working definition, it seems appropriate to consider a number of viewpoints of this concept. For instance, the *Concise Oxford Dictionary* (1990) described intuition as:

> "1. immediate apprehension by the mind without reasoning. 2. immediate apprehension by a sense. 3. immediate insight."

It appears that the term has a similar meaning in psychological circles, but it is viewed somewhat suspiciously by a number of psychologists, as the following extracts from psychological dictionaries indicate:

> "Immediate perception or judgement, usually with some emotional colouring, without any conscious mental steps in preparation; a popular rather than scientific term." (Drever, 1952)

> "1. direct or immediate knowledge without consciousness of having engaged in preliminary thinking. 2. a judgement made without preliminary cogitation. The term is more often used by laymen rather than by scientists." (Chaplin, 1968)

However, some psychologists, such as Carl Jung (1933) have described *intuitive* personality types in detail. Based on Jung's work, a personality type indicator known as *Myers-Briggs* has been developed. It divides perception activities into two categories: sensing and intuition. Jung described both types of perception as *irrational functions*, since neither operation is restricted by "rational direction" (Myers and McCaulley, 1985). Myers and McCaulley gave a description of each of these two perception functions:

> "Sensing ... refers to the perceptions observable by way of the senses. Sensing establishes what exists. Because the senses can bring to awareness only what is occurring in the present moment, persons orientated towards sensing perception tend to focus on the immediate experience and often develop characteristics associated with this awareness such as enjoying the present moment, realism, acute powers of observation, memory for details, and practicality. [In contrast] intuition ... refers to perception of possibilities, meanings, and relationships by way of insight. Jung characterized intuition as perception by way of the unconscious. Intuitions may come to the surface of consciousness suddenly, as a 'hunch', the sudden perception of a pattern in seemingly unrelated events, or as a creative discovery. ... persons orientated toward intuitive perception may become so intent on pursuing possibilities that they may overlook actualities." (p 12)

More recently, interest has increased amongst cognitive psychologists in a related area to intuition; that of explicit and implicit *memory*. Parkin *et al.* (1990) explained that:

> "Explicit memory refers to any test procedure that requires subjects to reflect consciously on a previous learning episode. ... Implicit memory tasks, in contrast, assess subjects' memory for a learning episode without any necessity for a conscious recollection of that episode."

In their experiment, Parkin *et al.* found that explicit memory of an episode was affected by an imposition of secondary processing demands whereas implicit memory was not. Similarly, the spacing of repetitions during initial learning affected explicit memory performance, but not that of implicit memory.

Piaget and Inhelder (1971) were aware of the existence of intuition. After observing that figurative aspects of thought are usually different from operational aspects, they wrote:

> "But there would appear to be an exception to this - the faculty known to mathematicians as geometrical 'intuition'. An adult subject who 'sees in space' ... does not stop at imagining static configurations in three dimensions any more than two. He [or she] is able to imagine movements and even the most complicated transformations thanks to a remarkable adequation of image to operation. This correspondence retains exceptional validity in spite of the well known shortcomings of intuition (such as the difficulty in visualizing curves without tangents, etc.)"
> (p 317)

The description of intuition, or similar notions, has not been restricted to the domain of psychologists. In 1952, for example, the famous mathematician Poincaré related in detail the differences he perceived between two types of mathematical mind, namely, *intuitive* and *logical* (Aiken, 1973). Similarly, Gagatsis and Patronis (1990) reported that:

> "Skemp (1971) draws a distinction between two levels of functioning of intelligence, that is, the intuitive and the reflective. The *intuitive* level involves awareness, through the senses, of data from the external environment which are 'automatically' classified and associated to other data. However, in this activity, the person is not aware of the mental processes involved. In contrast, at the *reflective* level, the mental processes become a focus of introspective awareness."

Of additional interest is Resek and Rupley's (1980) investigation of 'mathophobia'. Using the *Myers-Briggs Type Indicator* and ideas closely related to Skemp's (1979), such as *instrumental* and *relational* understanding, they found that a correlation existed between rule-orientation and sensation, as well as between concept-orientation and intuition.

Drora Booth (1975) has considered the intuitive use of symmetry operations in children's spontaneous pattern painting. In a personal correspondence (1991) with the researcher, she explained her own understanding of intuitive transformation concepts in children's or folk art work:

> "I take the term to mean any symmetry operation that can be identified in a work (painting, carving, weaving, block construction, etc.) that was created without the makers having formal knowledge of the mathematical concept."

This definition is very similar to the one eventually formulated in this thesis. However, in a cultural context, Grünbaum (1985) warned that:

> "Even if we were to believe ... that symmetries can be used to explain the ornaments, that has absolutely no implication on what the creators of these ornaments had in mind. Any of the periodic symmetry groups have as a prerequisite the infinite extent of the ornament; surely no Islamic artist would have dared even to think in such a sacrilegious way about the ornaments he can create. ... [Indeed], up to two centuries ago no artist or craftsman or *mathematician* defined regularity through symmetries. Equal parts - yes; equal position of parts with respect to their neighbours - yes; but equivalence with respect to the whole - never entered the picture."

A suspicion arising from Grünbaum's point is that some, or even all, of the symmetries able to be identified in a frieze pattern may not be intended, even *intuitively*, by the pattern's creator. Such symmetries in a pattern are therefore *accidental*, and labelling them as intuitive may be misleading. Naturally, the mathematical classification of patterns has the benefit of being systematic, but it *may* not provide a great deal of insight into a child's intuitive description or construction of a pattern. Lesh (1976) made a specific cautionary note:

> "...the researcher who begins with the assumption that children think in terms of slides, flips and turns may be just as naïve as the theorist who assumes flips come before turns and slides, just because flips are mathematically the most powerful. It *could* be that children do not conceive of rigid motions as compositions of slides, flips, and turns, but instead use some entirely different system of relations to describe spatial transformations." (p 234)

In conclusion, some of the key facets of the perceptual process of intuition seem to be that it is immediate, non-reflective, informal (independent of a formal framework), and associative (in the non-mathematical sense). While the identification of symmetry or transformations within a pattern may indicate an intuitive use of transformation geometry on the part of the creator, this isn't *necessarily* so. Thus, in this thesis, the working definition of *intuitive transformation geometry* in frieze patterns is the non-accidental presence of transformations or symmetry within a frieze pattern independent of a formal transformation framework. The property that the perception or creation of a design be *immediate* remains a secondary consideration throughout this study. However, in the analysis of survey results, one of the four measures employed to indicate the relative 'intuitive-ness' of the frieze groups addresses this concern also.

One final point: expressions such as 'more intuitive' indicate a comparison of one or more of the facets of intuition discussed above. It is hoped that the context of this phrase will make these facets clear.

1.2 The Motivation for Exploring Intuitive Transformation Geometry and Frieze Designs

"Perhaps more emphasis needs to be devoted to investigations exploring the intuitive (i.e., non-formalized) acquisition of systems of mathematical operations, relations and transformations. There is a popular misconception that concrete and intuitive mathematics is inferior mathematics and that the viability of a mathematical topic is measured solely in terms of its formalization and abstractness. In fact, the situation is often exactly the opposite." (Lesh, 1976, p 203).

"... linear patterns, sometimes called strip or frieze patterns, ... I believe are one of the great untapped geometrical treasure chests." (Williams, 1989)

Grünbaum and Shephard (1987) noted that the art of tiling and pattern-making appears to have begun very early in the history of civilization and, although the cultures emphasized different aspects of design, it seems that:

"Every known human society has made use of tilings and patterns in some form or another." (p 1)

They also claimed that many examples of artifacts from *all* cultures display a high degree of intricacy and complexity. Of particular interest to New Zealand is Knight's (1984a) observation that Maori rafter patterns, *kowhaiwhai*, suggest a well-developed geometrical intuition on the part of their creators.

Figure 1.2
Reproduced from Hamilton (1901).

These designs, when imagined to be infinitely extended along their length, are examples of frieze patterns. The relevance of this present study seems to be supported by Knight's conclusion:

> "The growing awareness of the importance of Maori Culture in New Zealand makes it particularly appropriate for students, both Maori and Pakeha, to relate the mathematics they learn to their cultural heritage."

Not long after Knight's article was published, the New Zealand mathematics syllabus for Forms 1 to 4 (1987) indicated that kowhaiwhai could be used to explore translation symmetry in Forms 3 and 4. However, it appears that little is known about the way in which students *perceive* these patterns, or if intuitive symmetry considerations form a part of this perception. If a teacher is employing a process-orientated approach to this topic (Skovmose, 1985), it may also be of interest to know the character of students' use and understanding of transformation geometry in their *constructions* of strip patterns.

Gagatsis and Patronis (1990) pointed out that intuition (non-reflective information processing) plays an important role in the development of reflective thinking, especially for children. In fact, they maintained that:

> "... intuitive thinking necessarily precedes reflective thinking and can help its evolution."

By implication, it would appear that exploiting a student's informal understanding of a concept may prove to be particularly valuable to mathematics educators. For instance, Bruner (1966), advocated the use of a "child's intuitive level as the starting point for teaching" (Booth, 1984). Booth (1985) herself concluded success in employing children's spontaneous pattern painting as the starting point for teaching art and transformation geometry.

However, only 13 years ago, Shultz (1978) indicated that:

> "Little is yet known about or agreed upon regarding children's cognitive abilities concerning transformation geometry." (p 195)

Today, this still appears to be the case. In addition, Lesh (1976) indicated that difficulties in teaching motion geometry may be a result of the fact that:

> "children make many mathematical judgements using qualitatively different methods than those typically used by adults." (p 186)

He also noted that such differences are not particularly well understood by researchers, particularly in the area of geometry. By focussing on the character of intuitive transformation geometry concepts, this thesis attempts to contribute towards the knowledge in this area.

1.3 An Overview of the Thesis

To explore this topic, we examine, in chapter 2, the mathematics education literature on the role of geometry in the development of spatial sense and consider the merits of the 'transformation approach.' A summary of the relevant psychology and mathematics education literature on the perception and learning of transformations and symmetry is subsequently undertaken. The literature review also considers a study of children's spontaneous pattern painting and its implications to intuitive transformation geometry. The review ends with a summary of some mathematical classifications of designs.

Chapter 3 outlines the design and execution of the surveys and interviews conducted, and describes the analysis methods used to examine the results. Chapters 4 and 5 include the results and discussion of the unrestricted and restricted frieze pattern construction activities. Chapter 6 characterizes the written and oral responses to the frieze pattern description activity. Summaries are given at the end of chapters 4, 5 and 6.

Chapter 7 concludes the study by discussing the implications of this thesis' findings for both researchers and mathematics educators at the Primary and Secondary school levels. To this end, appendix D includes a brief review of some relevant material for use in the learning environment.

2 *Literature Review*

2.1 Geometry and the 'Transformation Approach'

2.1.1 The Importance of Geometry and Spatial Thinking

The Nature of Spatial Sense and Ability

In the *Commission on Standards for School Mathematics* (1989), the National Council of Teachers of Mathematics (NCTM) stated that, in part, spatial sense is "an intuitive feel for one's surroundings and objects in them." In fact, as Del Grande (1990) pointed out, it has been found that spatial perception does not consist of one ability but many. He listed and described nine skills. Briefly, these are: visual copying, hand-eye coordination, left-right coordination, visual discrimination, visual retention, visual rhythm, visual closure, figure-ground relationships, and language and perception.

Owens and Stanic (1990) explained that spatial ability appears to consist of two key facets: *visualization* and *orientation*. These factors were derived from Bishop's (1983) and Halpern's (1986) material as cited by the authors. Visualization can be thought of as the ability to imagine how objects will appear after some change (usually affine) is imposed on them, while the orientation factor includes the ability to detect arrangement of elements within a pattern or the ability to maintain accurate perceptions despite a change of orientation.

Wheatley (1990) viewed spatial sense in terms of imagery. He summarised the aspects of imagery that Kosslyn (1983) proposed: *construction* (concrete or abstract images constructed from viewing, reading or reflecting), *representation* (the recalling of images, not necessarily the same as the original), and *transformation of self-generated images* (e.g., mentally rotating an object to compare it with another; recognising symmetry).

Geometry and Spatial Thinking

The NCTM (1989), in the United States stressed the importance of developing spatial sense in students:

> "Spatial relations are necessary for interpreting, understanding and appreciating our inherently geometric world."

Shaw (1990) shared this belief, and indicated that geometry and the development of spatial sense should be prominent in the elementary and middle school curricula. She gave several reasons to support her view, including a quote which indicates that geometry is as important as numbers in mathematics education. Some examples of the benefits she saw are that such development is practical because it relates to the real world; children already have inherent notions of space because they grow up in a three dimensional world; it encourages problem solving; and students attitudes are enhanced.

Hemmings *et al.* (1978) pointed out that in everyday life, people are faced with spatial problems more often than numerical ones. As well, they proposed that mathematics is a way of appreciating the environment. Therefore, a considerable portion of people's appreciation comes about through spatial awareness and understanding, since the physical environment is itself spatial. More generally, they stated that:

> "Intuitive awareness of spatial properties seems to be at the heart of most mathematical thinking."

Little wonder then, that Poincaré (the famous mathematician) maintained that geometers are more intuitive, whereas analysts are more logical, in their thinking (Aitken, 1973).

Wheatley (1990) indicated that a shift in mathematics curricula from *procedures* towards *relationships* is taking place, and therefore spatial sense takes on increased importance. Spatial sense, he surmised, is indispensable in giving meaning to our mathematical experiences. In a similar vein, the National Council of Supervisors of Mathematics (NCSM) warned in their report on *Essential Mathematics for the Twenty-first Century* (1989) that (American) graduates in 2001 presently face the prospect of a "computation-dominated curriculum more suitable for the nineteenth century." Among the twelve topics essential for students presently they listed geometry, with attention given to parallelism, perpendicularity, congruence, and symmetry.

Interestingly, Wheatley (1977) reported that research into the hemispheres of the brain suggests that the left hemisphere (in right-handers) excels in routine sequential tasks, logical reasoning, analysis and language processing, while the right hemisphere processes information more holistically and is superior to the left on spatial tasks. The right hemisphere 'thinks' in images while the left 'thinks' in words. One consequence of this specialisation of the hemispheres, Wheatley explained, is that it *is* possible to understand or 'know' a concept without being able to verbalise it. He continued (1977, 1990) by remarking that mathematics curricula in most (American) schools provides little opportunity for encouragement of right-hemisphere thought. Geometry, involving problem solving activities that require imagery, is therefore sorely needed. Jensen and Spector (1986) extended Wheatley's idea by noting that the 'right side' is suited for spatial relationships and music. They went on to outline a number of movement activities for primary school children. These are designed to "unleash cooperative efforts between the two hemispheres of the brain" (p 16).

Mason (1989) added another slant. He conjectured that the importance of geometry is "being aware of the fact that there are facts, rather than mastery of some particular few facts." It is the 'mustness' of many geometry relationships which is important for the teacher to convey and for the pupil to see.

Lesh (1976) gave yet another reason for focussing on geometry:

> "... most of the models and diagrams (e.g., number lines, arrays of counters, fraction bars, Cuisenaire rods, etc) that teachers use to illustrate arithmetic and number concepts presuppose an understanding of certain spatial concepts. Consequently, because of a lack of understanding of the spatial concepts, children sometimes experience misunderstandings about the models that are used." (p 186)

Geometry, in aiding the spatial development of an individual, furthers that individual's progress in other facets of mathematics. However, the degree to which this is true is uncertain. In some instances, the 'carry over' may not be particularly apparent. For example, in the abstract to their study on *Spatial Ability, Visual Imagery and Mathematical Performance*, Lean and Clements (1981) made an interesting deduction from their factor analysis:

> "...spatial ability and knowledge of spatial conventions had less influence on mathematical performance than could have been expected from recent literature."

Perhaps the last perspective should be left to Owens and Stanic (1990), who took Lesh's point a step further and argued that spatial abilities are involved, not only in other parts of the mathematics curriculum, but in other subjects and in parts of people's lives.

Obstacles to Learning Geometry

> "Many people, among them can be found a large number of our elementary school teachers, look back to this [deductive] geometry with fear and trepidation for they never really understood what deductive proof was all about. But this is not geometry! Geometry is not the study of proof! Geometry is the study of spatial relationships of all kinds; relationships that can be found in the three dimensional space that we live in and on any two-dimensional surface in this three-dimensional space." (Egsgard, 1970)

Bishop (1986) identified three main areas of difficulty in geometry education: learning about space, learning about 'mathematizing' space, and learning about geometry. His suggestions for the improvement of geometry teaching (and research) seem to summarize nicely many of the concerns about learning difficulties that mathematics educators in have in this area. References are given for each point, and where necessary, a few notes are made.

1. More use of the *child's* spatial environment needs to be made (Meserve and Meserve, 1986; Bishop, 1977). Bishop (1988) gave the specific example of mathematical enculturation. For many pupils around the world, the geometric ideas they are being taught are based on a view which is 'foreign' to their 'home culture'.

2. The status of geometry in elementary school needs to be raised (Shaw, 1990; Wheatley, 1990; NCTM, 1989; Lesh, 1976).

3. Children need to not only be engaged in spatial activities, but reflect on them as well (Lesh, 1976; Mason, 1989; Pappas and Bush, 1989; Fielker, 1973). Gagatsis and Patronis (1990) found a favourable outcome from the use of geometric models in a process of reflective thinking. However they did note that, for a model to be suitable for the classroom, it must *first* have been implicitly used by the children as a model of action. Dienes and Golding (1975) made a similar comment in a section on transformation geometry games (See appendix D1 for details). Also, Wheatley (1977) cautioned against an emphasis on verbalization as *the* method of reflection; see point 7 below.

4. A variety of representations needs to be given to young learners, together with tasks calling for representation (Bishop, 1977; Yerushalmy and Chazan, 1990; Mason, 1989). Fischer (1978) formed the following conclusion indicating that the first of Bishop's proposals is not, *by itself*, enough:

> "The presentation of a variety of shapes for a concept is not sufficient to prevent students forming limited concepts. The visual distinctions that students perceive in figures are often more compelling than the mathematical concept that is illustrated." (p 320)

Charles (1980) outlined strategies for dealing with some of the representation difficulties in geometry. In the light of Fischer's findings however, his suggestions seem to be necessary, but insufficient, for avoiding limited conceptualization:

(i) Identify the relevant and most frequently occurring irrelevant characteristics of the concept to be taught.

(ii) Select a variety of examples.

(iii) Select a variety of non-examples.

(iv) Draw the students attention to the relevant and irrelevant characteristics of the concept with questions and explanatory comments.

5. More use of 'scaling down' the environment is recommended (Hart and Moore, 1973). Bishop (1986) noted that geography educators recognised this importance earlier than mathematics educators.

6. Emphasis should be given to describing geometric properties and relationships, with a deliberate avoidance of labelling objects which can limit a child's understanding (Mason, 1989).

7. More use of visual imagery needs to be encouraged (Wheatley, 1977).

8. The teacher needs to expose the relationships between different aspects of spatial analysis (Mason, 1989). Fielker's (1973) point was a little different from Bishop's. He argued that the teacher should be aware of structural relationships in order to give direction to the mathematical activities that their students engage in. His concern was that many of the activities in the primary classroom look like a series of party tricks. He suggested a more structural approach because:

> "... primary school geometry ... seems to be nothing but detail, a patchy set of enjoyable experiences with no structure to hold them together other than the teacher's intuitive feeling that it is all part of mathematics." (p 12)

Little wonder, then, that geometry is often skipped or skimmed over by (American) elementary school teachers (Lesh, 1976). Fielker continued:

> "The pupils do not worry about the ... coherence. ... Perhaps even for the primary teacher the structure is not of primary importance. What is important is *what* activities to give the pupils. The advantage of the structure is that when you begin to see how things fit in you also see where the gaps are." (p 16).

Bishop (1986) concluded aptly:

> "Geometry learning needs to be taken much more seriously by research workers, and to be given greater priority by teachers and by curriculum developers."

2.1.2 The Development of Spatial and Geometrical Concepts

Much has been written on how the conceptualization of space develops. Detailed accounts of the relevant literature have been given by Hart and Moore (1973) and Downs and Stea (1977). Rather than present another comprehensive review, a brief description of the main points made by prominent researchers in this area is presented, based chiefly on more recent reviews by Dickson *et al.* (1984) and Bishop (1980).

Jean Piaget

It will come as no surprise that the work of Piaget *et al.* in this area is substantial. It includes studies such as *The Child's Conception of Space* (1956), *The Child's Conception of Geometry* (1960) and the lesser known *Mental Imagery in the Child* (1971) as well as summaries of these works. In all three books, the approach is developmental.

Piaget and Inhelder (1956) distinguished between *perception* and *representation*. Perception is "the knowledge of objects from direct contact with them", while representation evokes objects in their absence. Representation is often called *mental imagery*. They investigated corresponding tasks such as simple detection (perception) to identification and reproduction (representation). The development from simple perception to representation involves a progressive differentiation of certain geometric properties, and falls under three general headings which occur in the order: *topological*, *projective*, and *Euclidean*. This has become known as the 'topological primacy thesis' (Darke, 1982). Not until age eleven, stated Copeland (1979), does the child fully develop skills for metric geometry or *measurement*.

The 'topological' stage consists of global property distinctions, which are made independent of an object's size or shape. These include:

1. *Proximity* - the child can make a distinction based on which of two objects is nearer. For example, when drawing a face, the eyes may be close together but below the nose.
2. *Separation* - a distinction based on whether two objects touch or not. For example, the child separates the eyes and nose (no overlap).
3. *Order* - appreciating a sequence or 'between-ness'. For example, a nose is drawn between the eyes.
4. *Enclosure* - for example, drawing eyes inside a head.
5. *Continuity* - perceiving that an object's parts are connected, and which parts are connected. For example, drawing arms connected with the body but not the head.

'Projective' properties involve the ability to predict how an object will look when viewed from different angles. Straightness is an example of a projective property.

'Euclidean' properties relate to size, distance, and direction, and its development eventually leads to a metric geometry involving measurement (e.g., length and angle). For a more comprehensive review of these stages, the text *How Children Learn Mathematics* by Copeland (1979) is recommended. The development of transformations, as described by Piaget and Inhelder (1971) is omitted here and is outlined in subsection 2.2.1.

The significance of Piaget's work has meant that his theory has attracted a lot of study and, subsequently, criticisms of his ideas have arisen.. Dickson *et al.* (1984) gave three main reasons for this. Firstly, new perceptual theories in psychology maintain that the distinction between perception and representation is not particularly clear, that is, these processes differ only in degree of organisational complexity. Secondly, some of Piaget's experiments give quite different results when the procedure is modified in seemingly trivial ways. Thirdly, Piaget's use of logical mathematical structure is dubious.

Darke (1982) made a review of the research on this topic and formed the following conclusions about Piaget's theory:

1. There has been an imprecise use of the terminology related to topology. Schipper's (1983) article reinforces this point, stating that the use of mathematically correct criteria yields arguments against the acceptance of topological primacy.

2. "Experiments were often complicated by non-conceptual factors" and "the children's actions were not as coherent as Piaget's theory would suggest. Their concepts of space appeared to be influenced by factors such as language, schooling, social situation and the presence of a handicap."
3. It appears unwise to conclude that children should be taught topological concepts on the basis of psychological research.

Other Viewpoints

Bishop (1980) made the point that, despite the criticisms of Piaget, the Geneva school has had a sizable influence. However, Werner's (1964) work, which is recognised by geography educators, is not as well known to mathematics educators as it should be according to Bishop. He wrote:

> "Within his *orthogenetic principle* there is a progression from a state of relative globality and lack of differentiation to a stage of increasing differentiation, articulation, and hierarchic integration. ... Werner does not get into difficulties by using Piaget's 'mathematical terms' ... Werner's reminds us that insofar as we are concerned with spatial ideas in mathematics as opposed to just visual ideas, we must attend to large, full-sized space, as well as to space as it is represented in models, and in drawings on paper."

Werner hypothesized that there are three levels of representation: *sensori-motor*, *perceptual* and *contemplative*. Bruner proposed another view of psychological development in the context of algebraic mathematics. He also suggested that there are three levels of representation which he called *enactive*, *iconic*, and *symbolic* (see Hart and Moore, 1973 for details). Bishop remarked that there is some resemblance between the stages theorized by Piaget, Werner and Bruner.

Downs and Stea (1977) reviewed some of the literature on the development of *cognitive mapping* ("...an abstraction covering those cognitive or mental abilities that enable us to collect, organize, store or recall, and manipulate information about the spatial environment"), and noted the work of Jean Piaget and Jerome Bruner in particular. They also outlined some of the major spatial experiences (too numerous to list here) that occur from prebirth through to the end of primary school. Perhaps their most noteworthy point was their challenge to some developmental views which assumed that "the cognition of small-scale spaces must inevitably precede the cognition of large-scale spaces." They supported their claim by taking pains to emphasize one of Bower's (1966) findings in his study of six-week old infants. This study showed that congruence constancy doesn't emerge from more primitive perceptions. Bower himself surmised:

> "... *infants can in fact register most of the information an adult can register* but can handle less of the information that adults can. Through maturation they presumably develop the requisite information-processing capacity." (Bower, 1966) Note: italics added.

Finally, van Hiele's (1959) theory of spatial development has gained popularity (Coxford, 1976; Dickson *et al.*, 1984) and is more strongly related to Werner's ideas than Piaget's or Bruner's (Bishop, 1980). Briefly, it involves five levels (paraphrased from Dickson *et al.*, 1984):

Level 1: Figures are distinguished in terms of their individual shapes as a whole and relationships are not seen between these shapes or their parts. Wirszup (as cited by Dickson *et al.*) stresses that solid grounding in level 1 is a crucial prerequisite to levels 2 and 3.

Level 2: At this stage, an awareness develops of a figure's parts. This becomes realised through practical work, drawing, painting, etc. The child can still not see relationships.

Level 3: Relationships and definitions begin to be distinguished, and logical connections established, but only with guidance.

Levels 4 and 5: The development of deductive reasoning occurs and synthesis (theory construction) eventually gives rise to complete abstraction. Concrete interpretation is no longer depended on.

References to other more general learning theories are made throughout this text. For example, aspects of Piaget's theory of cognitive development and Dienes' theory of mathematics learning are used. Summaries of these theories can be found in sources such as Wadsworth (1979) and Reys & Post (1973) respectively. A few points about their ideas do seem important here and form part of this thesis' assumptions. Firstly, Piaget held that adults and children do not learn in the same ways. Children, especially young children, learn best from concrete activities. More generally, Dienes proposed the dynamic principle which occurs in three stages: the child should have initial (unstructured) *experiences* to relate subsequent experiences to, then engage in experiences structurally similar to the first in order to become *aware* of the concept, and then pursue the mathematical concept until *operational.*

Secondly, Reys and Post explained that Piaget's stages of cognitive development are sometimes misinterpreted. For example, it is *not true* that concrete materials are not needed by adolescent pupils. Until age 11 or 12, concrete operations represent the *highest* level at which a child can consistently operate. Therefore, in the development of new concepts, it is often necessary to start concretely before proceeding to abstract ideas. This last remark is similar to Dienes' constructivity principle.

Lastly, Dienes (1969) suggested that children need to build their own concepts from within, rather than having those concepts imposed on them. Indeed, the fundamental assumption of the more recent *constructivist theory*, based on the work of Piaget and others, asserts that:

> "... learners actively construct their own understandings rather than passively absorb or copy the understandings of others." (Simon and Schifter, 1991)

2.1.3 Curriculum and the Merits of a Transformation Geometry Approach

As mentioned above, the NCSM (1989) stressed the importance of geometry in the curriculum, and, among other things, they emphasized congruence and symmetry. They also advocated that:

> "... students should visualize and verbalize how objects move in the world around them using such terms as slides, flips and turns. Geometric concepts should be explored in settings that involve problem solving and measurement." (p 45-46)

But while there appears to be an increasing concern amongst various mathematics educators that spatial thinking be enhanced via geometry, debate still occurs over the relative merits of the 'transformation' approach and the traditional Euclidean approach. Sinha (1986) felt that this is unfortunate. He explained that:

> "an impression is gaining ground that transformation geometry runs counter to what Euclidean geometry stands for. [But] if one takes pains to look at it, one can say that Euclidean geometry is the study of those Euclidean transformations that keep the properties of geometrical figures invariant. ... Euclid's treatment is reinforced, reshaped and refashioned with the aid of Klein's [1870's] approach. Transformation geometry is, therefore, Euclidean geometry." (p 47-48)

Probably the main objection (at secondary school level) to the transformation approach is that a demise of the much cherished axiomatic treatment results and that 'the thinking is taken out of geometry.' Sinha faced this argument by suggesting an 'activity' instruction style for those up to 13 years of age; while "the formal axiomatic exercise can start with 14 year olds." In India, he claimed that good students have thrived on such an approach to transformation geometry.

Giles (1982) disagreed. He strongly advocated geometry for all pupils, not just the top students. He did admit that there were those who felt that his particular approaches lacked substance, but he continued:

> "... they are thinking of mathematics as static content, and their geometry is dead, entombed in textbooks. If the mathematics classroom is to be alive, then we must allow children to do their own mathematics. And a first step towards this is to set up activities that move problems out towards investigations." (p 37)

If Giles' message has any value then Küchemann's (1981) comment, aimed presumably at secondary level, seems particularly pertinent:

> "... it would seem pointless to [study transformations] in a didactic, expository manner. The fact that the transformations can be defined in terms of actions (folding and turning) and their results represented in a very direct manner by drawings means that the topic is ideally suited to a practical and investigative approach. The actions and the representations are both highly intuitable so that it should be possible to develop such an approach in ways that are meaningful to most children."

Furthermore, Sinha's 'reconciliation' of the two approaches to geometry may not be completely satisfactory to many New Zealand secondary school teachers who seem to prefer the benefits that *informal* transformation geometry can bring to students of *at least* 13 years of age. In the report on*Mathematics Achievement in New Zealand Secondary Schools* (IEA, 1987), the section on third form teachers' practices and beliefs stated that:

> "Teachers overwhelmingly favoured (51% emphasised, 31% used) an informal transformation approach as an important method, with good support also for a co-ordinate approach (35%, 38%) and an informal Euclidean approach (19%, 38%). Formal transformation or Euclidean approaches had little support...:" (p 123)

> "Teacher practice was entirely consistent with three other statements eliciting strong support. These, chosen from seventeen, were:
>
> - an intuitive approach to geometry is more meaningful to students at this form level;
> - geometry should be taught mainly through transformations (reflections, rotations, translations,etc.);
> - proofs of theorems should be delayed until students are at least 15 years of age." (p 126-127)

In support of the last point, Egsgard (1970) agreed with a similar British view that the axiomatic deductive geometry is suitable only for the top 5% of students.

Several mathematics education researchers have indicated that the intuitive nature of transformation geometry is of tremendous value to the learner. For example, Usikin (1974) felt that transformation geometry is more intuitive for a student because it uses simple symmetry explanations and familiar movements. And Thomas (1978) quoted from the 1967 report of the K-13 Geometry Committee of the Ontario Institute for Studies in Education:

> "The main value of motion geometry is in achieving the objective of an informal, intuitive appreciation of geometry."

A related reason was given by Küchemann (1981):

> "nearly *all* the children tested had some understanding of reflection and rotation, which means a basis exists for studying the transformations in secondary schools." (p 157)

It has been often argued that Piaget's findings support the transformation approach. This is not surprising. For instance, Piaget and Inhelder (1971) found that the essence of cognitive growth is the increasing ability to deal with invariant properties under more and more complex transformation systems (Perham, 1978). But, in another monograph, Lesh (1976) contended that Piaget's theory has often been used *unjustifiably* (or selectively) to foster support for transformation geometry in the primary curricula and for the 'laboratory' form of instruction in general.

Lesh (1976) himself gave a number of other reasons for teaching transformation geometry to primary students. One of these is that a greater number of figures can be explored using transformation geometry than other more traditional approaches. The 'Mira', for example, can construct many figures that a compass and a straight edge can not.

Another justification, often put forward by teachers, is that the primary curriculum prepares students for secondary geometry. However, Lesh argued that primary pupils are often asked to explicitly deal with basic rigid motions when some research indicates that young children do not think in terms of motions but in terms of "changes in certain properties of the end states of transformations." (It is interesting that this viewpoint corresponds more closely to the formal mathematical consideration of transformations than to the 'motions' perspective). More importantly, the rationales for teaching transformation geometry at secondary school don't necessarily apply to the primary level. For instance, Lesh (1976) maintained that the scope of transformation geometry at the primary level needs to broaden in order for it to relate to other subjects as it does in secondary school. His overall point was this:

> "Elementary school *is* different than secondary school. Its students are different; its classroom organization is different; its objectives are different." (p 192)

But hope is not lost for transformation geometry at the primary level. For example, Lesh also outlined activities for primary students which he sees as both important and fun, such as tessellations. This in turn provided another reason for learning transformation geometry, namely, it provides a set of enjoyable, and hence motivating, tasks for students (Lesh, 1976). Many of the authors of the articles reviewed in appendix D also supplied testimony to this postive effect on children.

As well as being important, enjoyable and motivating, one of the goals of geometry is to promote the spatial ability of the student. Perham (1978) proposed that for certain spatial tasks there may be a "link between transformation geometry and spatial ability." If this is the case, then this adds weight to the argument for including transformation geometry in the primary curricula.

One of the justifications for including transformation geometry in secondary school mathematics has been that combinations of transformations provide insight into mathematical structure. For instance, reflections and rotations *combined* provide examples of non-commutative systems (Thomas, 1978). Insight into the structure of a group has also been hoped for by many curricula (Küchemann, 1981). While these may be admirable aims, Küchemann (1981) made the following observations:

> "Most modern syllabi ... do not demand that this should be fully realised, with the result that teachers and their pupils are left working towards goals that are confused and incomplete. Moreover, many children's difficulties with single transformations pre-empts them from even considering *combinations* unless very simple transformations are used." Note: italics added.

Küchemann dispatched yet another rationale: that transformation geometry can lead to other topics such as vectors and matrices, which will demonstrate the unity of mathematics by strengthening the ties between algebra and geometry (Thomas, 1978). His opposition to this argument was again pragmatic:

> " ... it is extremely doubtful whether it is seen as meaningful by the majority of secondary school children."

He also felt that it was less than ideal to include the topic in secondary schools mathematics for purely negative reasons, that is, that traditional Euclidean geometry is unsuitable for most students (Egsgard, 1970). Indeed, transformation geometry has generally been received:

> "with a lack of conviction in many [British] schools and a reluctance to abandon traditional, expository methods of teaching." (Küchemann, 1981)

So why teach transformation geometry at secondary level if the aims for which it was put in place are not being attained? Küchemann (1981) concluded that two positions must arise:

> " ... either transformation geometry is accepted as relatively unimportant ... or the transformations are studied *singly* and are seen as valuable in their own right."

It appears as if the second of these perspectives is the most accurate. Küchemann (1980, 1981) reported findings that support the intrinsic worth of the single transformations; see subsection 2.2.1 for details.

In summary then, the strongest arguments for including transformation geometry in curricula seem to be:

1. At all levels, it appears to be a more intuitive approach to geometry.
2. The study of single transformations has intrinsic worth at secondary level, and probably at primary level.
3. The activities associated with transformation geometry can be enjoyable and motivating at all levels.
4. There is some evidence that transformation geometry can contribute to a child's sense of space (see subsection 2.2.1 for details).

Perhaps a further observation can be made. According to Gribble (1980), the education philosopher R. S. Peters set out three criteria for an activity to be considered as educational: (a) the activity should be valuable in itself; (b) it should "relate to other ways of understandings and experiencing", (c) the learner should feel that it is worthwhile. It seems that the more convincing justifications for transformation geometry satisfy two of these criteria (at least, in Britain and the United States). If it is to satisfy the remaining requirement, (b), that there should be a "wide cognitve perspective", a variety of connections to other interests is needed at both primary (Lesh, 1976) and secondary level. This certainly appears to be possible. The appendix D entries, for example, clearly display the extent of the links which transformation geometry has with play, art, culture and computing.

A Digression: The Pervasiveness of Symmetry

> "'Symmetry' is the classical Greek word ΣγΜ–ΜΕΤΡΙΑ, 'the same measure', due proportion. Proportion means equal division and 'due' implies that there is some higher moral criterion. In Greek culture due proportion in everything was the ideal. The word and the usage have been taken over as a technical term into most European languages. The Chinese word, also embedded deeply in Chinese culture, indicates reciprocity." (Mackay, 1986)

When viewed 'dinergically' (reconciling opposites), as Doczi (1986) did, symmetry appears to permeate nature and art in a paradoxical, numinous and creative way. Also, symmetry is an aspect of many areas of study. Any reservations about this fact are soon dispelled upon a perusal of other journal articles in Hargittai's (1986) *Symmetry: Unifying Human Understanding*. Topics included: the nature of symmetry, crystallography, aesthetics, systems, geometry, music, computing, art (e.g., Escher, Bartnig), physics, cell biology, literature, chemical reactions, biochemistry, graph theory, design, biology, classifications, dance, anthropology, fractals, archaeology, inorganic chemistry, philosophy, cosmology, and chaos theory.

Many other earlier works on symmetry exist. For example, Senechal and Fleck (1977) also made interesting links between various subjects via symmetry. Hofstadter (1980) used symmetry throughout his comparison of the work of Gödel, Escher and Bach in order to illustrate the similar structures present in various systems of thought and action. Weyl (1952) explored symmetry in a number of contexts, as did an updated version by Rosen (1975). Shubnikov and Kopstik's (1974) book explained, in some detail, their analysis of symmetry in both science and art. More recently, Hagen (1986), who is a perceptual psychologist, described some of the relationships between various geometries and representational art. Perhaps one of the most well known books involving symmetry is by Birkhoff (1933) in which he developed aesthetic measures for various objects. Lastly, Wood's (1935) contribution should be mentioned here, because he introduced non-mathematicians to the analysis of symmetry in patterns.

These works represent a small portion of the literature on symmetry. This fact points to the salience of symmetry, and adds weight to the argument for including it in mathematics curricula (Kappraff, 1986), as well as other curricula, at all levels.

2.1.4 The Present State of the New Zealand Mathematics Curriculum

A brief overview of the transformation geometry content in New Zealand mathematics syllabi at the primary and secondary school levels is presented in this subsection.

The Primary (Year 1-6) Syllabus

The main philosophy behind the geometry component of the year 1-6 curriculum is outlined in the *Syllabus for Schools* booklet (1985):

> "In this syllabus, children's ideas of geometry are seen as originating in their immediate environment and forming part of their everyday experience. The emphasis is on increasing their awareness of spatial elements such as points, curves, planes, shapes and solids, and on exploring the relationships between them. ... It follows that geometry is at all times to be approached practically, with children being encouraged to use their powers of observation to the full and to communicate their findings. Geometry should not be seen as an isolated topic." (p 14-15)

From the point of view of this thesis it is encouraging to see that, *from year 1*, the transformation geometry which the children are to study and explore includes:

1. Symmetrical and repeating patterns
2. Movement and position in space
3. Various ways of covering surfaces (including tessellations)
4. Symmetry in the environment.

In addition to the above activities, from year 3, children are to investigate ways of packing objects in containers and explore symmetry in plane figures. Given the pervasiveness of symmetry in a child's world, all these topics seem to be highly conducive to the aims sketched above.

It is worth mentioning that a great deal of geometric intuition and spatial thinking is required in other parts of the year 1-6 mathematics syllabus. The physical, iconic and symbolic representations of sets, whole numbers and fractions and the use of graphs are just some examples. A teacher should be conscious of this state of affairs because, according to Bishop (1977, 1986) and Lesh (1976), it is all too easy to believe that children extract the same information from diagrams that adults do. Bishop (1977) contended that, in fact, many diagrams contain quite arbitrary visual conventions that a teacher may believe to be obvious but a child may not see. Lesh (1976) added that children (and adults) do not simply 'read out' but must 'read in', that is, they impose their own rules of organisation to perceive the world.

The Form 1-4 (Year 7-10) Syllabus

The Form 1-4 syllabus also emphasises activities and working with mathematical apparatus wherever appropriate, but it is considerably less specific than the year 1 to 6 syllabus in indicating the purpose of exploring geometry. It states in its general objectives that:

> "students should gain a knowledge of geometrical relations in two and three dimensions, and recognise and appreciate their occurrence in the environment." (p 5)

The following information (with a focus on rigid transformations) is extracted from the transformation and symmetry section of the Form 1 to 4 geometry syllabus (p 26-27). The syllabus for the Form 1 to 4 students rests on the assumption that, by Standard 4, children can explore movement and position in space as well as recognise figures of the same shape and size. Furthermore, new Form 1 students should have had experience in exploring symmetry in the environment and in plane figures.

In Form 1, students should be able to perform, use, and state the properties of, elementary reflections in a mirror line and recognise examples of reflection in the environment. They should also collect and draw examples of objects having line symmetry and be able to identify lines of symmetry.

In Form 2, they should be able to perform, use, and state the properties of, rotations (multiples of 90^{o}) and translations, and recognise examples in the environment. In addition, activities such as finding or drawing objects with rotation symmetry should be undertaken, and the student should be able to identify centres of rotation.

In Form 3, the concepts of transformations and invariance are introduced via the environment, for example, plane figures to their shadows, objects to plans, maps, or diagrams. The properties of reflections and rotations and their invariant points should be explored with attention given to the special case of the half-turn. Using informal methods, students should also be able to find the angle and centre of a rotation. A classification of symmetrical objects according to the number of lines of symmetry and the order of rotation should also be made. It is suggested, in brackets, that an exploration of translation symmetry via wallpaper patterns, kowhaiwhai and weaving patterns could be made.

In Form 4, as well as reiterating the concepts from earlier years, students are required to explore the properties of translation and represent two dimensional translations with Cartesian vectors. Successive translations can be described by adding vectors. The magnitude and direction of a vector, their addition and subtraction, and their multiplication by a real number should also be explored.

The Form 5 (Year 11) Syllabus

The purpose of the geometry component in school certificate for Form 5 students was not explained in the *School Awards Prescriptions* booklet (1991). General aims of the whole syllabus were given instead. The approach taken appears to be at an intermediate level of formality. It was explained that, in the examination, reasons (but not proofs) for geometrical conclusions may be asked for and these reasons should clearly indicate the appropriate geometric principle. Interestingly, paper folding is listed among the acceptable construction techniques.

The (rigid) transformation component of the geometry section consists of:

(i) Geometry of the plane based on the transformations - reflection, rotation and translation, but not glide reflection.
(ii) Properties that are invariant under these transformations.
(iii) Combinations or composition of transformations.
(iv) Simple deductions based on transformation properties.
(v) Line symmetry and rotation symmetry.
(vi) Applications of vectors to translation.
(vii) Applications of 2x2 matrices to transformations, inverse transformations and combinations of transformations.

Perhaps two observations could be made. Firstly, despite the emphasis on working with concrete materials in the two previous syllabi outlined, no specific comment was made to this effect for the Fifth Form students. It has been noted above that students, from the concrete operational stage *onwards* , can benefit from this approach. In particular, this thesis argues that the exploration of patterns and figures in the environment and their symmetry is important in its own right and links mathematics with other subjects. Secondly, and more specifically related to transformations, given the aspersions cast by Küchemann (1981) on the use of transformation matrices to illustrate groups as an example of unification in mathematics (see subsection 2.1.3), it would appear that the last item on the list (vii) may be of little value to students at this level (if this is its motivation). Küchemann's assertion is reinforced by the observation that this relationship between matrices and transformations may be viewed only fleetingly, since it no longer appears in subsequent secondary mathematics courses. Indeed, as we shall see, the consideration of both groups and transformation geometry seems to disappear completely.

The Form 6 and 7 (Year 12 and 13) Syllabi

One of the aims of the Sixth Form Certificate mathematics course is "to consolidate and extend the work of fifth form mathematics." But while the present Sixth Form Certificate mathematics prescription aims to provide a course appropriate to the needs of the wider group of students at this level, it also has a strong 'top-down' influence since it is designed to prepare students for tertiary level study as well. It seems likely that this presents a conflict of interest in choosing suitable topics to include in the curriculum at this level, one result of which is that transformation geometry is not in the Sixth Form Certificate course at all. Instead, coordinate geometry is seen as the appropriate extension of the Fifth Form geometry section. Consequently, the syllabi for seventh form mathematics courses do not furnish a transformation geometry component either.

Finally, mathematics education researchers, teachers, and the curriculum (for, at least, pre-Fifth Form students) all agree in *theory* that spatial relations are important and that "the use of concrete models and instructional aids is essential for teaching geometry", yet the *practice* of many New Zealand secondary school teachers is inconsistent with this philosophy. For instance, the second IEA study (1987) reported that:

> "62% of teachers said that they ignored specific teaching of spatial relations ... and almost 65% of the teachers made no use of 'commercially or locally produced materials for students' and 80% of teachers found little or no use for 'commercially or locally produced films, filmstrips, teacher demonstration models, or overhead projector masters'." (p 126)

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With the publication of the new *Mathematics in the National Curriculum: Draft* (1992), an emphasis on the relevance of mathematics and practical problem solving may influence the degree to which patterns are used in the exploration of transformation geometry. Of course, the success of new curriculum development is dependent on satisfactory implementation.

Conclusion

In conclusion then, it seems that as students progress through Primary and Secondary school, the stress in the present New Zealand mathematics curriculum on the use of concrete materials diminishes considerably. Correspondingly, emphasis appears to shift from a concrete, 'intuitive' approach (up until Form 4 or 5) to a more symbolic, formal approach in later courses. What is peculiar about transformation geometry in this progression is that not that it is developed intuitively up until Standard 4, nor that it is formalized somewhat from Form 1 to Form 5, but that it suddenly vanishes in Form 6! This disappearance seems a pity (in the view of this thesis) since the symmetry of patterns, for instance, is by no means trivial and is a natural extension of earlier work on transformation geometry. Furthermore, in a New Zealand context, this topic is highly relevant to students in that it is one practical and aesthetic way of connecting mathematics with an aspect of Maori culture (Knight, 1984). Such a topic may also provide stimulus for some teachers to use materials as part of the facilitation of their students' learning. With the advent of a new syllabus, this approach may become more common.

2.2 Transformation Geometry: Learning and Perception

2.2.1 Mathematics Education Studies in Transformation Geometry

By the end of the Fourth Form, students in New Zealand have had exposure to most types of affine transformations, apart from glide reflection. In this section, investigations are outlined which describe the child's conception of those transformations. Emphasis is given firstly to rigid transformations and then to symmetries.

On a more general note, Lesh (1976) warned education researchers against of the pitfalls of employing Piaget's style of using of mathematical structures to model and analyse the learning of mathematics concepts with. Lesh did this by drawing a parallel between Piaget's topological primacy thesis and the evolution of isometries. The gist of the argument is that, because *all* rotations are the composition of two reflections, and *all* translations are the composition of two rotations, the evolution of these concepts must necessarily evolve in the order: reflection, rotation, translation. But Lesh indicated that this mathematical conclusion is probably unhelpful from a cognitive point of view since:

1. The most general mathematical relation is not necessarily the most psychologically basic.
2. Children and adults use different rules on which they form mathematical judgements.
3. "If operationally isomorphic tasks vary too much in difficulty, it may be meaningless to equate tasks on the basis of operational structure." (e.g., some research indicates that translations are easier than reflections, yet it is also quite easy to design a task where a reflection is easier than a translation). In short, classifying rigid motions as 'slides', 'flips' and 'turns' may not be the most revealing analysis method.

The last point not only presents a challenge to Piaget's topological primacy thesis, it also raises questions about investigations into the learning of transformation geometry. One implication of Lesh's observations seems to be that, if the argument for transformation geometry to be included in the curriculum is to have any validity, the *character* of the research in this area needs to be considered carefully. Two years after Lesh made his warning, Moyer and Johnson (1978) raised a specific challenge:

> "It will be difficult to devise maximally effective instructional activities concerning transformation geometry concepts until research can discover what children consider to be the essential characteristics of rigid transformations" (p 277)

It is to the studies addressing these concerns, and the conceptualization of symmetry, that we now turn.

Transformations

Moyer and Johnson (1978) reinforced Lesh's warning that cognitive development appears to depart from predictions made from mathematical structure by concluding that, for young children, "the flip is not primitive." In fact, they claimed that the developmental order appears to be translation, reflection, and lastly, rotation.

Let us digress for a moment. While certainly not mathematics educators, *per se*, Piaget and Inhelder (1971) drew some interesting conclusions about children's conceptions of transformations from *Mental Imagery in the Child.* An explanation of their terminology {such as *image*; *reproductive* and *anticipatory* images; *executional* and *evocational* anticipation; *static*, *kinetic* and *transformation* reproductive images; *gesture* and *drawing*;

immediate and *deferred* images, *product* and *modification*} can be found in their book on pages 1-6. Among their conclusions they included statements such as:

> "... it is easier for the child to imagine the product than the process, i.e., the movement as a trajectory." (from chapter 4 on kinetic anticipatory images, p 160).

> "... when the subjects imagine and draw in detail the arc-straight-line transformation, they represent the end product less well than when it is the main object of the test ... [And] while symbolic action bears directly on the transformation, imaginal representation bears first and foremost on the product of the transformation rather than on its successive stages. The image of the end product is even somewhat better when there is no attempt to imagine the transformation itself." (from chapter 5 on reproductive images of transformations, p 172-173).

We can now return to the concerns of mathematics educators. Lesh (1976) noted that another study in mathematics education bears out the truth of Piaget and Inhelder's comments. Perham (1978) also noted that another important finding from *Mental Images in the Child* is that:

> "children are unable to comprehend reflections (flips) until age eight and rotations (turns) until age nine."

However, she also pointed out that some other education studies suggest that these two isometries can be understood earlier than this. Booth's (1985) study of pattern painting with young children also supports this finding (See subsection 2.3.1 for details).

Karen Shultz, F. Richard Kidder, Faustine Perham and Diane Thomas all contributed articles in a monograph entitled *Recent Research Concerning the Development of Spatial and Geometric Concepts* (Eds., Lesh and Mierkiewicz, 1978). Dickson *et al.* (1984) have summarised this material for teachers, so there is no need to reproduce it here in the same detail. Instead, some of the conclusions from the articles themselves are mentioned.

This research by Lesh *et al.* was carried out almost entirely within the Piagetian tradition (Bishop, 1980). For instance, two studies investigated the conservation of length under various transformations. Thomas (1978) gave children aged 6, 9 and 12 the classic length conservation test and then conducted a set of tasks which involve the transformation (translation, reflection and rotation) of a triangle. Her conclusion follows:

> "Results ... showed that Piagetian stage (rather than the age of the students) seemed to make an important difference when elementary students considered the question of invariance of length of geometric figures under given transformations. For rotations and flips, most students seemed to believe that length remained invariant, but for slides, the non-conservers saw the length of a geometric figure as changing. Similarly, for tasks involving a before-after comparison of only one figure, most students said that length would stay the same; however, when there was a congruent copy close by with which the student could make a visual comparison, non-conservers were significantly more apt to believe that a transformation changed the length of a figure." (p 191)

Kidder (1978) studied a group of 8, 9 and 10 year olds who were successful with Piaget's length conservation test. They were given operational definitions of translation, reflection and rotation. Each child was asked to select, and use, one of five sticks (all different lengths) to indicate how the 'object' stick would look after a particular motion. Only 7 out of 31 classical conservers could consistently perform the task correctly. Two main conclusions follow:

1. "... classical conservation of length is not sufficient to ensure length conservation on more complex mental operations."
2. Classical conservation of length is probably a *prerequisite* for conservation of length in more complex situations (Dickson *et al.*, 1984).

In a similar study of 9, 11, and 13 year olds (using triangles instead), Kidder found a large percentage of errors occurred due to a failure to conserve length. He therefore conjectured that, before a child is at the formal operations stage, she or he can only operate by focussing on one aspect of the task at a time, ignoring the remaining components.

Thomas also constructed a 'conservation of position' task for the same subjects from her previous task. She positioned a penny on the 'object' triangle and asked the child to determine where a second penny should go on the 'image' triangle after a particular translation, reflection or rotation. She reported that while the data collected appears to indicate that the reflection and translation tasks were easier than rotation, the results aren't statistically significant. What *is* significant is the type of error made by each age group. The 6 years olds errors revealed that they tended to focus on the *sides* of the triangles, whereas the 9 year olds' errors were influenced equally between vertices and sides. The 12 year olds could successfully negotiate both together resulting in almost no errors at all.

Perham (1978) examined the influence of the *orientation* of the mirror line on 6 year olds' ability to perform reflection tasks. Similarly, the *direction* of translation and the *angle size* of rotations were investigated. Before- and after- instruction comparisons were also made, using a control group. The 'before' results showed that students understand horizontal and vertical translations quite well, being able to construct an image themselves. Oblique translations, and all reflections and rotations were not understood at either level of representation (multiple choice or image construction). After instruction, gains were made on horizontal and vertical translations, and horizontal and vertical reflections (both external and internal). Performance on rotation tasks increased only for the multiple choice level of representation. Overall, Perham felt that the results reinforced Piaget and Inhelder's (1971) proposed order of learning transformations, that is, translation first, then reflection, then rotation. Secondly, the orientation of a transformation rather than its type appeared to more important in effecting learning; oblique transformations were particularly difficult for 6 year olds, even after instruction.

Shultz (1978) investigated a number of variables that might affect the difficulty of transformations for 6 to 10 year olds. She considered the three 'basic' types of transformations, various displacement sizes, horizontal and diagonal transformations, meaningful or non-meaningful configurations, as well the size of the object transformed. Several conclusions were made (see Dickson *et al.*, 1984), some of which are listed here:

1. Translations were easier than reflections, and reflections were easier than rotations.
2. Short translations were easier than long or overlapping ones, and horizontal translations were much easier than oblique ones.
3. Overlapping images was difficult.
4. The youngest children often changed a non-meaningful configuration into a meaningful one.
5. Large configurations were usually easier to translate than smaller ones.
6. Oblique reflection errors showed a fixation with vertical or horizontal displacements.
7. Children tended to turn an image so that it faced the direction of the reflection or turn.

Fischer's (1978) study on *Visual Influences of Figure Symmetry on Concept Formation in Geometry*, although not specifically on transformation geometry, seems particularly relevant. Among her conclusions are included statements such as:

> "Students do form concepts that are biased in favor of upright figures. [Also] students can more easily recognise upright figures for a concept than tilted figures, regardless of instructional experience. [In addition,] the visual distinctions that students perceive in figures are often more

compelling than the mathematical concept that is illustrated. [But] upright biases do not appear to be a reflection of poor conceptualization. ... there is a direct relationship between a bias *for* upright figures of a concept and success in learning the concept." (p 319-320)

Thomas' (1978) also considered the effect of a figure's symmetry on the difficulty of reflection and rotation tasks. The figures used were letters of the alphabet. The subjects were 9, 12, 15 and 17 year olds. Some of her findings were:

1. Rotating a figure with rotation symmetry was difficult to visualise.
2. Horizontal reflection with the asymmetric figure J was also difficult.
3. The 9 year old group scored lower on all the tasks than the other three groups.
4. There was no difference between performances on vertical reflection and horizontal reflection tasks, nor between direction of rotations.

Lesh (1976) himself made a number of related points about primary school children's difficulty with transformations. For instance, the difficulties in rotating simple objects and simple configurations seem to be roughly the same. However, if the object becomes too complex, then the task is more difficult. Secondly, a complex figure's properties may be preserved under a 'simple' transformation like translation but not for a more difficult transformation (like rotation). Thirdly, transformations can prove difficult if they are too large or too small (overlapping). Also, Lesh also reported that vertical and horizontal translations, reflection or rotations are all easier than oblique transformations. Lastly, and most unexpectedly, he claimed that a single transformation (e.g., reflection) is often only slightly easier than a composition of two transformations (two reflections). Furthermore, "children are sometimes more confused about the terminal configuration of a single transformation than they are about the terminal configuration of several transformations." Lesh added that the *order* of two compositions may also be a factor in a task's difficulty.

Education research on primary children's conception or performance of transformation tasks since the 1978 monograph seems to be scarce. Dickson *et al.* (1984) listed one other investigation: the first APU study in 1980 shows that 11 year olds were more successful in reflecting a figure in a vertical rather than oblique (45^o) mirror line.

In the journal *Arithmetic Teacher*, a number of articles on the use of logo for exploring transformations at the primary school level appear. While not presented as research studies in themselves, they all point to the motivation which computers induce in their students. For more details, see appendix D.

Finally, in Brazil (Nasser, 1989) and in Spain (Jaime and Gutierrez, 1989), more recent investigations have considered transformation geometry learning from the perspective of the van Hiele levels. In fact, much earlier, Coxford (1978) concluded at the end of the Lesh and Mierkiewicz's (1978) monograph that such a model be used for geometry education research in general. While extensive work by Russian researchers has occurred on the van Hiele levels in general (Wirszup, 1976), it is encouraging to see that specific ventures are now being made into the applicability of van Hiele's work to the learning of transformation geometry. Unfortunately, at the time of this thesis' submission, these articles are still unavailable and, consequently, they are not reviewed here.

Overall, it seems that the amount of research in the area of children's transformation geometry learning remains small and Thomas's (1978) comment still appears to be true:

> "... there is still a need for a more basic type of research concerned with assessing the understandings children have of transformation geometry concepts before they are exposed to any formal instruction in that area." (p 178)

Unfortunately, research on learning transformations at the *secondary* school level may be even more uncommon. Küchemann's (1980, 1981) reports on the Concepts in Secondary Mathematics and Science (CSMS) project appear to be the best known studies in this area. In the more detailed report (1981) of the two, he examined the effect of the presence or absence of a grid, the complexity of an object, and the slope of an object on the difficulty of reflections. He also considered variables affecting the difficulty of rotations (quarter turns). The subjects were 449 third year secondary school pupils, around 14 years of age. He also described the different kinds of answers given to the test items and the associated levels of understanding. Finally, combinations of reflections and rotations were explored as well as inverses. Some of his conclusions are listed below:

1. "Children found it much easier to cope with a vertical (or horizontal) mirror-line than one that was slanting... A common error when the mirror line was slanting was to ignore its slope and simply reflect across or down the page."
2. The grid was often a powerful guide to the choice and direction of reflection. However it did not help students overcome overt errors such as the horizontal reflection fixation when the mirror line was oblique.
3. "An increase in the complexity of the object can have a marked [detrimental] effect on facility and also on the quality of children's answers."
4. "... the tendency to reflect horizontally or vertically ... is particularly strong when the slope of the object is itself horizontal or vertical."

5. Overall children found it difficult to coordinate the slopes (of object and mirror line) so as to "conserve the angle between them".
6. The rotation results showed that, "in contrast to the reflection items, the presence of a grid did not necessarily make the items easier."
7. Horizontal and vertical starting positions were easier, and significant difficulty arose when the centre of rotation for the quarter turns was not on the flag.
8. "For combinations of transformations the results indicate that if children are to cope with these at all, the constituent transformations will have to be very much simpler than the ones used in the test." While nearly all the children had some understanding of the single transformations, the nature of some of the conceptual difficulties indicate that the study of reflections and rotations individually "need not be trivial."
9. "If transformations are to be studied in their own right, it would seem pointless to do this in a didactic, expository manner. ... The approach being advocated here is one that directs the children towards discoveries from which the rules and properties of the transformations can be surmised and against which they can be tested."

Some of the IEA (1987) results may be of interest here, especially to New Zealand educators. The study concluded that most of the questions set were too difficult or foreign for the New Zealand Form 7 students. Since transformation geometry is no longer present in the syllabi at that level (if it ever truly has been), there seems little point in reviewing their findings. The summary of the Form 3 study reported that New Zealand students obtained an average success rate of 55% with the six transformation geometry questions despite teachers expecting a low performance. Interestingly, in a multiple choice question involving the translation of a scalene triangle, only 32% of the students could correctly identify the appropriate transformation, compared with 88% who correctly recognised a reflection in another question. The study suggested that a bias in teacher presentation may have been responsible for this. Also surprising was that the boys did significantly better than the girls on this question. A pre- and post- test analysis was also done, with the following conclusion:

> "Generally, students showed an 11% growth from pre-test to post-test with the most marked differences (26% and 11%) arising on the two questions involving half-turn rotations. Boys performed a little better than girls, with an exception evident in a question where the object to be turned was not a shape, but the students themselves. The girls were 4% more successful than the boys with this question." (p 180)

Over the last two decades, the use of microcomputers for education purposes has increased dramatically. Two studies are discussed which investigate adolescent transformation geometry learning in a computing environment. The earlier of the two is a short communication by Ernest (1986) on an experiment designed to "provide motivated practice for the retraining of transformation geometry." Subjects were fifteen year olds of "below average mathematical ability." After 6 weeks of classroom instruction, the students were given two pre- and two post- computer gaming tests; Test 1 was specific to the game, Test 2 more general. Comparing the results of the control (no gaming) group with the experimental group, Ernest found that the computing gaming improved the results of the specific tasks in Test 1 but no gain was apparent in the related (but distinct) skills required in Test 2. While there was no evidence to show that non-computing practice of these specific skills would not have produced the same effect, Ernest concluded:

> "the computing gaming also resulted in a positive affective response, which the paper and pencil practice might not match, especially with the below average students of the sample." (p 207)

The second article by Edwards (1988), which formed part of a larger ongoing study into learning mathematics in the microworld environment in general, discussed the performance of 12 to 14 year students on tasks such as transformation identification, drawing the result of a transformation, finding inverses, and combining transformations. She found that the identification task was the most difficult, especially when finding the centre of a rotation. Two interesting 'erroneous' strategies arose for this particular task: (a) "selecting the midpoint of an imaginary line connecting the starting vertices of the two shapes", thereby missing some of the problem's constraints (or possibly generalizing from the half-turn case); (b) interpreting rotation as "a composite motion which moves the shape *to* a specified point and then turns it around its starting vertex." In fact, this second interpretation is not erroneous; it is simply a composition of two transformations instead of using only one. Neverthelss, Edwards believed that it reflects a naïve conception that it is only the figure, and not the plane, which is being transformed. Consistent with Küchemann's (1981) findings, Edwards also found that rotations, where the centre of rotation lay outside the figure, were difficult for some students. Finally, almost all the students were successful at predicting images and finding inverses. Most could also manage the transformation combinations, somewhat unexpected in the light of Kuchemann's study (1981). The most difficult combination was that of two parallel reflections; some students found it surprising that the result was not a reflection but a translation.

Symmetry

So far the discussion has considered transformations, leaving the consideration of symmetry, which is easily a topic in its own right, until now. The key difference between symmetry tasks and transformation tasks, in Piagetian terms, is that the former usually involves a static identification of an image, whereas the latter may involve kinetic images (reproductive or anticipatory). The number of studies in mathematics education on the identification of symmetry is small, especially when compared to the bulk of literature to be found on this topic in the field of perceptual psychology. Our summary begins with the Primary school sector.

Dickson *et al.* (1984) briefly noted the findings of the first APU Primary Survey. It discovered that the nature of a figure could affect the ability of 11 year olds to identify a vertical line of symmetry. On the item that featured a rectangle, only 19% were successful in identifying both lines of symmetry.

Arithmetic Teacher reported that the *Results and Implications of the Second NAEP Mathematics Assessment* (1980) of elementary schools in the United States found that over 80% of 9 year olds could identify a pair of figures that were the same size and shape (congruent). The results decreased only slightly when some transformation of these figures was required for matching. Secondly, almost 80% of 9 year olds could determine that a shape did not have reflection symmetry with respect to a given line. The *language* used for this task also had significant effect on these tasks. For example, 80% of the 9 year olds could "select a letter that could be folded so that both sides would match", but only 18% could answer the same question when the term 'symmetric' was used. (While the details aren't clear, it certainly appears that the survey has assumed that the word symmetric means reflectionally symmetric).

At the Secondary level, only a few indications of students' understanding of symmetry seem to exist. The first APU Secondary report (cited by Dickson *et al.*) required students to indicate all lines of symmetry in various figures. Dickson *et al.* (1984) summarized their material in a table which suggests that the 15 year olds (and the 11 year olds) found identifying a single vertical reflection line easier than identifying a single horizontal or oblique reflection line. Furthermore, the proportion of students who could identify a horizontal line of symmetry was the same as the proportion who could identify an oblique line of symmetry. Not surprisingly, the group of 15 year olds performed better on all such symmetry tasks than did the 11 year old group.

The second NAEP (1980) study also reported that over 90% of 13 year olds could pick out a pair of figures with the same shape and size. Again, the success rate dropped only slightly when a figure needed to be transformed in some way to make a match. However, the term 'congruent' was unfamiliar to many 13 year olds.

Given this lack of information on symmetry identification at all levels, it seems appropriate to at least be aware of the perspective that perceptual psychology brings to this subject. We take recourse to this approach in the following subsection.

2.2.2 Symmetry: A Perceptual Psychology Viewpoint

A Word of Caution

One approach to the question of symmetry perception and preference is via perceptual psychology but it can be difficult to draw implications for the formal learning environment from laboratory experiments. For instance, most of the studies' subjects are adults; most students are not. It is also important to be cautious about making pedagogical inferences from such studies because the psychologist's interests is not the same as the mathematics educator's (Bishop, 1980):

> "Certainly no currently available psychological theory, including Piaget's, is ready for wholesale adoption by mathematics educators. ... It is time for mathematics educators to forge ahead to investigate new concepts and new tasks that have not yet been considered by psychologists. ..... [A goal of space research is] to clarify some of the relationships between figurative and operative thought. Even if psychologists believe they can ignore variations due to figurative content, the issue is highly important to educators who must use concrete materials and figurative models to teach mathematical concepts." (Lesh, 1976)

> "More recent school experiments show that didactical results derived from psychological research do not agree with the results found in classroom research." (Schipper, 1983)

In some instances, Washburn and Crowe (1988) warned that the conclusions of these studies may be mathematically misleading. For example, Julesz (1975) did not consider repetition to be a form of symmetry, which is indicative of a larger problem with several experimental psychological studies, that is, that symmetry is assumed to mean mirror symmetry. Washburn and Crowe also reported that few articles even defined symmetry and none had controls for cultural differences, rendering some conclusions useless for anthropological consideration.

Similarly, throughout Howard's (1982) book are examples of some of the shortfalls of psychological studies in orientation and symmetry. Consider, for instance, the attempts to illustrate that mirror image orientations are more easily confused than are the other orientations. Incredibly, the shapes used in *all* such studies were not asymmetrical so that certain arrangements of two such congruent figures could be regarded as 180^{o} relative rotations rather than reflections. This observation has implications which are discussed in subsection 6.3.1 (see 'undifferentiated transformations').

Despite these shortcomings, Bishop (1980) argued that there is still value in considering the work of psychologists, factor analysts and the like because they can sharpen the ideas of mathematics educators and provide stimulus for "the development of teaching material for the classroom." Perhaps, then, the most natural and fundamental question to begin with is: "what is perception?"

Perception

Essentially, perception is a *process* by which an individual obtains information from the environment. In almost all settings however, there is usually more information than a single person can assimilate. Washburn and Crowe (1988) concluded in their review:

> "Thus perception involves selection. Through socialisation, an individual in a particular culture learns to focus on features which will enable him [or her] to predict events, reduce uncertainty, and make appropriate responses."

The first stage of perception involves the encoding of the stimuli, called *pattern recognition* (Reed, 1973). Foster (1984) summarized four kinds of shape recognition that experimental psychologists have discovered:

1. *Local features* like whether points in a pattern are on a straight or curved line.
2. *Local spatial relationships*, for example, whether points in a pattern are left, right, above or below the vertical/horizontal reference.
3. *Global features* associated with the pattern such as symmetry and orientation.
4. *Global spatial relationships* like the position of the pattern in the field.

These features are used in two different learning processes: *discrimination learning* and *generalisation learning*. Respectively, these processes involve learning the features which distinguish one pattern from another or which are used to group patterns as equivalent. The second of these processes can be thought of as analogous to the acquisition of

Howard's (1982) *descriptive domains* which are the rules a person uses (often subconsciously) to relate an object to a class of objects, other members of which are not in view. Such a class of objects is called the *stimulus domain*. Howard noted that ambiguities about stimulus and descriptive domains plague most experiments in perception. He also introduced the idea of *response domains*, that is, the type and level of complexity of a response. Responses can vary from simple *detection* to *distinguishing between*, *identifying* and *describing*.

Actually, several years beforehand, Bruner (1957) made the assumption that "all perceptual experience is necessarily the end product of a categorization process." He described four facets of the (unconscious) classification process. These are:

1. The critical attribute values (properties) for an object to be included in a category.
2. The way in which the properties can be combined in order to make inferences about the category.
3. The assigning of weights to various properties.
4. "The acceptance limits within which properties must fall to be criterial."

Howard's (1982) work on human visual orientation contains some other useful terminology which is used in chapter 6 of this thesis. Firstly, the *direction* of an object is specified by selecting a reference point O through which is drawn a reference line. The direction of a point P on the object, from O, with respect to the reference line, is the signed angle between line PO and the reference line. The *orientation* of an object is its rotational position in space relative to some fixed axis of reference. There are three types of *directional* and *orientational judgements*: *proprioceptive* (the direction or orientation of one part of the body with respect to another part of the body), *egocentric* (the direction or orientation of an external object with respect to the body of the observer), and *exocentric* (the direction or orientation of an external object with respect to an external point and/or axis). Other helpful vocabulary is included in Howard's discussion of visual polarity:

> "Objects like trees, people, and houses are normally seen in one orientation with respect to gravity and with respect to the normally-vertical retinal meridia. Such objects have an intrinsic top and bottom and will be referred to as *mono-oriented* objects, to distinguish them from *poly-oriented* objects, such a scissors, toothbrushes and tennis rackets, which have no preferred orientation" (p 571)

Symmetry Detection and Preference

Freyd and Tversky's (1984) article found that the relevant literature supports the notion that humans have an effective and efficient symmetry *detection* ability. A natural question to ask is whether symmetry detection leads to biases in (mental) representations of visual forms. The answer is yes. Freyd and Tversky found, using a similarity-judgement task and a matching figures task, that:

> "... nearly symmetric standard forms are judged to be more similar to, and are more confusable with, even more symmetric forms than they are with less symmetric forms. The pull toward a more symmetric form does not occur for standard forms of lower symmetry....[Also], if the form is perceived as having overall symmetry, the form is assumed, sometimes incorrectly, to have symmetry at the local level as well"

In an article on symmetry in visual art, Molnar and Molnar (1986) note that many objects are 'seen' as symmetrical when they are not according to the mathematical definition of symmetry. Luckiesh (1965) illustrated this tendency to distort images to 'see' symmetry with the following example. The figures "S" and "8" appear to be approximately 'even'. However, invert the sequence 8888SSSS, and the half-turn symmetry of each figure is shown to be an illusion. Other tests conducted by Eisenman and Rappaport (1967) showed that symmetrical shapes (taken from Birkhoff, 1933) are preferred (and preferentially produced and reproduced) over asymmetrical shapes, whatever the complexity of the shapes.

With many studies (see Howard's work for a critical overview) displaying the importance of visual orientation, some of the findings with respect to reflection symmetry are not surprising. For instance, Fox (1975) suggested that bilateral symmetry is a *diagnostic* feature, that is, an appropriate response (identification of an object) is made without any other analysis of the stimulus. Zusne and Michels (1962) found that their subjects always used bilateral symmetry to order a set of figures according to any of the three terms 'geometric', 'regular' or 'familiar'. Goldmeier (1972, originally 1937) demonstrated that not only is reflection symmetry a very important feature of a form's appearance, but *vertical* reflection symmetry is the most salient. Rock and Leaman (1963) supported this finding:

> " ... this is the only orientation in which a symmetrical figure will spontaneously be perceived as symmetrical. ... This fact can be observed in everyday life. Vertically symmetrical figures do

look symmetrical, whereas the symmetry is often not noticed in horizontally symmetrical figures..."

Rock and Leaman (1963) continued to explore Goldmeier's findings by testing for differences in the phenomenal (visual frame) orientation of an object and its retinal (ego- and oculo- centric) orientation. Their results showed that there is little change in appearance of a *novel* figure when only the retinal image is changed. If the orientation of a novel figure in the environment is changed it is usually perceived as different. Rock (1973) explained that this is due to individuals assigning the directions top, bottom, left and right on the basis of cues from the visual or gravitational frame. The greatest differences in perceived form occur for rotations of 45^o or 90^o, with the least for left-right reversals. However, Howard (1982) maintains that the ability to recognise *familiar*, mono-oriented objects is more disrupted by a disorientation of 180^o with respect to the retina than to gravity or the visual frame. In fact:

"A disorientation of 90^o or less with respect to the extrinsic axis has very little effect on the ability to recognise mono-oriented objects under ordinary conditions."

Finally, from the point of view of intuition, it is interesting to note that preference for the phenomenal vertical orientation, and symmetry, is often subconscious:

" ...this effect occurs without the subject being in any way aware of what it is that determines his [or her] selection, or that he [or she] is behaving consistently with respect to any determinant. Symmetry is never mentioned and, as far as could be determined, did not enter as a conscious basis of choice." (Rock and Leaman, 1963, p177)

2.3 Patterns and Symmetry

The amount of reference material on the creation of patterns or designs and its relationship to *learning* transformation geometry appears to be small, but growing. Sources such as *Arithmetic Teacher* and *Mathematics in Schools* provide some excellent ideas for use in the primary or secondary classroom. For instance, Oliver (1979) presented an effective way of investigating symmetries through tessellations of various shapes. Also, Renshaw (1986) discussed how reflection and rotation symmetries can be taught using some familiar trademarks. However, Williams (1989) made the point that while the symmetry of finite designs and wallpaper patterns gets considerable treatment in (American) schools, the use of one-dimensional (frieze) designs for this purpose is somewhat overlooked. One benefit of exploring frieze patterns, Williams contends, is that their complexity lies between that of finite designs and infinite two-dimensional designs.

But while the authors of various articles extol the virtues of transformation geometry and encourage a 'design approach', very little education research appears to exist which explores the relationship of visual patterns to transformation geometry from the perspective of students. Two natural questions arise: Firstly, do primary or secondary school school students intuitively use symmetries to produce patterns, and if so, which symmetries? Secondly, are any of the symmetries of a pattern perceived by primary or secondary school students, and if so, which symmetries? In both cases, it is also of interest to consider *how* the symmetries are used or understood by the students.

Not surprisingly, the studies of visual pattern perception are chiefly psychological. Unfortunately, the patterns used in these investigations were almost always 'finite' (trivial). Some of these articles are reviewed in subsection 2.3.2. In the next subsection we examine the conclusions of an extensive study by one researcher on the paintings of 5 and 6 year old (Australian) children and the presence of symmetry within them.

2.3.1 Young Children's Spontaneous Pattern Painting and Transformation Geometry

Perhaps the most closely related education study to this thesis topic is the work of Drora Booth which includes a masters thesis (1975), a Ph.D. thesis (1981) and a set of papers published in a variety of education journals from Australia. Booth's work has involved studying the spontaneous patterns painted by five-to-six year old children. In all her studies:

> "The children were instructed not to overload the brush with paint but merely to load the tip of the bristles, and they were shown how to wash the brush clean of one colour before using another. *No other instructions were given. They were completely free to paint how and what they liked.*" (Booth, 1982). Note: Italics added.

> "... conditions [were] free from teachers' unconscious pressures towards pictographic representation." (Booth, 1980)

> "[The paintings] were retained in the schools until they were analysed and a record of their date and content made. They were then sent home in bundles at the end of each term in order to

minimise the cumulative effect of repeated parents comments and criticisms on paintings taken home individually." (Booth, 1982)

Booth's earlier treatment of young children's pattern painting (1975) was essentially a cognitive (and aesthetic) development approach. Her concern was that of an art educator. She interpreted the patterns *post hoc* in terms of Piaget's cognitive development theory and showed that:

"... pattern making parallels the development of logico-mathematical structures of the child's knowledge of geometry and mathematics." (p ix)

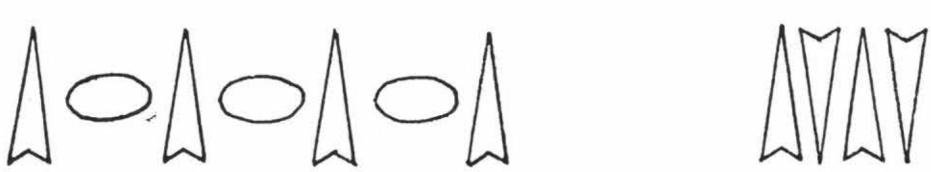
One of her main findings was that:

" ...pictographic representation is not a spontaneous discovery by the child [whereas] non-representional geometric pattern making is a spontaneous development and does not depend on learning by imitation." (p viii)

In her discussion of the theoretical implications of her masters thesis, Booth contends that the essential shortcoming of young children's art education, even today, is that importance is given to representational drawing at the expense of design. She emphasized this conclusion with a rather pointed quote:

"There can be no doubt ... that the average child has extraordinary inventiveness in design and the average adult has none whatever, and that in between these two states there occurs the process known as art education." (Robert Fry, as cited by Booth)

An interesting feature of Booth's analysis is the use of qualitative principles that she engaged. She listed five simple operations: translation, rotation, reflection, alternation and inversion. What makes the last two terms noteworthy is not that they are redundant (mathematically), but that they appear to be closely related to the *intuitive conceptualization* of some transformations made by children interviewed in this present study on frieze designs.



'Alternation' 'Inversion'

Figure 2.0 (from Booth, 1975, p 18)

In a personal correspondance (1991), Booth explained that inversion can be thought of as a half-turn. This may be correct, but her *diagram* (fig. 2.0) suggests that inversion can also be interpreted as a glide reflection, a rigid transformation that Booth does not include in this or her subsequent studies.

While Booth's earlier concepts may still be useful for describing a child's 'intuitive understanding' of relationships within a pattern, the redundancy present can make methodical classification difficult. Systematic research of children's spontaneous non-representational ('geometric') art doesn't appear to have begun until well into this century (Booth, 1975, 1982), leaving room for revision and refinement. For instance, Booth now prefers to classify 'geometric patterns' using a transformation geometry framework, as she explained in personal correspondence (1991) with the researcher:

> "You will have noticed that I no longer include these ["inversion" and "alternation"] in my articles. The reason being that since my masters thesis I have attempted to simplify my description of patterns to accord with transformation geometry or symmetry concepts."

More generally, from her work (1975, 1976, 1977, 1980, 1981, 1982, 1984, 1985, 1987, 1988, 1989), Booth has reported that three broad types of pattern were identified: non-representation, pictographic representation, and a mixture of the two previous categories. The non-representation patterns were classified according to three developmental stages (scribble, topology and geometric) and two intermediate stages (pre-topology and quasi-pattern). The mixed category was also classified according to the non-representation content present in the paintings.

The geometric patterns were divided into two main groups, the first arising from a systematic repetition of an element, the second arising from a division of the plane. It is interesting that these categories correspond quite closely to Grünbaum and Shephard's (1987) mathematical definitions of *patterns* and *tilings*.

Booth (1982) summarized her 'geometric' results by stating that the first pattern to emerge from the topological stage is a 'translation pattern' usually of vertical lines, or occasionally dots. After that, the development sequence is not linear but complex and tree-like. For example, an old structure may disappear only to reappear several patterns later. However, it emerged clearly that:

> "... the symmetries operations underlying the construction of patterns develop in the order of (i) translation in one dimension, (ii) translation in two dimensions and one-fold reflection, (iii) two-fold reflection, and (iv) rotation."

Booth (1982) drew several other conclusions. Firstly, children's intuitive symmetry in spontaneous pattern making could provide a natural foundation from which to teach mathematics, science and art. This is because symmetry is an important link between these subjects. Secondly, the development of the symmetry operations underlies aesthetic development. Lastly, Booth argued that knowledge of the developmental sequence in pattern painting contributes to our understanding of human cognition.

Booth also tested out some of these conclusions in a longitudinal study on the use of young children's spontaneous pattern painting as the starting point for teaching art and transformation geometry. This was done by raising to a conscious level a number of intuitive concepts embedded in the paintings, the vocabulary including the names of lines, shapes and patterns. Booth (1984) reported encouraging results:

> "Children whose intuitive use of symmetry operations in spontaneous pattern painting had been raised to a conscious level appeared to have a better grasp of symmetries than children taking part in formal lessons only."

2.3.2 The Role of Symmetry in Pattern Perception - A Psychological Perspective

Given the presence of patterns, and in particular kowhaiwhai and tukutuku (Maori rafter and weaving patterns), in the New Zealand Fourth (and possibly Fifth) Form mathematics curriculum as examples for exploring symmetry, it is somewhat surprising that little is known about the extent to which symmetry is *perceived* in such patterns. Little or no mathematics education literature exists on this topic as far as the researcher can ascertain so, as before, we make an initial step towards that goal by examining some of the relevant perceptual-psychology literature.

Symmetry as 'Goodness'

Fred Attneave was one of the pioneers of research in the field of symmetry perception. Among other things, he conducted tests to provide a "clarification of the Gestalt doctrine that 'figural goodness' is favourable to memory." (1955) He did this by measuring symmetry as a form of redundancy. His (1955) article indicated that symmetrical patterns are more easily remembered than asymmetrical patterns occupying the same number of cells (his experiments used patterns of dots within rectangular matrices). He suggested that symmetric patterns contain less information.

Locher and Nadine (1973) argued that symmetric *shapes*, being redundant, should therefore take less time to examine, but subsequently found that this was not the case. While the distribution of eye fixations was different for symmetrical and asymmetrical shapes, no differences were found in the number of fixations or the length of the fixation times.

Nevertheless, in a discrimination study, Gardner and Sutcliff (1974) found that *patterns* of equal goodness (as defined by rotation/reflection equivalence set size) were equally encodable, and the better ('more symmetric') the pattern, the faster it encoded. Szilagi and Baird (1977) also found that subjects preferred and preferentially produced symmetrical patterns in one-, two- and three-dimensional space. The 'goodness' of patterns was inversely related to the quantitative degree of asymmetry. Furthermore, Howe (1980) reported that:

> " ... the less the partial symmetry in a pattern, the lower was the judged goodness of the pattern. [Also], correct reproduction improved with degree of partial symmetry [and] concordant results were found in a free recall learning task."

The studies above, while indicating a strong preference for 'symmetry' in patterns, are not specific about *which* symmetries are preferred and *what* factors influence those preferences. It is to these considerations we now turn.

Factors Influencing Symmetry Preference in Patterns

The perception of reflection symmetry in a pattern (with a single mirror axis) is affected by its orientation. Takala (1951) found that subjects were better at finding a figure embedded in a complex pattern if the figure's axis of symmetry was aligned vertically rather than horizontally. Corballis and Roldan (1975) also obtained results which suggested that the symmetry of a pattern about its vertical axis is easier to recognise than horizontal reflection symmetry. In addition they found that symmetry about a diagonal (45^{o} or 135^{o}) axis was of intermediate difficulty. But in a similar study (with the axes not displayed) Palmer and Heneway (1978) concluded that while vertical reflection symmetry was easier to recognise than horizontal reflection symmetry, oblique (or diagonal) reflection symmetry was the most difficult to detect.

In a study of young (American) school children's *recall* and *preference* for patterns, Paraskevopoulos (1968) formed conclusions parallel to those of Palmer and Heneway above. He also maintained that the structure necessary to decode symmetry begins to become effective for:

1. Double symmetry (i.e., vertical and horizontal reflections *together*) at the age of six.
2. Vertical bilateral symmetry at the age of seven.
3. Horizontal symmetry at the age of eleven.

However, Paraskevopoulos explained that his results are more in agreement with Piaget's (1947) theory that perceptual organisation is a result of a prolonged and sequential development rather than the Gestaltist view that the principles of perceptual organisation are intrinsically and innately determined.

In Bruce and Morgan's (1975) study, observers were asked to detect small violations in (finite) patterns with vertical symmetry or a translation. They found that:

> "The decision time for symmetric patterns [i.e., with vertical reflection symmetry] tended to be shorter than for repeated patterns. ... [Also], the saliency of [vertical reflection] symmetry seems to a considerable extent to depend upon the ease of comparing spatially-contiguous elements *near* the midline of the pattern" Note: italics added.

Therefore, in some instances, symmetry perception is also affected by proximity as well as orientation. Corballis and Roldan (1974) also discovered this when they required their subjects to judge patterns as symmetrical (i.e., vertically symmetric) or asymmetrical (i.e., repeated). The patterns consisted of either dots or arrowheads. From the data given it appears that the recognition of 'symmetrical' patterns is faster if the figures are close together. Also, the *form* of the patterns' motifs is also a contributing factor to the symmetry perception 'equation', since the perception of 'symmetrical' patterns was faster for arrowheads than dots. Other aspects of the experiment such as instructions and the locus of fixation also played a part. The experimenters concluded:

> "These factors and the interplay between them can be reduced to a single common principle: If the patterns are perceived holistically, [reflection] symmetry is more salient than repetition, but if they are perceived as two separate figures to be matched, then repetition is judged more rapidly than [vertical] symmetry."

The obvious omission from this discussion is that of rotation and glide reflection symmetries in patterns and the factors which affect their perception Unfortunately, relevant material seems to be very sparse indeed. The only source of information (discovered by the researcher) is in Washburn and Crowe's (1988) review of symmetry reproduction where they formed the opinion that the various studies were conflicting! In this review they noted:

> "Tekane (1963) found that Bantu subjects could complete patterns with bilateral symmetry better than patterns with rotational symmetry." (p 23)

2.4 Mathematical Classifications of Finite and Border Designs

In chapter 1 a rough sketch of the four rigid transformations, and the seven possible strip designs that can be generated from various combinations of these transformations is made. In this section we examine those concepts more closely through mathematical definitions. Considerations such as the classification of designs, patterns and pattern construction are also introduced. Almost all the terminology presented is used throughout the following chapters. However, *some modifications to the meanings of the terms have been made in chapter 3* in order to better reflect the 'intuitive' *use* of transformations by the subjects of this study. Some of the theory below is found in Martin's (1982) introduction to transformation geometry or Schattschneider's (1986) article on the creation of patterns, although most is extracted from Grünbaum and Shephard's work (1981, 1983, 1987).

2.4.1 Isometries

An <u>isometry</u> or <u>congruence transformation</u> is any mapping of the Euclidean plane E^2 onto itself which preserves all distances. That is, α is an isometry if, for any points P and Q in the plane, PQ = P'Q' where P' = α(P) and Q' = α(Q). (By implication, an isometry also preserves angles and area). A <u>symmetry</u> on a set S, is an isometry that maps S onto itself, that is $\alpha(S) = S$. For brevity, the term 'transformation' is used throughout this thesis instead of 'congruence transformation'; that is, it excludes non-rigid affine-transformations unless stated otherwise. It can be shown that every isometry is one of four types (see Martin, 1982):

1. Translation in a given direction through a given distance. More formally, given two fixed points A and B in the plane, the translation τ_{AB} is the transformation $P \rightarrow P'$ where PP' is parallel and equal to AB.

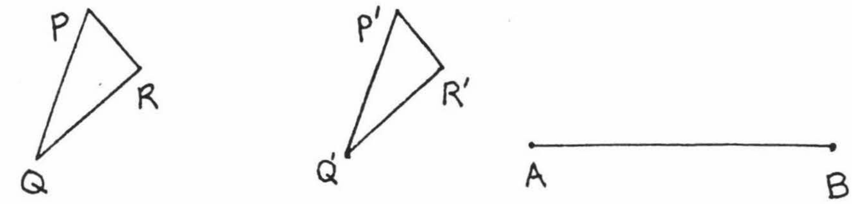


Figure 2.1

2. Rotation about a single point O through a given angle θ. The point O is called the *centre of rotation*. The geometric definition states that if A is a fixed point in the plane, and θ is a fixed directed angle then the rotation $\rho_{A,\theta}$ is the transformation $P \rightarrow P'$ such that $AP = AP'$ and $\angle PAP' = \theta$. When $\theta = \pi$, the mapping is often called a *half-turn*.

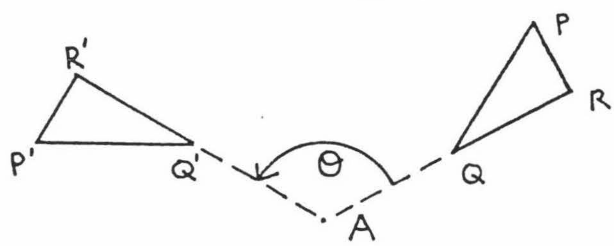


Figure 2.2

3. Reflection in a given line ℓ (the *mirror* or *line of reflection*). This means that if ℓ is a fixed line, a reflection in ℓ, σ_ℓ, is the transformation $P \rightarrow P'$ such that if P is not on ℓ then ℓ is the perpendicular bisector of PP', and if P is on ℓ then $\sigma_\ell(P) = P$.

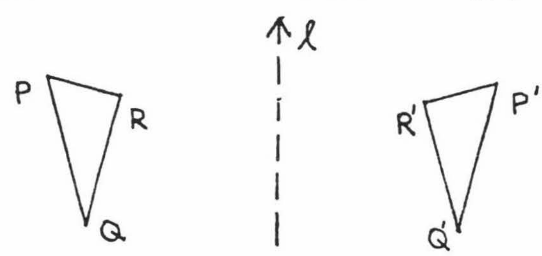


Figure 2.3

4. If ℓ is a fixed line then a glide reflection γ with axis ℓ is the product $\sigma\tau = \tau\sigma$ where $\sigma = \sigma_\ell$ and τ is a translation in the direction of ℓ.

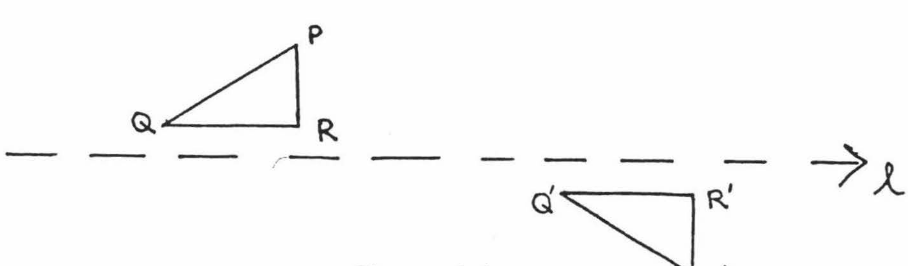


Figure 2.4

Isometries of the types (1) and (2) are usually called *direct* or *even* isometries because if a triangle has its vertices labelled ABC clockwise, its image under either of these isometry types retains the same orientation. Isometries of the types (3) and (4) are usually called *indirect, reflective* or *odd* isometries, since they reverse the orientation of the labelling described.

Two more points: note that the identity transformation maps every point of the plane onto itself and is a symmetry of every set. It can be considered as a trivial translation or a trivial rotation. Secondly, no distinction is made between a counterclockwise rotation of θ and a clockwise rotation of $2\pi - \theta$, nor between a rotation of θ and a rotation of $\theta + 2\pi k$, for any integer k. As symmetries these are regarded as identical. Indeed, from a mathematical point of view, only the final result of a transformation is relevant, not the means of arriving at that result.

2.4.2 Symmetry groups

By convention, the horizontal direction is considered to be in the direction of the frieze group's translation symmetry; the vertical direction is perpendicular to the horizontal direction.

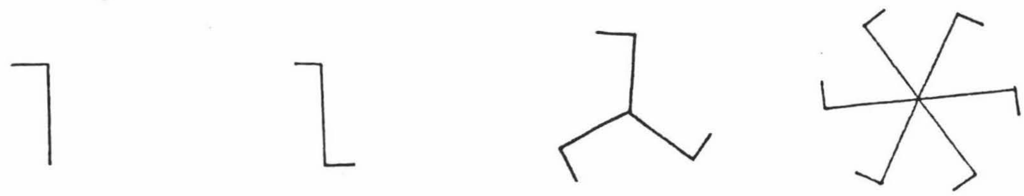
Probably the most common mathematical way of classifying a design, A , is according to the collection of all symmetries of A, S(A). Two such collections are of the same *class* if one can be transformed into the other by an affine transformation. It is well known that S(A) forms a group under composition and the number of symmetries in S(A) is called the *order* of the group.

The notation for symmetry groups of finite order is well established. Such groups arise when A is a plane set with the property that:

SG.1a There is a point invariant under all symmetries in S(A).

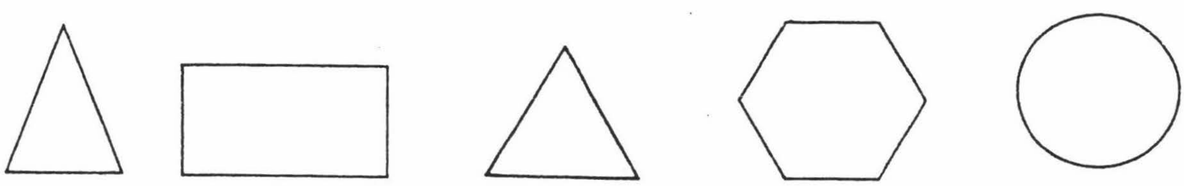
The group consisting of one isometry only (the identity) is denoted as C_1 (or "e"), and C_n is used for the group consisting of rotations through an angle $2\pi j/n$ ($j = 0, 1, ..., n-1$) about a fixed point. This is called the cyclic group of order n. This can be thought of as the symmetry group of the 'n-armed śwastika'.

The second kind of finite symmetry group, D_n, is called the dihedral group of order 2n and consists of all the isometries of C_n together with n reflections equally inclined to one another. For $n \geq 3$, D_n can be considered to be the symmetry group of the regular n-gon. The infinite dihedral group, D_∞, consists of all rotations about a given point and all reflections in lines through that point. It is the symmetry group of a circle. A small list of illustrative figures follows:



C_1 C_2 C_3 C_6

Figure 2.5.1 (Examples of Cyclic Groups)



D_1 D_2 D_3 D_6 D_∞

Figure 2.5.2 (Examples of Dihedral Groups)

Instead of imposing property SG.1a, we can obtain infinite border designs by letting A satisfy the following conditions:

> **SG.1b** S(A) contains non-trivial translations, and all translations in S(A) are parallel.
>
> **SG.2** S(A) does not contain arbitrarily short translations.

Under these requirements it turns out (see Martin, 1982) that there are 7 classes of frieze or strip groups. An example of a design from each class can be found together with its nomenclature in section 1.2. Equally well known is that if SG.1(b) is replaced by the condition that S(A) contains translations in two independent directions, there are exactly 17 classes, called wallpaper groups. However, in this thesis, attention is generally restricted to border, frieze or strip designs.

It is possible that designs created in this thesis may not completely satisfy SG.1 and SG.2. Of interest then is Grünbaum and Shephard's (1981) comment that suitable modifications of SG.1 and SG.2 can produce interesting designs and add to the number of classes. The mathematics of such considerations is far from trivial. For instance, the possible classes of symmetry groups S(A), when A is assumed to satisfy SG.2 but no topological restriction is made, have never been determined according to Grünbaum and Shephard (1981).

2.4.3 Tilings and Patterns

Grünbaum and Shephard (1987) organised their comprehensive book on the basis of two categories: tilings and patterns. The definition of tiling that they give is of the plane, but it can be easily modified to apply only to a strip, $\mathcal{S}$, which is a plane set enclosed between two parallel lines (the border lines, or edges, of $\mathcal{S}$). We restrict our attention here to a strip $\mathcal{S}$ with finite width, w, infinite length, and a centreline, l, which is equidistant from the edges.

A loose, general definition of a tiling of strip $\mathcal{S}$, is a countable family of closed sets $\mathcal{T} = \{T_1, T_2, \ldots\}$ which cover $\mathcal{S}$ without gaps or overlaps. The sets $T_1, T_2, \ldots$are called tiles of $\mathcal{S}\mathcal{T}$. In most investigations it is convenient to consider normal tilings of the strip. Put formally, the union $\bigcup_{i \geq 1} T_i$ is the exactly the strip $\mathcal{S}$, and:

N.1 Every tile, T_i, of $\mathcal{T}$ is a topological disk.

N.2 The intersection of every two tiles of $\mathcal{T}$ is a connected set, that is, it does not consist of two (or more) distinct and disjoint parts.

N.3 The tiles are uniformly bounded.

In contrast to a tiling of the strip, a *strip pattern* can be loosely thought of as repetitions of a motif, within a strip, in a 'regular' manner with no overlaps. A motif means any non-empty plane set. A mono-motif strip pattern, $\mathcal{P}$ with motif M is a non-empty family of sets in a strip, labelled by index-set I, such that the following conditions hold:

P.1 The sets M_i are pairwise disjoint

P.2 Each M_i is congruent to M and called a copy of M.

P.3 For each pair M_i, M_j of copies of the motif there is an isometry of the plane that maps $\mathcal{P}$ onto itself and M_i onto M_j.

Those symmetries of $\mathcal{F}$ which map a copy of M_i of the motif M onto itself form a group denoted by $S(\mathcal{F}/M_i)$. All groups $S(\mathcal{F}/M_i)$ are isomorphic and $S(\mathcal{F}/M)$, the *stabiliser* of M in $S(\mathcal{F})$, is considered to be *equal* to all each of these groups. If $S(\mathcal{F}/M)$ consists of the identity alone, the pattern is called *primitive*; otherwise it is *non-primitive.*

2.4.4 Further Refinements

A mono-motif strip pattern, $\mathcal{F}$ with motif M is called a discrete pattern if the following conditions hold:

DP.1 The motif M is a bounded and connected set.

DP.2 For some i there is an open set E, which contains the copy M_i of the motif but does not meet any other copy of the motif; that is, $M_j \cap E_i = \varnothing$ for all $j \in I$ such that $j \neq i$.

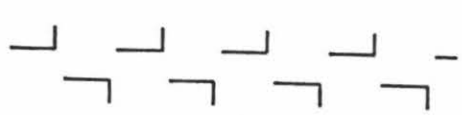
Such a discrete pattern is non-trivial if it satisfies another condition:

DP.3 The pattern contains at least two copies of the motif.

There exist 15 discrete types of frieze pattern, shown below. Patterns of the same discrete type are said to be *henomeric.*


PS1


PS5


PS2


PS6


PS3


PS7


PS4


PS8

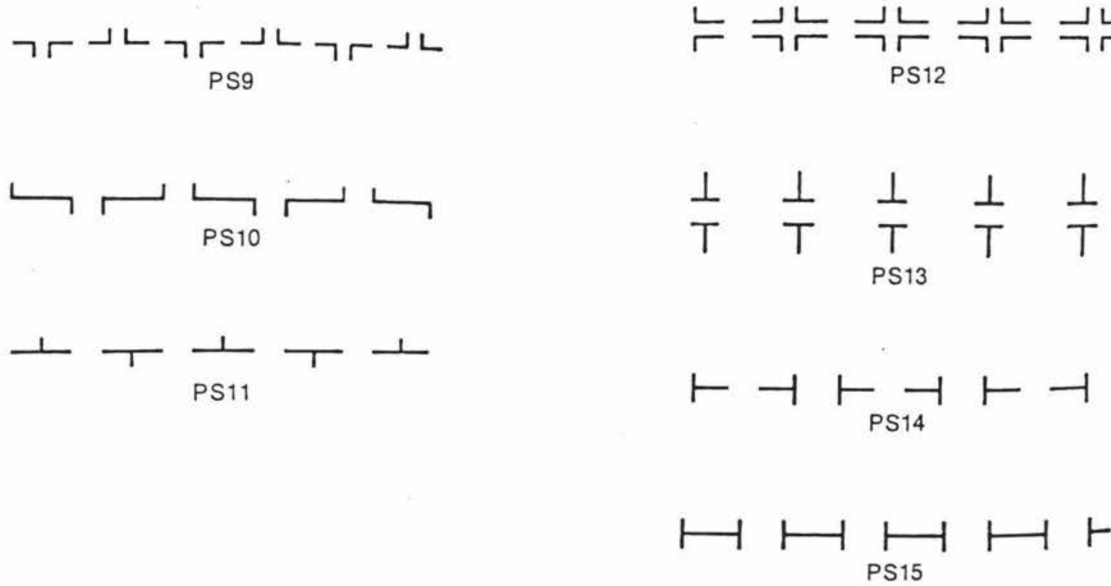


Figure 2.6 (Examples of the 15 types of Discrete Strip Patterns)
Reproduced from Grünbaum and Shephard (1987), p 244.

Further refinements of the discrete patterns (using such concepts as homeomerisms and diffeomerisms) are possible but are not defined here. For a synopsis of such classifications see Grünbaum and Shephard (1981, 1983).

Some pattern 'anomalies' can also be described successfully if sufficient restrictions are placed on them. For instance, we can consider mono-motif patterns with ***unbounded motifs***. A particular example of this is a curve pattern which satisfies conditions P.1, P.2, P.3, and DP.2. Instead of imposing condition DP.1 we also require that:

> **UP.1** The motif M is a closed set obtained as the image of a straight line under a homeomorphism of the plane onto itself.

This (roughly) means that M is a simple curve which stretches to infinity in both directions. A special type of curve pattern is a filamentary one which has a motif (called the *filament of* $\mathcal{F}$) with the property that:

> **UP.2** M is contained in a (two way) infinite strip between two parallel lines, but is not contained in any (one-way infinite) half-strip.

Put another way, a filament is the image of the midline of a strip under a homeomorphism (a one-to-one, bicontinuous mapping $\phi: E^2 \rightarrow E^2$) onto itself. One example is a sine curve. Other examples from Grünbaum and Shephard's (1987) work are given below.

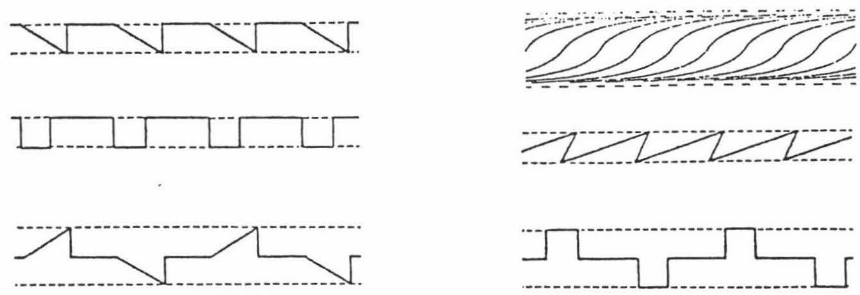


Figure 2.7

2.4.5 Adjacency Relationships

> "The various kinds of symmetry groups are useful in the description of many of the artifacts, but more general approaches (based on 'adjacency relations' or other 'local' criteria) are necessary for a better understanding of the ornaments and artwork, and of the ways their creators thought about them." (Grünbaum *et al.*, 1986)

Besides the considerations of the previous subsections, other definitions have been developed by the researcher (Kelly, 1990) for the two purposes. The first aim was to classify *finished* frieze designs in a way that reflects the relationships between adjacent motifs. These relationships are often 'ignored' by the frieze classification. The second aim was to provide a *basis* from which to develop a classification of the behaviour which actually occurs to produce a pattern, that is, to provide a blue print of the intuitive use of transformation geometry as described by the creator. (see subsections 3.1.3 and 3.2.3 for the resulting analysis methods) This aim is a similar pursuit to that of Schattschneider (1986) whose emphasis is on:

> "... the creation of designs through the use of isometrics, rather than the classification of completed designs."

The classification of adjacency relationships lead to an infinite number of classes (Kelly, 1990) called incidence categories. The practical outcome of this result is that a frequency analysis of adjacency relationships becomes difficult, and is not attempted formally in this thesis. Therefore, rather than pursue this theory of pattern classification, we examine its implications for describing pattern construction in subsection 3.5.3.

Schattschneider's (1986) article described a design technique to *generate* tilings or patterns of a rosette (a design inscribed in a circle), a border (a frieze design) and a wallpaper design. She argued that this approach allows the designer to control the artistic 'input' by creating the tile or motif. She also used the idea of a minimal set of generators, and constructs a table of them for each frieze group. A similar table appears in subsection 4.3.4. She explained that:

> "Using this table, a designer can choose a motif, decide the symmetry type of design to create, place the motif in a suitable generating region and use a minimal set of generators (properly positioned) to create the desired border design."

Schattschneider makes an interesting comment which concludes this brief discussion of adjacency relationships:

> "Although the *number* of isometries in a minimal set of generators for a design is unique, the *choice* of these isometries is not always unique."

This means that there is often more than one way of using symmetries to generate a particular frieze design and furthermore, not all the symmetries present in a pattern are needed to produce it. Some implications of this observation are discussed in more detail in chapter 4 of this thesis.

3 *Methodology and Analysis*

3.1 The General Approach

3.1.1 An Overview

In this thesis, the working definition of *intuitive transformation geometry in frieze patterns* is the informal, non-accidental presence of transformations or symmetry in a frieze design. Two research methods were employed to explore this topic, and involved subjects from three age groups: Standard Three and Four students, Form Four students, and First-Year Tertiary students. Firstly, a *survey* was conducted to identify any norms and help raise relevant questions with respect to the topic (Johnson, 1980). The overall objective of this data collection was to set tasks which revealed the presence of transformations in these patterns. This approach had two different, but related facets: 'perception' and 'representation' activities.

The 'perception' survey, activity (c), involved the written description of seven frieze patterns, each pattern an example of a different frieze group. Two representation surveys were conducted, both of which involved the construction of frieze designs. The first representation survey, activity (a), had very few restrictions, the second, activity (b), confined the students to the use of a particular asymmetrical shape. For each age group, the representation tasks were set *before* the perception task in order to prevent the patterns presented in the perception task influencing the construction activities.

The second type of research method employed was the use of *case studies*; specifically, interviews were conducted with ten Primary school children who had participated in the surveys. Explanations of their intentions and understandings of the survey tasks were encouraged and further construction and 'matching' tasks were set.

3.1.2 The Subjects

Given that a key factor of intuitive concepts is that they are informal, it seemed natural to choose subjects who were as young as possible and yet were capable of the tasks required. In this respect, Standard Three and Four children appeared to be the most appropriate group on which to focus.

The justification for this choice is as follows. While most children can distinguish one orientation from another, Piaget and Inhelder (1956) claimed that the skills required to make differential responses or *explicit reference* to the orientation of an object in space do not develop before the age of *nine* years (Howard, 1982). More importantly, although some research has suggested that the age may be younger, Piaget and Inhelder (1971) also proposed that it was not until the age of nine that children are capable of understanding the three 'basic' operations of translation, reflection and rotation. Hence, the ability to *describe* most transformations (in some form) may not be available to children before this age, thereby thwarting the aims of activity (c).

Secondly, the restriction in activity (b) required the subjects to use only right-angled (scalene) triangles, a shape too difficult to draw for many eight year olds. Indeed, the geometry component of the *Syllabus for Schools* (1985) booklet indicated that children are not taught the concept of 'perpendicular' until the age of nine, and the pilot study conducted at Colyton Primary School suggested that even some nine and ten year old children struggle with the concept of a right angle. This finding is also consistent with Kerslake's (1979) conclusion that the orientation of a right angle affects children's identification of it up to ten years of age.

Obviously, Standard Three and Four children are not as intuitive (i.e., informal) in their conceptualizations as younger Primary students (for example, in Booth's 1975 study). Nevertheless, while the exploration of symmetry and the use of transformations is not new to children of this age, the consideration of transformation geometry by Standard 3 and 4 students has not been particularly reflective or formal, as both Lesh (1976) and the mathematics syllabus for the year 1-6 children (1985) indicated.

Older groups at the 'during-formal' instruction stage (Form 4) and 'post-formal' instruction stage (Year 1 Tertiary students) were also surveyed to evaluate the effect of the more formal transformation framework, as well as cognitive development, on the patterns. Due to the limited time available at the post-Primary institutions, only activities (b) and (c) were conducted at these levels, with the exception of one Form 4 class that participated in activity (a) also. No interviews were conducted for the during- and post-formal groups.

3.1.3 Organisation

Letters were sent to the principals of various Primary and Secondary schools to explain the purpose of the study and to seek permission to survey students of the required age. In addition, consent was obtained to select a few Primary school students for interviews of approximately 40-60 minutes. At the Tertiary level, license to survey first year Tertiary students was procured by approaching a number of mathematics education lecturers at the Palmerston North College of Education. In short, a class only took part if the teacher was willing to participate in the study. For the post-Primary groups, a one-hour block was usually granted, which allowed time for activities (b) and (c) to be easily completed without any time pressure imposed on the students.

The schools and classes participating in the study were from around the Palmerston North/Manawatu region and are listed below:

1. Colyton School (pilot study), Standard 3 and 4's; teacher: Sue McDowall.
2. St. James' School, Standard 4's; teacher: Barry Slade.
3. Winchester School, Standard 4's; teacher: Jane Gilliland.
4. West End School, Standard 3 and 4's; teacher: Ross Richdale.
5. Takaro School, Standard 3 and 4's; teacher: James Kendrick.
6. Awatapu School, Form 4's; teacher: Jim Wilkinson.
7. Awatapu School, Form 4's; teacher: Michelle Wheatley.
8. St. Peter's College, Form 4's; teacher: Marion Roser.
9. Queen Elizabeth College, Form 4's; teacher: Neroli Field.
10. Palmerston North College of Education, Division A Year 1's; lecturer: Gus Hubbard.
11. Palmerston North College of Education, Division A (bilingual) Year 1's; lecturer: Gus Hubbard.
12. Palmerston North College of Education, Division AS Year 1's; lecturer: Gus Hubbard.
13. Palmerston North College of Education, Division A Year 1's; lecturer: Vivienne Bryers.
14. Palmerston North College of Education, Division A Year 1's; lecturer: Barry Brocas.
15. Palmerston North College of Education, Division A Year 1's; lecturer: Ian Stevens.

3.1.4 Environment and Data Collection

Physical Environment

All exercises were conducted in a classroom setting, with children sitting at their desks.

Data Collection

Time was allowed for the everyone to finish. The activity sheets were collected shortly after the last person had finished. Anyone who finished before this time was asked to turn their sheet face down (to avoid others 'copying' their efforts), and were set other tasks so as not to disturb those who were still creating patterns.

In sections 3.2, 3.3 and 3.4, a more specific outline of the methodology of the survey activities is given. Details of the interviews can be found in section 3.5.

3.2 Unrestricted Pattern Construction: Activity (a)

3.2.1 Subjects

The Primary school subjects were 99 nine-to-ten year olds (F = 54; M = 45). The Secondary subjects were 19 fourteen year olds (F = 8; M = 11).

3.2.2 Procedure

Design

One of the aims of this exercise is engage the creative processes of the participating individuals so that a full range of patterns is produced. Wallach and Kogan (1965) explained that the essential character of creativity may be contained in two facets, namely, "the production of associative content that is abundant", and a "playful, permissive task attitude." Thus, the appropriate variables to index individual differences in creativity are numeracy and relative uniqueness.

While these measures were recorded, they are not of primary interest here. What is important is how to enhance the creative process in the construction of the frieze patterns. Wallach and Kogan offered the following advice:

"... the assessment context must be quite different from the kind utilized in the studies reviewed; there should be freedom from time pressure and there should be a playful, gamelike context rather than one implying that the person is under a test." (p 289)

From this suggestion, a number of features were built into the design of activity (a). Firstly, the students were allowed as much time as they wished to complete the exercise. Secondly, the exercise was presented and reiterated to the participating students as a fun *activity*, not a test. Almost all responded well to this approach. Related to this point, the strips given were not numbered, and the students were encouraged to do as many as they could with the proviso that they could stop whenever they wished. Furthermore, the students were encouraged to create any pattern they wished, so long as it had some repetition in it. The freedom that the subjects had in their designs was reinforced by written instructions on the activity sheet and by oral instruction several times throughout this exercise.

Materials

Students were provided with two sheets of paper with lines denoting the borders of the strips (See appendix E1). Between successive strips, asterisks were placed to avoid any confusion in the subjects about where it was on the page that they were supposed to construct their patterns. A total of 16 empty strips were provided, with the instruction at the top of the sheet to "fill in as many of the following strips as you can (in any way you wish) to make different repeating patterns." It should be pointed out that the word 'pattern' seems to be quite vague to many students; it means something like 'regular design'. Therefore, the word 'repeating' was also included to distinguish it from some random mess (which was also explained to the students orally), and had the second motivation of encouraging students to make designs with translation symmetry, that is, frieze patterns.

Instructions to Subjects

All the Primary school sessions began with an introductory game, allowing time for the children and the researcher to gain some familiarity with one another. At the Secondary level, this introductory component was more brief. The sheets were handed out and the instructions explained as follows:

1. Only a pencil and rubber were to be used. This was to avoid colour symmetry, an aspect of frieze patterns not considered in any depth in this study; as a result of

shading, some two colour patterns did emerge. Also, from the pilot study, the use of rulers appeared to slow the rate of progress considerably so these were excluded from the process.

2. The students could fill in the strips in any way they wished, provided the resulting pattern had some repetition in it. This point was repeated to the students as the activity progressed. The students were asked not to copy another student's work but to create their own patterns.

3. The subjects were encouraged to make as many patterns as they could. Each pattern drawn was to be different from any others they had made. They were to take as much time as they wanted. They did not have to fill in all the strips, and they could stop whenever they wished. Those who finished early were given another design task to keep them occupied, and to avoid distracting the other students. Generally, all students had stopped working on their patterns after 20 minutes.

4. Students were asked to put up their hand if they had any questions so that any ideas they had wouldn't influence the whole class.

Instructions to Teachers

The instructions given to teachers were minimal. They were asked to answer any student's question during the activity by reiterating the points that: the student can fill in the strip however she, or he, wishes, so long as there is repetition in the pattern; the student was to make as many patterns as possible and; the student could stop whenever she, or he, wanted to. Furthermore, in a similar manner to Booth (1975), any comments on students' patterns were to be positive and devoid of criticism or phrases that might suggest either representation or non-representation.

3.2.3 General Quantitative Analysis

A frieze group classification of the patterns drawn was made. Since a frieze group is made up of various types of symmetries, it was hoped that this classification would give some *clues* to the intuitive use of symmetry in the patterns drawn. Two other categories were also included to allow for the possibility that designs may not display translation symmetry *(No Translation Symmetry)* or that the underlying frieze group of the design was ambiguous *(FG Unclear)*. These, and other anomalies are discussed with extensive illustrations.

Four measures were devised, each an attempt to indicate 'how intuitive' a pattern's symmetries were. Respectively, these were:

1. *Frequency*. This indicator examined the total numeracy of each frieze group. If students produced a lot of examples of one frieze group in comparison to another frieze group, then it seems reasonable to conclude that the particular set of symmetries in the frequently produced pattern are likely to be more intuitive than those of the infrequently produced pattern.

2. *Commonality*. This indicator examined the number of students producing each frieze group. If a lot of students produced examples of a particular frieze group, then the associated set of symmetries are likely to be more intuitive than the set of symmetries from an unpopular frieze group.

3. *Average number produced**. The asterisk indicates that the average was calculated on the students who actually produced an example of the frieze group in question. This measure indicated whether students who made a particular frieze group tended to make only one of them, or whether they tended to make more. The implications to the relative 'intuitive-ness' of symmetries is similar to the comments given for the frequency measure.

4. *The first three patterns produced*. Intuition, as noted above, often connotes an *immediate* apprehension, indicating that the first patterns produced *could* be the most intuitive to the child. Differences between this measure and the frequency measure were therefore noted.

Further refinements were made to the frieze group classification in an effort to gain clues about the processes which *may* have occurred to produce the respective frieze patterns. The broadest categorization was the distinction made between disjoint and connected patterns, *suggesting* that a pattern was made as repeating 'parts' or as a 'whole' respectively . Examples are given below:

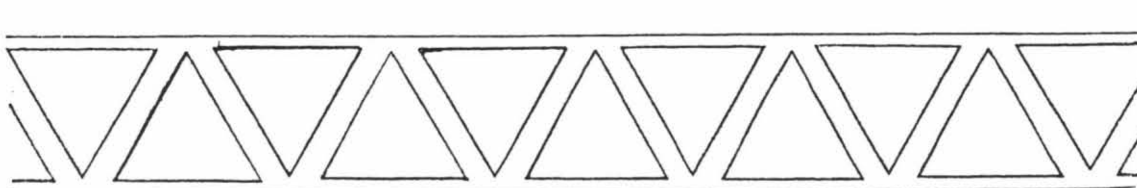


Figure 3.1 (Example of a Disjoint Frieze Pattern)

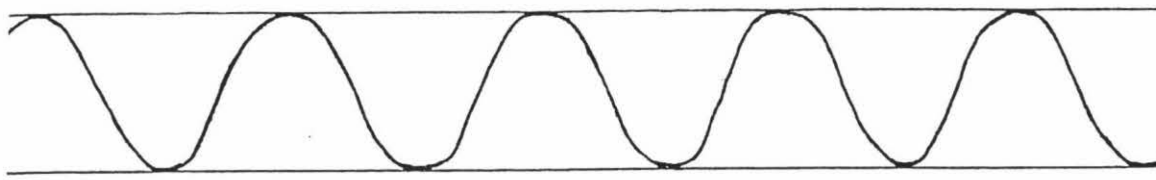


Figure 3.2 (Example of a Connected Frieze Pattern)

A *connected* frieze pattern is one which has a filamentary subset or, more loosely, some part of it extends the length of the strip and can be traced without lifting one's pencil. Any frieze pattern without this property is considered to be *disjoint*. A separate category *(Not Sure)* was made to allow for ambiguities between these two pattern *kinds*. From the mathematical classifications outlined in section 2.4, it certainly appears as if further refinements could be made to this classification. However, a substantial percentage of the activity (a) patterns displayed ambiguity at this level of categorization, so this refinement was abandoned. The frieze group/kind classification was determined by the following steps:

Step 1. Does the pattern display some evidence of translation symmetry? (i.e., are there two translation units?). If so, goto step 2. If not, then record it under the 'No Translation Symmetry' category and goto step 3.

Step 2. Can the underlying frieze group of the pattern, when imagined to be extended infinitely in each direction, be identified unambiguously? If so, identify and record the underlying frieze group and goto step 3. If not, then record it under the 'FG Unclear' category and goto step 3.

Step 3. Can the pattern kind be identified unambiguously? If so, identify and record the kind (disjoint or connected) of pattern.and stop. If not, then record it under the 'Unsure' category and stop.

3.3 Restricted Pattern Construction: Activity (b)

3.3.1 Subjects

The Primary school subjects were 99 nine-to-ten year olds (F = 54, M = 45). The Secondary subjects were 85 fourteen year olds (F = 42; M = 43). The Tertiary subjects were 69 First-Year students of various ages (F = 51; M = 18). The ages of the College of

Education students varied between 18 and 45 years, and their mathematics background ranged from Form 4 to Second Year university courses.

3.3.2 Procedure

Design

The overall design of this activity was very similar to that of activity (a). The difference was that in activity (b), the subjects were required to only use right-angled scalene triangles in their construction. On the surface, the purpose of this restriction may not seem obvious, but it had a definite motivation. One of the possibilities in activity (a) is that symmetries of the whole pattern can be present as a consequence of quite arbitrary choices of variables such as the spacing, the number, the position, and the combination of elements in the translation unit. This theoretical prediction was verified in the interview material (see subsection 4.3.4). Therefore, to reduce this occurrence, it was decided to allow the students to only use a C_1 shape (i.e., an asymmetrical figure) in the construction of their patterns.

The shape of the right-angled, scalene triangle was chosen for the following reasons:

1. Unlike some asymmetrical figures, it can be drawn by nine and ten year olds.

2. Unlike some asymmetrical figures, it is easy to orientate its sides vertically and/or horizontally, or align the sides with the border lines or other triangles. These possibilities appear to be natural preferences according to much of the literature reviewed in this thesis.

3. Unlike most asymmetrical figures, it can be used to make a variety of different *styles* of patterns, that is discrete, non-discrete, touchings, tilings and filamentary patterns. See subsection 3.3.3 for further details.

4. Unlike some other asymmetrical figures, the spacing of the right-angle triangles does not influence the overall symmetry of the resulting frieze pattern.

From a mathematical point of view, it may well be that no other shape satisfies all four properties listed above. However, while the analysis is stronger than that for activity (a), in that 'accidental symmetries' are avoided to a greater extent, it is also weaker in the sense that the *particular* choice of a C_1 figure (a right-angled, scalene triangle) may affect the construction methods and, therefore, the patterns produced.

Materials

(See appendix E2). Materials for activity (b) were identical to those used in activity (a) except that the printed instructions included the added restriction that the subjects could only use the shape:

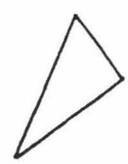


Figure 3.3

Instruction to the Subjects

Instructions were almost identical to those for activity (a) with a few exceptions related to the restriction imposed. For example, the session began with a series of diagrams displaying examples and non-examples of right-angled, scalene triangles, which were presented and discussed in an attempt to familiarise the subjects with the concept. The main motivation, of course, was to prevent other *symmetrical* triangles from being drawn. Students were free to ask if their triangles were right-angled or not, and this was checked by the researcher during the activity.

Instruction to the Teachers

Instructions to the teachers were very similar to activity (a) except that questions relating to the 'correctness' of the right-angle triangle were to be answered by the researcher. In order to prevent suggestive comments, the range of answers to these questions was limited. Most explanations didn't comment on the specific triangle that had been drawn in order to allow the subject to make the final evaluation for themselves. Instead, comments usually referred to the diagrams discussed in the introduction to the activity.

3.3.3 General Quantitative Analysis

The quantitative analysis employed for the restricted pattern construction exercise used the measures of numeracy, commonality, average number drawn*, and the first three drawn to indicate the relative 'intuitive-ness' of the seven frieze groups. The first three of the four measures were also applied to a refinement of the frieze group classification, namely pattern *styles*.

The analysis of activity (a) considered pattern kinds, that is, disjoint and connected. This analysis divides the disjoint category into discrete and non-discrete patterns, and the connected category into touchings and tilings/filamentary. It is a fairly straightforward task to derive the formal definitions of these terms from Grünbaum and Shephard's (1987) theory. These, and the more informal explanations of the patterns styles, are listed below:

Discrete frieze patterns are defined and illustrated in subsection 2.4.3. Loosely, it can be thought of as a frieze pattern with connected motifs that aren't arbitrarily close to one another and are pairwise-disjoint. In addition, all the isometries that map one motif onto another must also map the pattern onto itself. The particular henomeric type was also recorded in this case.

A *non-discrete frieze pattern* has the special meaning that it is a disjoint strip design which fails to satisfy one or more of the conditions P.3, DP.1 or DP.2, as defined in subsections 2.3.3 and 2.4.4. This roughly means that the pattern's motifs are disconnected or can map onto one another without necessarily mapping the whole pattern onto itself; the other possibility is that the pattern contains motifs which are arbitrarily close to one another. An example follows:

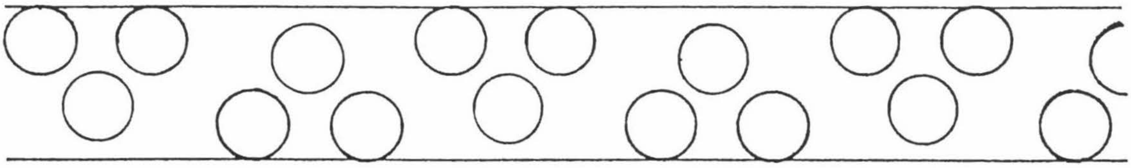


Figure 3.4

A *touching frieze pattern* is a little difficult to define precisely, and a loose definition is given. Essentially it is a connected pattern, made up of bounded motifs which do not form a tiling of the strip, and are not filamentary patterns. An example is given below:

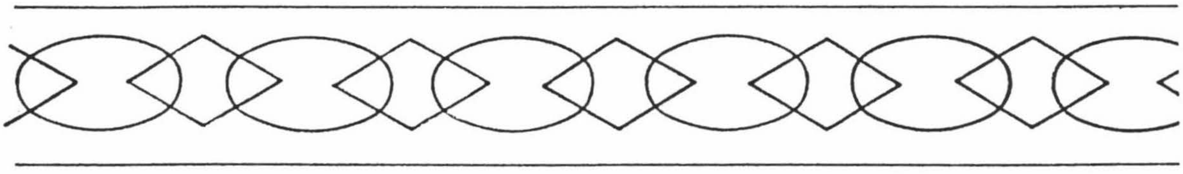


Figure 3.5

Tilings of the strip and *filamentary frieze patterns* are described in subsections 2.4.3 and 2.4.4. The following ambiguity, which can be described as a filament contained in an asymptotic strip, requires the latter two categories to be grouped together:

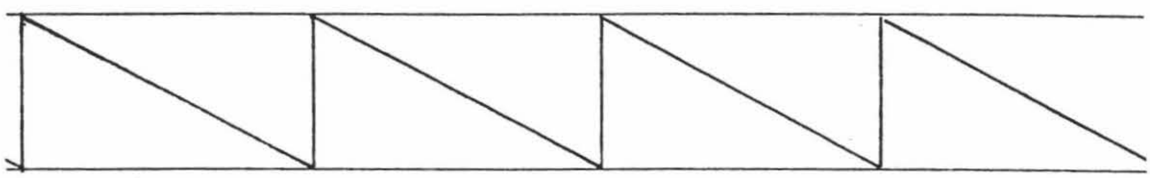


Figure 3.6

Mathematically, this is a tiling of the strip and not a filamentary pattern, since the definition of a filamentary pattern requires the filaments to be pairwise congruent and pairwise disjoint. However, this definition obviously ignores the fact that a 'filamentary' process can be used to construct the pattern shown above. Since it is not possible to know the intention of the drawer with any certitude, and since 'filamentary' patterns can not arise without the use of the boundary lines (because of the particular restriction imposed), tilings and filamentary patterns were not distinguished in activity (b). Finally, when the occasional ambiguity arose between styles, it was recorded as '*Unsure*.'

The steps for classifying an activity (b) pattern were similar to those for activity (a), except that the 'No Translation Symmetry' and 'FG Unclear' categories were not split up into style categories. Thus the steps were:

Step 1. Does the design display evidence of translation symmetry? If not, record this as 'No Translation Symmetry' and stop, otherwise goto step 2.

Step 2. Is the frieze pattern's underlying symmetry group clear? If not, record this as 'FG Unclear' and stop, otherwise record the pattern's underlying frieze group and goto step 3.

Step 3. Is the pattern's style clear? If not, record this as 'Unsure' and stop, otherwise record the patterns style. If the style is discrete, record its henomeric type and stop.

3.4 Frieze Pattern Description: Activity (c)

3.4.1 Subjects

The Primary school subjects were 79 Standard Three and Four students (F = 40; M =39). The Secondary subjects were 91 Form Four students (F = 52; M = 39) and the Tertiary students were 69 Year 1 College of Education students (F = 51; M = 18).

3.4.2 Procedure

Design

In this activity, the focus is on the manner in which a subject perceives a frieze pattern and relationship this perception has to intuitive transformation geometry. In order to do this methodically, an example of each of the seven frieze patterns were presented to the subject and a written description was asked for. While a 'written' description approach has obvious shortcomings, particularly for the Primary school subjects, it appeared to be the only way of collecting the data efficiently.

The selection of the seven frieze patterns was made carefully. Firstly, it was decided to reduce 'distractors' as much as possible. For example, a standard asymmetric motif (i.e., a right-angled, scalene triangle) was used in all seven patterns, and the spacing between adjacent motifs was as uniform as possible, so that any perceived grouping of the motifs would be more likely to be based on some other criterion. Also, all the patterns were discrete and primitive. These properties were chosen so that all symmetries underlying a pattern had the same 'status', that is, all transformations or symmetries identified must be 'between' motifs, and not 'within' a motif.

Materials

Materials consisted of 4 sheets, each with two patterns on it (apart from the last sheet). Below each pattern were 8 lines which could be used to write on. This proved to be ample space for all but a few students. The printed instructions simply said "Please describe the following patterns." (See appendix E3).

Instructions to the Subjects

Again, instructions were kept to a minimum. The subjects were asked to describe the patterns on the sheet. Once one pattern was finished, the subjects should move on to the next one. If subjects showed any confusion about the task, supporting comments such as "write what you see" or "imagine you're describing it to someone" were used.

Instructions to the Teachers

Teachers were asked to avoid using suggestive comments such as "write down what it is *like*", or "what does it *remind* you of?", or even "how would you *make* it?"

3.4.3 Analysis of Frieze Pattern Descriptions

The analysis method employed was devised *after* the exercise was completed so as to best reflect the descriptions made. The categories that arose showed similarities to some of the relevant literature, and so terminology from Booth (1975), Howard (1982) and Piaget and Inhelder (1971) was utilised to describe the classes. A brief description of the categories follows:

A. Explicit Phrases: These expressions indicate one or more of the four rigid transformations - translation, reflection, rotation, and glide reflection. They may refer to these transformations as motions of some set of points or as the symmetry of some set of points. These sets of points may be the whole pattern or some finite subset of it. Expressions which are ambiguous with respect to the transformation they are indicating are labelled undifferentiated.

A1. Differentiated expressions indicate particular transformations or symmetries in such a way that they can be distinguished clearly from other transformations. For example, "then it repeats" implies translation, and "fold it over" implies reflection.

A2. Undifferentiated expressions describe a relationship between two congruent objects, but not in a way that a transformation can be unambiguously identified. For example, "turn it upside down" is clearly a transformation, but it may mean a horizontal reflection, a half-turn or even a glide reflection. These phrases report the relative orientation or relative direction of two objects; usually two right-angle, scalene triangles. The recognition of congruence is also included. Some of these descriptions are more suggestive of transformation geometry than others, but no distinction is made here.

B. Implicit Phrases: Sentences in this category make a comparison of a set of points in the pattern with another object or set of points not in the pattern. They characteristically share the same symmetries or even symmetry group. The most implicit examples occur when the objects used for comparison are drawn from the stimulus domain, that is, when a simile or metaphor is used to describe the pattern.

B1. Comparison of Whole Pattern to another object. In this case, the 'whole' pattern (at least, the portion displayed on the page) is given a entity as another object. For example, "this pattern looks like a chain saw blade".

B2. Comparison of Parts of Pattern to another object. This comparison gives each part of the pattern, generally the motif or base pattern, an entity as another object. For example, two 'proximate' triangles with a reflection between them may be described as a "bird's beak".

C. Orientation and Directional Judgements involve the description of the orientation or direction of individual objects in an egocentric or exocentric fashion. These expressions attempt to establish the orientation of each triangle individually, most often within the base pattern. Because no mention is made of the relative orientation of objects, the presence of intuitive transformation is implicit at best, and may not be present at all. For example, "In the first triangle the right-angle faces to the left, and in the second triangle it faces to the right."

D. Position Judgements describe the estimations or measurements of distance between objects. They also include the position of triangles relative to the strip, or relative to each other. Often expressions like "top", "bottom", "in the air", "on the ground", "above" and "in rows" are used.

E. Miscellaneous Descriptions include a number of different descriptions such as the enumeration of the triangles in the pattern, or stating the intrinsic properties of a triangle (e.g., right angle, scalene). Other responses include the use of criteria such as the grouping the triangles, order, and separation as sketched by Copeland (1979) of Piaget's topology thesis. See part one of Holloway's (1967) summary for further details of Piaget's work in this area. Finally, subjective opinions of the pattern are also included (e.g., "very, very fancy"). All these descriptions have been grouped together since the likelihood that they indicate intuitive transformation geometry seems small. However, in the comparison of the three age groups, this category was split into three classes (topological judgements, motif properties, subjective evaluations), since the number of responses in each class was large.

3.5 Interviews

3.5.1 Subjects

After all the survey activities had been completed, the patterns and descriptions of the Primary school children were broadly analysed, and a total of ten students (F = 6; M = 4) were selected. These were students which showed some evidence of intuitive transformation geometry, such as producing a number of different frieze groups or using phrases in their descriptions which suggested some form of symmetry or transformation. The number of students selected from each school was in rough proportion to the number of students (n) who were surveyed in all three exercises.

A list of the interviewed children's names (pseudonyms) and their respective schools is given for reference in chapters 4, 5 and 6:

St James' School (n = 22): Mary, Alice, and Carla.
Winchester School (n = 25): Kate, Amy and Toni.
West End School (n = 24): Rachel, Mark and Aden.
Takaro School (n = 6): Richard.

3.5.2 Aims and Design of Interviews

Many patterns do not require a great deal of examination before it becomes obvious that the manner in which they were constructed is not clear. Indeed, in *most* cases, a certain amount of guesswork is required to describe a likely construction method. Furthermore, many factors within a pattern can affect its symmetries, and the intentions of the creator may not be obvious with respect to these factors. Therefore, the purpose of the interviews was to discern the *intentions* behind some of the patterns constructed, and to determine the *process* by which a pattern was created. Behind both of these aims was the overall objective of gaining some insight into the consideration of how children conceptualize transformation geometry in frieze patterns.

For activity (c), the pattern description exercise, the purpose was to ask the subjects what they meant by their descriptions and to expand or modify their descriptions if they wished. One reason for doing this is that the possibility that the students did not write everything they perceived was probably high, as Wheatley suggested (1978). In fact, Choat (1974) made a similar observation in an interview situation with 'Johnnie':

> "He had grasped the configuration of the relationship but did not possess the language necessary to express his thoughts" (p 9)

Given these aims, the natural interview technique appeared to be that of the clinical interview, or the indirect approach (Lesh, 1976). By employing this technique, we were not so much interested in the actual cognitive processes of a child as we were with the nature of a child's concepts (Lesh, 1976). Mulhern (1989) explained that an important function of the clinical interview is to evaluate understanding in children. He continued:

> "... clinical techniques are ... seen as particularly useful for studying young children. ... these methods allow the researcher to probe children's mathematical thinking much more deeply than other techniques, such as pencil and paper methods." (p 48)

The character of a clinical interview is quite different to that of other procedures such as the 'thinking aloud' technique. It is chiefly an *indirect* research method. Lesh (1976) explained that an approximate analogy of the indirect method to twenty questions (or to the process a police artist uses to construct a picture of a criminal) could be made, in that the approach centres in on an idea by finding out what the idea is not. He supports the use of the indirect method by reiterating a theme of his article:

> "... it is important to occasionally use indirect research techniques. Otherwise, it is very easy to impose inappropriate mathematical structures on the thought processes of children." (p 234)

However, he finished by noting that the direct approach eventually becomes necessary if generalizations are to be made.

The clinical interview is based chiefly on the methods of Piaget, and does not employ standardized questions so much as following the thoughts of the subject wherever they may lead and asking questions related to the responses of the interviewee. Thus it involves a large amount of flexible questioning, including the use of various materials. Indeed, as a 'follow-up' to activity (c), three new matching tasks were pursued. The first task involved asking the subject to indicating if any of the seven frieze groups in activity

(c) appeared similar and to say why if possible. For ease of comparison, the frieze patterns were presented to the subject on (green) cards so that she, or he, could shift them around and 'explore' different combinations; the seven patterns can be seen in appendix C1. The second task involved the matching of the 'green' patterns with another set of seven frieze patterns, also mounted on cards (red). The 'red' examples were discrete and primitive, with only the motif differing from that of the 'green' patterns (see appendix C2). Similarly, the last task involved the matching of seven examples of filamentary frieze patterns, mounted on beige card (see appendix C3), with the 'green' patterns.

Another activity component of the interviews was the construction of vertical patterns, that is, the strip to be filled in was aligned vertically with respect to the drawer.

In conclusion, while the clinical method has strengths, and is probably the most suitable method for this study, the obvious pitfall of such a technique is that:

> "standardization is sacrificed because not all the subjects are asked the same questions." (Lesh, 1976, p 216)

3.5.3 Analysis of Interview Material

Analysis of Interview Responses to Activities (a) and (b)

The analysis of the interview material from activities (a) and (b) aimed to describe the method employed to make the patterns, and how the children understand the process they used. Thus the analysis was a description of the *order* in which a pattern was constructed, as well as an examination of the child's language used to describe the process. In the former case, a number of pattern construction trends emerged, and the use of mathematical terminology was used to describe and group the common design methods.

In particular, the pattern construction analysis was a synthesis of applications of Kelly's (1990) and Schattschneider's (1986) theory, as well as terminology drawn from Grünbaum and Shephard's (1987) book (see section 2.4 above). Other definitions were created where necessary. These modifications, parallels and new definitions may not be 'standard', but seem justified, as Grünbaum and Shephard (1983) write:

> "Could it be that we mathematicians ... have cowed generally reasonable people into unreasonable attitudes? If so, would it not be desirable to change the situation and to give the various 'users' of

mathematics the theories applicable to their concerns? ... if we are to provide the mathematical tools for their disciplines we should develop the appropriate 'geometries'."

Two types of construction algorithms ensue from the incidence theory mentioned above: *base pattern construction* and *incidence construction*. Essentially, the first method involves the construction of some generating region (Schattschneider, 1986) and then translation of this region. The second approach uses a motif, copies of which result under some set of transformations. For both constructions, we let $\mathcal{S}$ be a strip in the plane of width w and chose t* to be the shortest translation symmetry of the design. The algorithm for the respective methods are as follows.

A Base 'Pattern' Construction

1. Draw some motif, A_1, so that it lies in $\mathcal{S}$.

2. For $i = 1, 2, ..., n-1$ draw (in $\mathcal{S}$) the image $A_{i+1} = \lambda_i(A_i)$, where n is the smallest positive integer such that $A_n = t^*(A_1)$ and λ_i is an isometry.

3. The collection of images $A_1, A_2, ..., A_{n-1}$ is called the base pattern. This set is translated by t* to generate a frieze design.

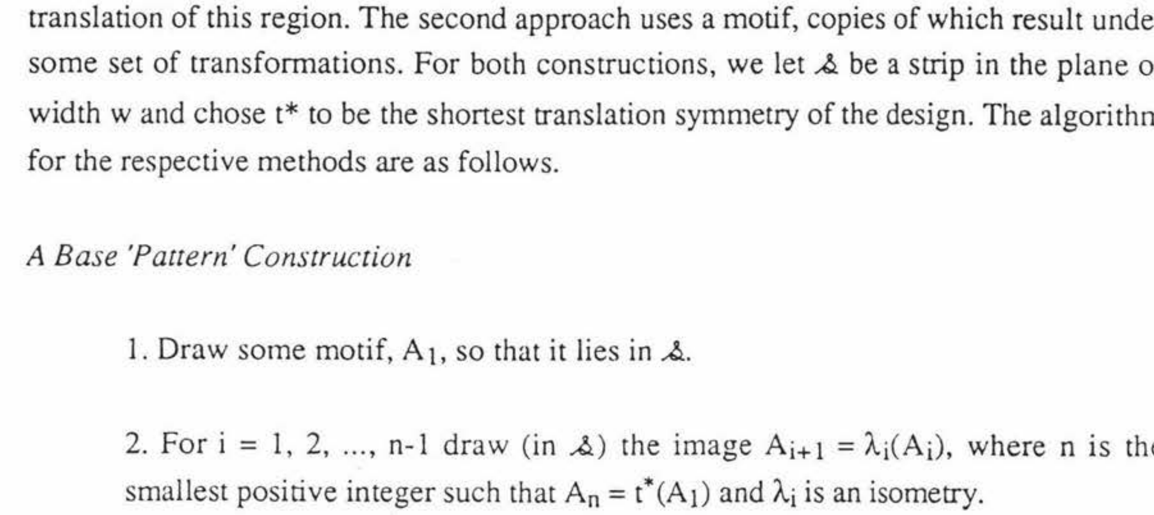


Figure 3.7 (A Base Pattern Construction)

An Incidence Construction

1. Draw some motif A_1, so that it lies in $\mathcal{S}$.

2. For $i = 1, 2, ...$ draw (in $\mathcal{S}$), the image $A_{i+1} = \lambda_i(A_i)$ where λ_i is some isometry.

If n is the smallest positive integer such that $A_n = t^*(A_1)$, then the ordered n-tuple $(\lambda_1, \lambda_2, ..., \lambda_n)$ is termed an *incidence sequence* of the resulting frieze pattern.

If λ_i and λ_{i+1} are the same type of symmetry, for i = 1, 2, ..., n-1, then the resulting frieze pattern is said to have *λ incidence of order n*. For example, if, in an incidence construction, the isometries are all vertical reflections, then the resulting pattern is said to have vertical reflection incidence of order 2.

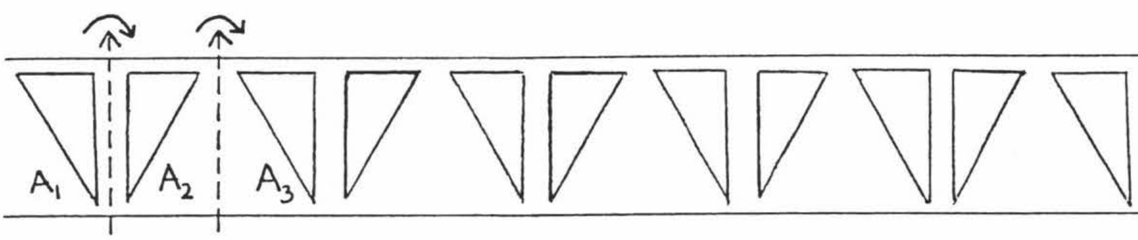


Figure 3.8 (An Incidence Construction)

In one special case, the two constructions methods are identical. This occurs when the base pattern is the motif A_1, and the incidence construction (with the same motif A_1) has an incidence sequence consisting of t* only.

It also turns out to be useful to describe information about a motif's orientation if its underlying symmetry group is dihedral. We shall adopt the following notation, assuming that the direction 'horizontal' is parallel to the direction of translation symmetry of the (resulting) frieze design. A dihedral motif D_n and its orientation is denoted by:

$D_n(v)$ if it has a vertical line of symmetry;
$D_n(h)$ if it has a horizontal line of symmetry, but no vertical line of symmetry;
$D_n(n)$ if it has neither a vertical nor a horizontal line of symmetry.

It should be noted that there are three possible orientations when n is odd, and only two if n is even (since (h) is not possible in this case).

Other construction methods are possible, such as a tiling or a filamentary construction. The first method involves the copying of some closed shape in order to fill the strip in regular manner, and is closely related to the incidence construction described above. The second method involves drawing a closed curve (from left to right) across the strip without the designer lifting her, or his, pencil.

Combinations of the four construction methods are also possible. A more comprehensive description of the construction methods and the associated ambiguities and subtleties is given in sections 4.4 and 5.4.

Besides the construction analysis of the unrestricted and restricted patterns, a simple frieze group classification of the vertical patterns drawn by the interviewed subjects was made. The results were compared to the corresponding frieze group distributions of the horizontal patterns. With such a small sample (n = 10), the comparison was only intended to be suggestive.

Analysis of Activity (c) Interview Material

The language used to describe the construction processes above, and the children's oral explanations of their activity (c) written descriptions, were analysed using the categories and terminology from the survey analysis of activity (c).

A tabulation of the matching exercise was made. The descriptive analysis of the matching activities was based on a modified version of the classification used by Piaget and Inhelder (1971) in a chapter entitled *The Spatial Image and 'Geometrical Intuition';* their matching activity differed in that it involved the estimation of the relative lengths of inscribed and circumscribed shapes. The list of criteria is given below; it should not be confused with the classification of the language used in the patterns construction or description activities:

A. No justification, invalid justifications, or straight forward descriptions or tautologies.

B. Criteria based on position or order.
- B1. Evaluations based on surroundings and interiority.
- B2. Evaluations according to boundaries: order of extremities (projection or convergence) and points or proximity.
- B3. Evaluation based on upper and lower position.

C. Criteria based on the general shape, or on the composition of the parts.
- C1. Evaluation based on shape or global area.
- C2. Evaluation based on free spaces and gaps.
- C3. Evaluation in terms of number and size of constituent parts.

D. Criteria based on orientation or direction of parts.
- D1. Evaluation based on orientation or direction judgements.
- D2. Evaluation based on relative orientation or direction.

E. Criteria based on transformations.

4 *Results of the Unrestricted Pattern Construction Activity*

In this chapter trends in the unrestricted construction of frieze patterns of the Primary and Secondary students are examined[1], and an attempt is made to link these trends to intuitive transformation geometry. This is achieved by a general presentation and quantitative analysis of the patterns drawn in *activity (a)*. This approach is supplemented by a descriptive analysis of the patterns, as well as selected interview material from 10 case studies that strengthens or qualifies the survey results.

The unrestricted exercise, activity (a), required the participants to fill in as many of the sixteen empty strips as they could with 'repeating patterns', each strip having a different pattern from the rest. For further discussion of the instructions and design of this activity, see subsection 3.2. Detailed tables of the results for this chapter (and the next) can be found in appendices A (and B). Several important aspects of the survey are selected from this data summary and are presented in the form of column charts throughout chapters 4 and 5.

Unlike Küchemann's (1981) generic use of the word, 'children' refers only to the Standard Three and Four subjects; not to the Form Four or College of Education students. When a specific group is being referred to this will be made explicit.

4.1 Survey Results from the Primary Schools

4.1.1 Frieze Group Analysis

Table A1.1 can be interpreted as follows. For each of the seven frieze groups, as well as the 'Frieze Group (FG) Unclear' and 'No Translational Symmetry' classes, there is a corresponding row of figures. Respectively, these indicate the numeracy or each frieze pattern, the percentage that number was of the total number of drawings, the number and percentage of students who actually produced them, and the average number of patterns produced by those who drew them (as well as by the class as a whole). For instance, 254 (18%) of the 1385 patterns produced by the children can be classified as pmm2. Of the 99 Primary students surveyed, 75 (76%) drew at least one example of this symmetry group.

[1] For convenience of presentation, the patterns displayed in this chapter are 80% of their original size.

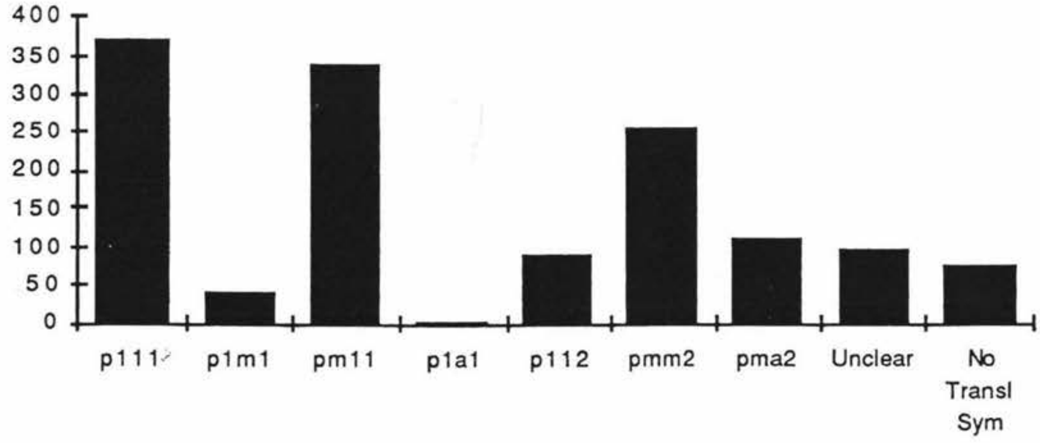
Those who did draw a p112 tended to draw more than 3 on average, whilst the average over the whole class was a little under 3 since most, but not all, of the Primary students made patterns in this symmetry group.

Graphs 1, 2, 3, and 4 summarize and display four attempted 'indicators' of intuition present in tables A1.1 and A1.2, namely: frequency, commonality, the average number[2] drawn*, and the first three patterns drawn. *All four* of these column charts furnish a similar 'skyline' profile, suggesting a relative 'intuitive-ness' of the seven frieze groups. This ordering for the frieze groups seems to be:

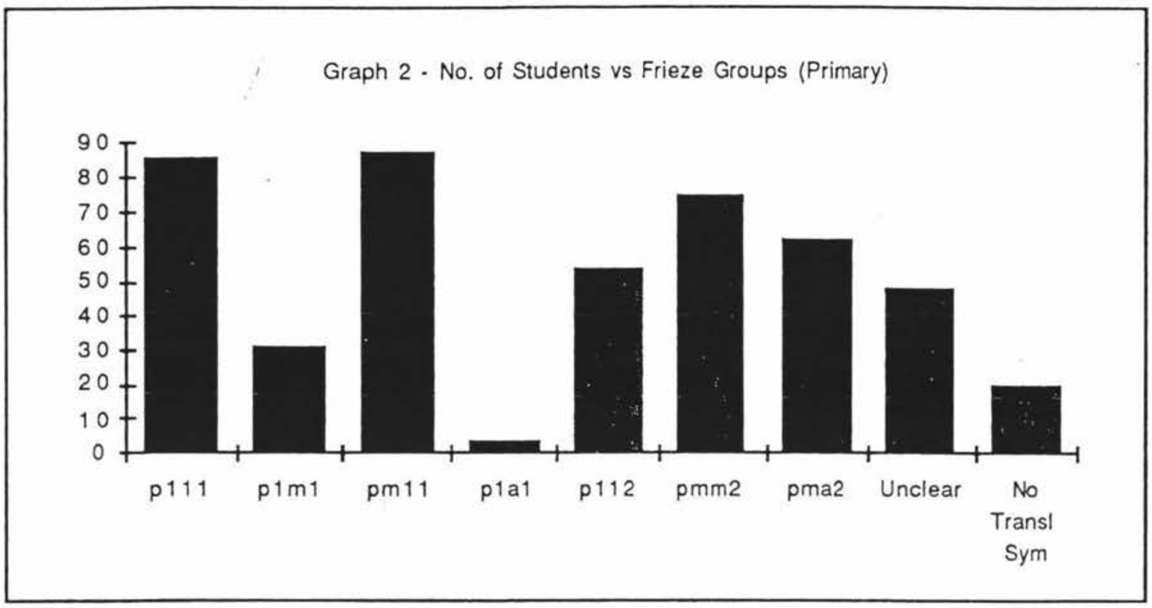
1. p111 and pm11 (common and frequent)
2. pmm2
3. pma2 and p112
4. p1m1
5. p1a1 (uncommon and infrequent)

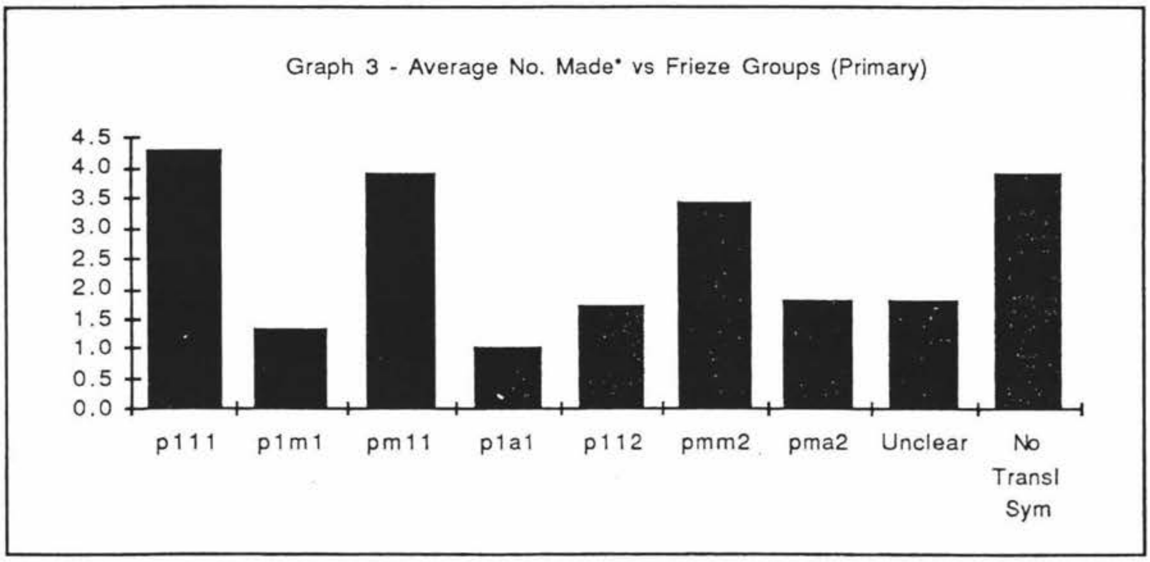
The 'Frieze Group (FG) Unclear' and 'No Translation' categories have not been included in this list since the former doesn't reveal anything quantitatively about symmetries of patterns (because the symmetry groups are ambiguous), and the latter doesn't include frieze patterns which are the focus of this research. Note that the class 'Can't Classify' (in Graph 4) indicates those patterns which fall into either the FG Unclear or No Translation categories.

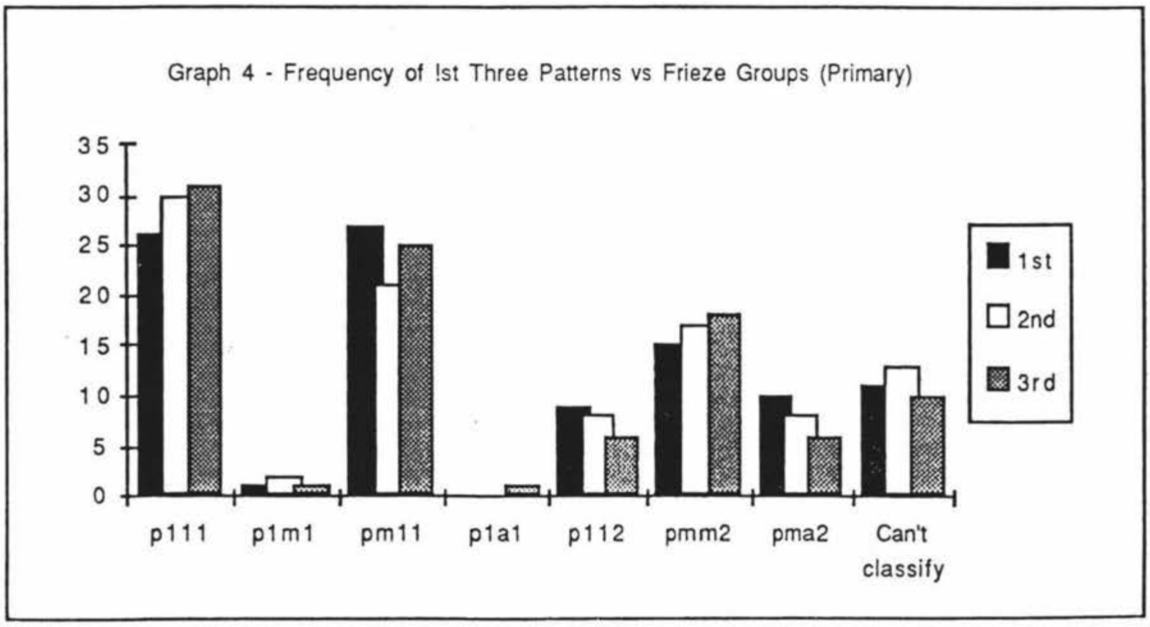
Graph 1 - Frequency vs Frieze Groups (Primary)



[2] See subsection 3.2.3. for an explanation of the asterisk shown for this measure.


Graph 2 - No. of Students vs Frieze Groups (Primary)
90
80
70
60
50
40
30
20
10
0
p111
p1m1
pm11
p1a1
p112
pmm2
pma2
Unclear
No Transl Sym


Graph 3 - Average No. Made* vs Frieze Groups (Primary)
4.5
4.0
3.5
3.0
2.5
2.0
1.5
1.0
0.5
0.0
p111
p1m1
pm11
p1a1
p112
pmm2
pma2
Unclear
No Transl Sym


Graph 4 - Frequency of Ist Three Patterns vs Frieze Groups (Primary)
35
30
25
20
15
10
5
0
p111
p1m1
pm11
p1a1
p112
pmm2
pma2
Can't classify
1st
2nd
3rd

From graph 1, we can see that 13% of the activity (a) patterns made in the Primary school survey were unable to be classified as frieze groups. However, only 6% of the patterns seemed to lack evidence of translation, indicating that the instructions seem to have been followed most (94%) of time; that is, the children usually produced *repeating* patterns as oppose to single figures or random sketches. It is also revealing to note that 1 in 5 of the Primary students made patterns with no translation symmetry, and those who did produced 4 of them on average (see graphs 2 and 3). This means that most of the Standard 3 and 4's found the instructions clear, but the minority who didn't usually displayed this fact more than once.

By far, the most frequently drawn frieze patterns were p111, pm11 and pmm2. Of the 1385 patterns drawn, 70% were one of these three patterns. Correspondingly, each of these three patterns was commonly drawn by Primary students. From table A1.1 it can be seen that 86% of the Primary students produced p111, 87% produced pm11 and 75% produced pmm2. In fact, on average, students who drew p111 tended to produce 4 of them; students who produced pm11, 4; and those who made pmm2 constructed 3.4 on average.

On the other hand, two of the symmetry groups, p1a1 and p1m1, were constructed quite infrequently by the Primary students and, correspondingly, the number of students making them was also low. The p1a1 symmetry group was particularly rare; of the 1385 patterns drawn, only 3 belonged this frieze group.

Of additional interest, is the frequency of the first three patterns drawn (graph 4) in comparison to the frequency column chart (graph 1). The comparison indicates that, on the whole, the nature of the patterns drawn didn't change with respect to one dimensional symmetry groups as the Primary classes progressed through the activity sheet. Nevertheless, with respect to one aspect of intuition, namely *immediacy*, the best indicator may be the frequency distribution of the patterns made first. Therefore, whilst the overall profiles of graphs 1 and 4 are very similar, there are slight differences that may be worth noting.

For instance, the proportion of patterns made first that were examples of p111 is slightly smaller than the proportion of the total number of patterns that were p111. This indicates that p111 may be slightly *less* intuitive than the total frequency distribution graph indicates. This can be verified by the increasing number of p111 patterns drawn second and third (see graph 4). Using a similar argument, the pm11 pattern may be slightly *more* intuitive, p112 may be slightly *more* intuitive, pmm2 may be slightly *less* intuitive, and pma2 may be slightly *more* intuitive than the total frequency graph (graph 1) indicates.

4.1.2 Transformation and Symmetry Analysis

The discussion of the results has so far been concerned with symmetries in terms of frieze groups. These results can also provide some information about the four rigid transformations within the frieze groups. It is perhaps redundant to mention that translation is common to each of these commonly produced symmetry groups. Furthermore, since the instructions required the participants to make 'repeating patterns', the inferences about its intuitive use in the context of patterns that share symmetries is questionable. Perhaps the best that can be said is that most of the 99 Standard Three and Four children who were surveyed found it relatively easy to use translation to make their patterns with little explanation from the researcher. Its placement at the top of the list which follows should therefore be considered provisional.

Besides translation symmetry, the frequently drawn pm11 and pmm2 groups have vertical reflection symmetry in common. It is therefore not surprising to find that the next most commonly drawn frieze pattern was pma2, which also has vertical reflection as an element of its symmetry group. This common use of vertical reflection symmetry accords closely with the perceptual psychology literature, as well as the findings of Fischer (1978) and other mathematics educators (see subsections 2.2.1 and 2.3.1).

A helpful approach to see which transformations (besides translation) are the most natural to use is to look at the frieze groups with translation symmetry and only one other type of symmetry. These groups are p1m1, pm11, p1a1 and p112. This analysis method furnishes some very clear results. The number of p1a1 was very low, and hence the number of students who drew them was also small. This frieze pattern is characterized by glide reflection. Furthermore, only a third of the Primary students made the p1m1 pattern which has translation and horizontal reflection symmetries only. Twice as many children produced frieze patterns with just half-turn (p112) although still not frequently, and almost all the students produced patterns with vertical reflection only in them (pm11). Thus, a rough ordering of the frequency and commonality of the symmetries present in the unrestricted frieze patterns from activity (a) seems to be:

1. Translation (common and frequent)
2. Vertical Reflection
3. Half-turn
4. Horizontal Reflection
5. Glide Reflection (uncommon and infrequent)

It is quite clear, then, that the seven one-dimensional symmetry groups were not equally likely to be drawn and, given their distribution, some symmetries of the whole pattern arose more naturally (consciously or not) than others in the production of these patterns. In particular, it seems that translation and vertical reflection were the predominant transformations present in the patterns drawn. Conversely, glide reflection and horizontal reflection were not readily used by themselves in these patterns.

There is also an obvious correspondence between these observations and the ordering of isometry learning (in finite situations) proposed by Piaget and various mathematics educators. The list does, however, differ in that a *particular* rotation, the half-turn, occurs above horizontal reflection. We gain, therefore, a possible refinement and hence further insight into some of the rigid transformations intuitively present in patterns. This does not, however, necessarily imply anything about the actual motivations of the drawers, since symmetry may not have been a consideration in any explicit way, as Grünbaum (1984) pointed out.

The ordering above does beg the question: which symmetries, in patterns with more than translation and one other type of symmetry, were used intuitively, and which were merely an accidental by-product of the other symmetries? In other words, are any of the symmetries present in pmm2 and pma2 incidental?

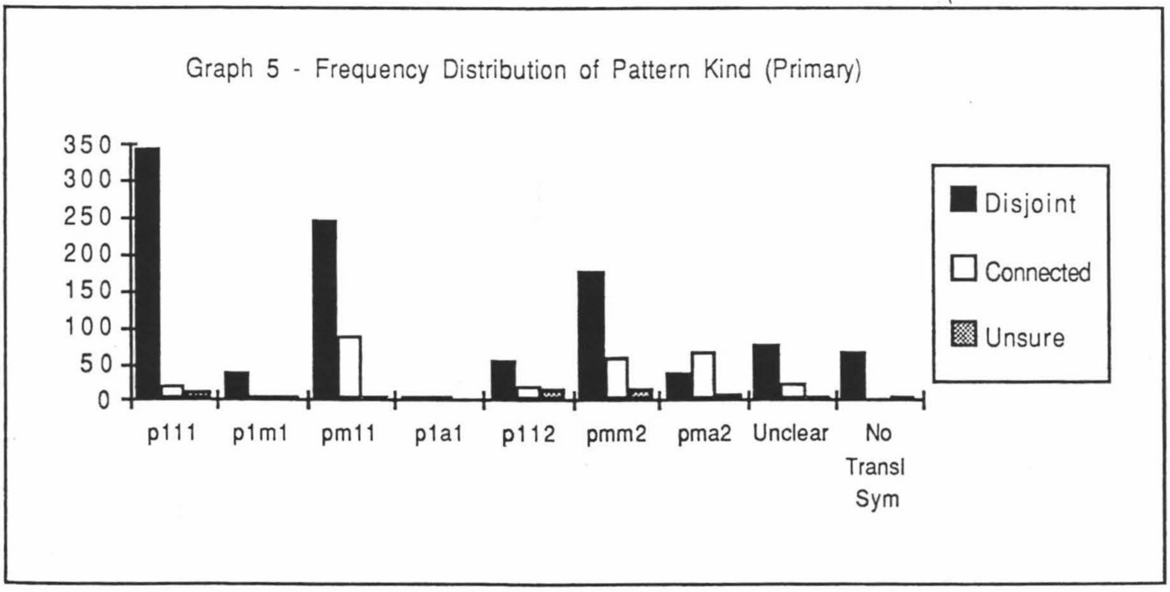
This question has proved to be difficult to answer satisfactorily. An initial attempt could be made from the frequency of the presence of symmetries in patterns p111, pm11, p1m1, p1a1, and p112, as summarized by the list above. Thus, whilst pmm2 has translation, horizontal reflection, vertical reflection, half-turn and trivial glide reflection symmetries, if any have been used deliberately to any degree, the likely intuitive ones are translation, vertical reflection and half-turn whose combination which will generate the pattern. Similarly, pma2, which includes translations, vertical reflections, half-turns, and glide reflections, also seems likely to be the result of intuitive translation, vertical reflection and half-turn (using a different combination of the same symmetries).

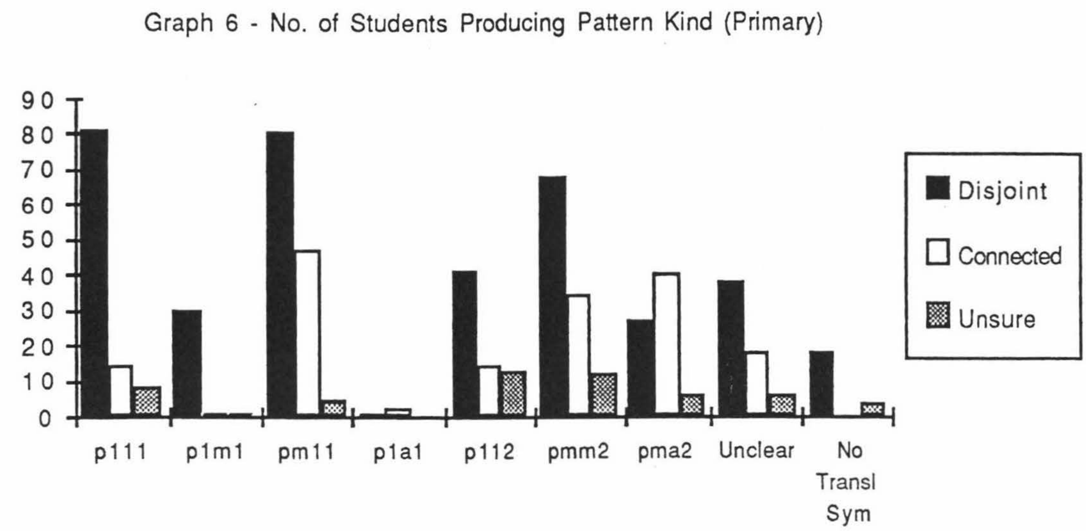
Not surprisingly, this rough explanation lends itself to criticism. For instance, it contains the assumption that the symmetries within pmm2 and pma2 were used *independently*. If they were not, there is little validity in using the frequency distribution of the 'less symmetric' patterns to make inferences about the manner in which pmm2 and pma2 were made. It also does not include important factors such as the construction methods and the intentions of the creators themselves. The quantitaive analysis in the next subsection makes a first step towards describing the likely use of transformations in the activity (a) exercise.

4.1.3 Construction Analysis

Using the findings of the interviews (see section 4.4) a further refinement to the frieze group sorting of the activity (a) patterns can be made. This new analysis emphasizes the two main probable methods of construction by making a distinction between two *kinds* of pattern:*disjoint* and *connected.* The categories add another dimension to the way the patterns are viewed, and became of importance when considering how the Standard Fours probably constructed their frieze patterns. (See subsection 3.2.2 for details of these categories)

The important measures seem to be the relative *frequency* of disjoint and connected patterns for each symmetry group and, at least as significant, how *common* each of the *kinds* is for each symmetry group (see appendix A1.4). Graphs 5 and 6 are shown for each measure respectively.





In addition to the information from graphs 5 and 6, the table in appendix A1.3 indicates that virtually all of the Primary school subjects produced disjoint patterns, while only half of them constructed a connected pattern. The 'Unsure' category indicates an ambiguity between the disjoint and connected categories; therefore the number of students who made disjoint patterns and the number of students who drew connected patterns have been underestimated. However, during the classification process, it seemed to the researcher that most of the patterns put into the 'unsure' category were in fact disjoint ones.

While most of the patterns drawn were disjoint, it is interesting to see that the pma2 symmetry group was most frequently and most commonly drawn as connected. This result suggests that it was only this frieze group which may have been 'understood' as a connected whole by the majority of the standard fours surveyed. This *may* have been true of the p1a1 group as well, but the number of students and the number of patterns falling into this category was too small (3) to make any inferences about disjoint and connected ratios. The number of patterns making up the p1m1 category was also relatively low (41). However, almost all (39) of these were disjoint so it seems safe to conclude that the disjoint kind was the more intuitive way of constructing a p1m1 pattern.

The fact that many children did make patterns of both kinds suggests that a different intuitive use of transformations may be used by the same child from pattern to pattern. For example, the pma2 patterns, which are usually made as connected (and probably continuous as well) seem to have been created with a sense of the 'whole'. The pmm2 pattern are usually made as disjoint (and hence have a sense of 'repeating parts').

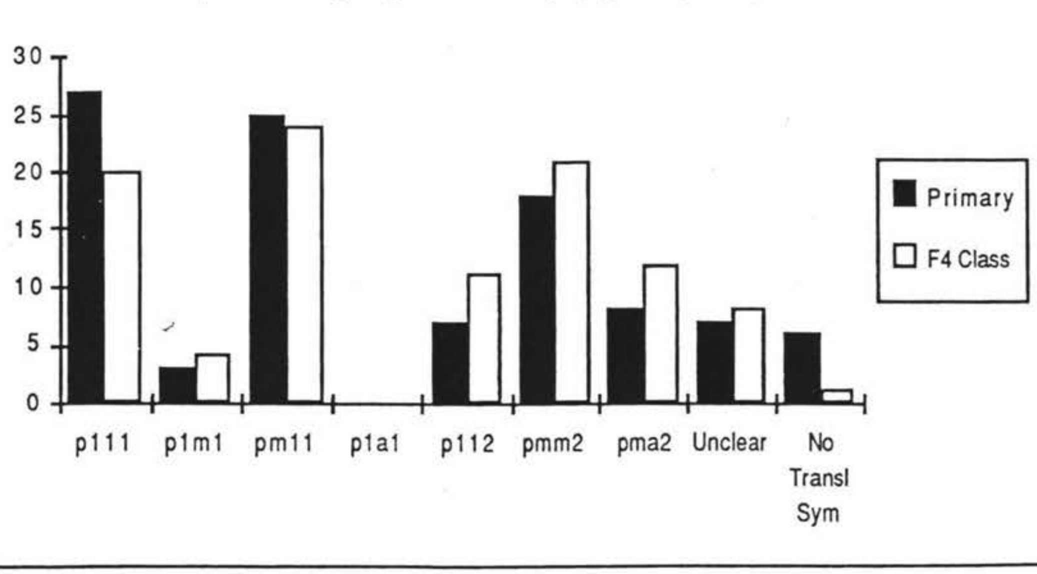
This new consideration of the construction method also adds further doubt over extrapolating results for the intuitive use of symmetry from some symmetry groups to others. This is why the explanation in subsection 4.1.1, suggesting which symmetries had been intuitively used in the constructions of pmm2 and pma2, is presented tentatively.

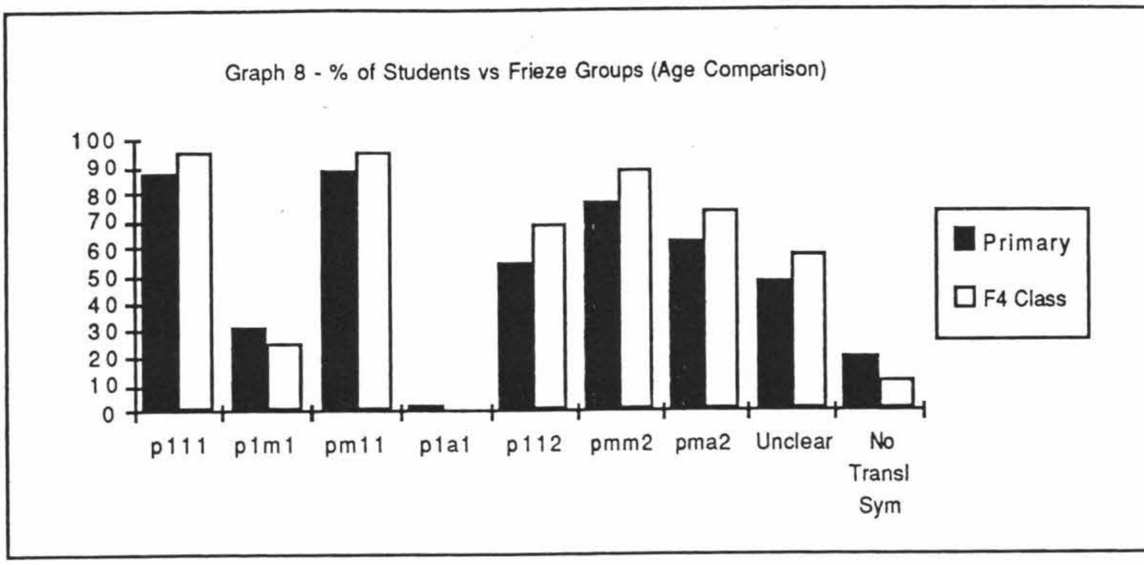
4.2 Age Group Comparison

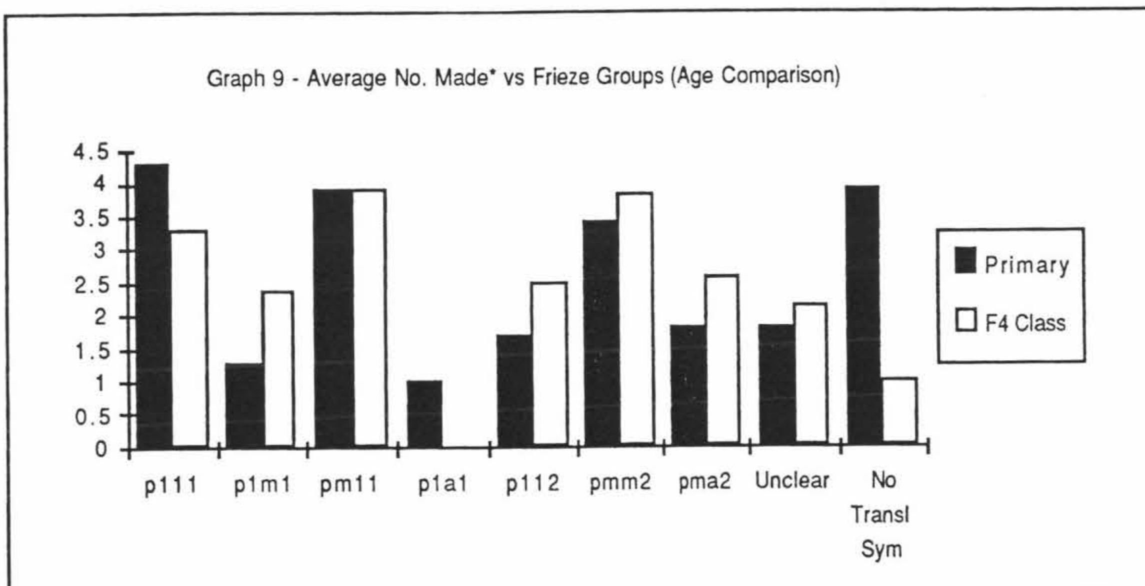
4.2.1 Frieze Group Analysis

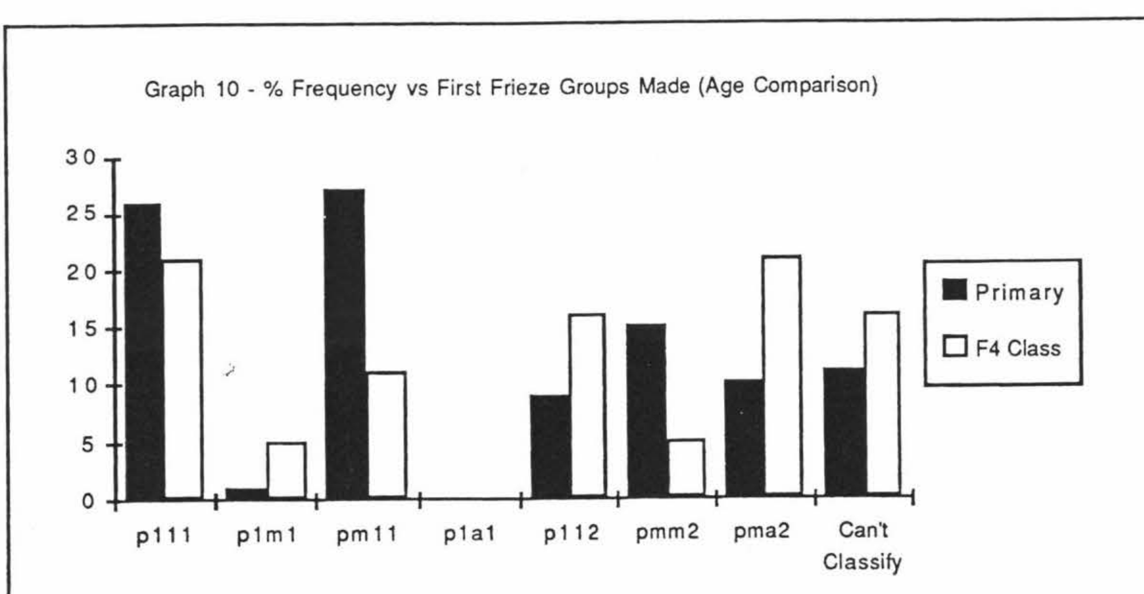
The two age groups considered here are Primary school (Standard Three and Four) children, and Secondary school (Form Four) students. These groups correspond to pre-formal and during-formal transformation geometry learning. Due to a lack of available class time, it was not possible to survey any post-formal groups. Furthermore, the number of Form Four's that participated in activity (a) was only 22 (one class) whereas the number of Primary students involved with this activity was 99. Therefore the results presented here are included mainly for interest; graphs 7, 8, 9, and 10 are simply suggestive.

Graph 7 - % Frequency vs Frieze Groups (Age Comparison)




Graph 8 - % of Students vs Frieze Groups (Age Comparison)
100
90
80
70
60
50
40
30
20
10
0
p111
p1m1
pm11
p1a1
p112
pmm2
pma2
Unclear
No Transl Sym
Primary
F4 Class


Graph 9 - Average No. Made* vs Frieze Groups (Age Comparison)
4.5
4
3.5
3
2.5
2
1.5
1
0.5
0
p111
p1m1
pm11
p1a1
p112
pmm2
pma2
Unclear
No Transl Sym
Primary
F4 Class


Graph 10 - % Frequency vs First Frieze Groups Made (Age Comparison)
30
25
20
15
10
5
0
p111
p1m1
pm11
p1a1
p112
pmm2
pma2
Can't Classify
Primary
F4 Class

The aim of this discussion is twofold: firstly, to highlight the factors which seem to be *common* to both groups; such factors indicate the nature of the patterns which are intuitive (that is, independent of a formal transformation geometry framework). The second objective is to examine some of the *differences* between the groups in order to obtain clues about the changes in pattern construction methods which may occur as a person moves from age 10 to age 14. In Piagetian terms, this transition corresponds roughly to the development from 'concrete operational' to 'formal operational' stages. Any contrasting properties of the patterns made by each of the groups may be helpful to teachers, since it provides information which is specific to those levels.

In graph 7, we can observe that the three most frequently drawn frieze groups were p111, pm11 and pmm2 for both groups. Also, the p1a1 and p1m1 patterns were drawn in low frequency by both age groups. In fact, as observed in the previous section, a rough 'intuitive ordering' can be constructed from the three measures of frequency, commonality, and average number of patterns drawn by the Primary and Secondary students surveyed:

1. p111, pm11 ≥ pmm2 (common and frequent)
2. pma2 ≥ p112
3. p1m1
4. p1a1 (uncommon and infrequent)

Whilst the symmetry groups in one line of the list may be roughly equally intuitive, slight differences that occurred in all three measures for both age groups suggest the ordering indicated by the symbol '≥'. For instance, both p112 and pma2 have been ranked 2nd on the intuitive ordering list. However, on all three graphs for both age groups, the columns for pma2 are marginally, yet consistently, higher than the columns for p112. A comma (,) between two frieze groups signifies that the differences are not consistent between the measures or the age groups. For example, p111 is higher than pm11 on two measures, and is lower on the other two measures.

The most obvious difference between the age groups was the proportion of the patterns that were p111's. Graph 7 indicates that 27% of the Primary patterns were p111 whereas only 20% of the Secondary patterns were p111. This 'drop' in the proportion of translation-only patterns yielded small increases in the percentage of p112, pmm2 and pma2 symmetry groups produced. This shift indicates that other non-translation transformations were being used more often by the *during*-formal transformation geometry learning group than by the *pre*-formal students.

The difference in the numeracy of p111 was not paralleled in the proportion of students making this pattern; the percentage of Fourth Form students who made this frieze group was a little greater than the percentage of Primary pupils who made it. Indeed, this was the case for the five most common frieze groups. The number of patterns, and the number of students producing them, was very low for the other two symmetry groups.

The distribution of the first patterns made by each age group is shown by graph 10. Two observations seem to be clear:

1. The 'skyline' profiles of graph 7 and graph 10 for the Fourth Form were different.
2. The 'skyline' profile of graph 10 differed for each age group.

These observations suggest the following conclusions. Firstly, the distribution of frieze groups and hence the symmetries used by the Fourth Formers changed as they progressed through activity (a). Particularly noticeable was the high occurrence of pma2's made first relative to the proportion of all pma2's of the total number of patterns produced. This was a pattern which commonly 'leapt to mind' first but subsequently became less frequently drawn as the class continued filling in the empty strips on activity sheet (a). This suggests that the pma2 pattern may have been more intuitive to the Fourth Form class than the other three measures indicate. Conversely, the frieze groups pm11 and pmm2 occurred far less frequently at the beginning of the exercise than later, which indicates that the pm11 and pmm2 symmetry groups may not be as intuitive as the other three measures suggested.

Secondly, the patterns that initially 'leapt to mind' to the Primary students were of a different character to the patterns that the Fourth Form subjects drew first. Whilst both groups drew a large proportion of p111, they differed a great deal with respect to the pm11, p112, pmm2 and pma2. For instance, the proportion of Primary students who made a pm11 first, or who made pmm2 first, was twice as great as the corresponding proportion of Fourth Form students.

4.2.2 Transformation and Symmetry Analysis

The rough frieze group 'ordering' can in turn give clues to the relative frequency of some of the symmetries intuitively used by both groups. It should be remembered that this ordering does ignore the results of graph 10 which focuses on the first patterns drawn only. Using a similar argument to that in subsection 4.1.2, we can infer the relative 'intuitive-ness' of the transformations (which is the same for both age groups):

1. Translation (common and frequent)
2. Vertical reflection
3. Half-turn
4. Horizontal reflection
5. Glide reflection (uncommon and infrequent)

The analysis of this list is the same as the previous section. If (a) the symmetries are used without affecting each other (i.e., independently) *and* (b) there is no difference in the use of symmetry depending on construction methods (e.g., disjoint and connected patterns), then we might infer that the symmetry groups pmm2 and pma2 are both constructed using translation, vertical reflection and half-turn in an intuitive fashion. The glide reflection that exists in the pma2, and the horizontal reflection that exists for the pmm2 are probably incidental or even accidents. However, the two assumptions necessary to make this conclusion are open to question and thus the inference is probably not valid. This leaves the question of which symmetries present in pmm2 and pma2 were accidental still unanswered.

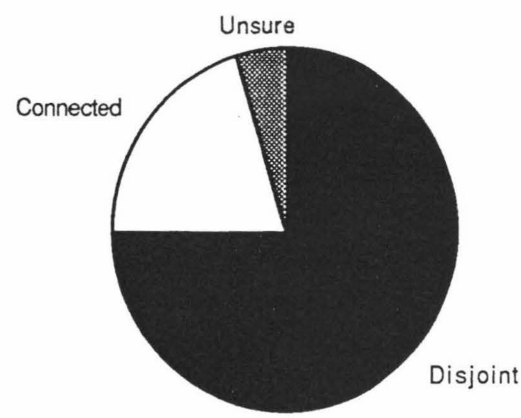
4.2.3 Construction Analysis

We can almost certainly deny the second of the two premises above by considering the distribution of *pattern kind* (i.e., disjoint and connected patterns) for all the frieze groups, particularly pma2 and pmm2. It is quite clear, from tables A1.4 and A2.4 that the distribution was not the same for each symmetry group. For example, the patterns in the pma2 category occurred far more frequently as connected than disjoint. In contrast, the p111's were almost always disjoint. Since the construction method and the associated conception of a pattern were probably different for each of these kind, the way symmetries were used was probably different for some frieze groups.

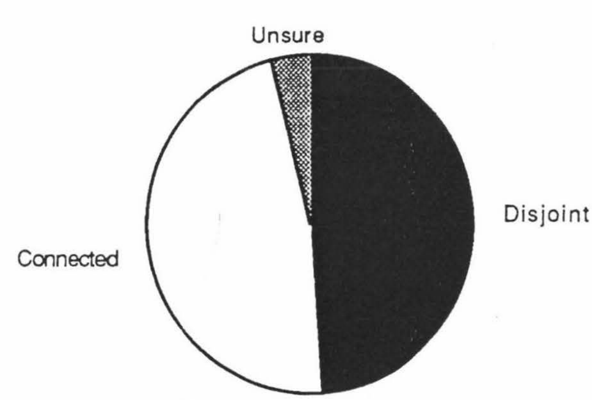
If we consider the proportion of patterns that were either disjoint or connected, there seems to be a clear difference between the two age groups. About 75% of all the Primary patterns are disjoint; whereas only 49% of the Secondary patterns are. Virtually all of the Primary and Secondary students produced a disjoint pattern, yet only 55% of the Primary students, compared with 100% of the Secondary pupils, made a connected pattern.

Now it doesn't necessarily follow that the construction of a connected pattern implies a sense of the whole on the part of the creator, whereas the sense was usually one of 'repeating parts' for the disjoint patterns (see subsection 4.4.1). Furthermore, even the Primary children who did make connected pattern made far fewer of them than disjoint patterns. This means that, *at most*, only half the Primary students use both construction methods associated with each kind (disjoint and connected) of pattern.

Graph 11 (i) - Kind of Patterns Produced (Primary)



Graph 11 (ii) - Kind of Pattern Produced (Form 4 Class)



4.3 Further Observations

This section describes five aspects of activity sheet (a) not already covered by the quantitative analysis in the previous subsections. These include the 'No Translation', 'FG Unclear' and 'Unsure' categories, symmetries present in the patterns not indicated by the symmetry group classification as well as seemingly accidental symmetries. Whilst classification as a frieze group was occasionally not possible, many of the drawings in the

'Can't Classify' category showed evidence of transformations. The examples of design anomalies are extracted only from the Primary school subjects. The discussion concludes by considering the results of 10 'vertical' patterns drawn by the interviewees.

4.3.1. No Translation Symmetry

By definition, a pattern fell into this category if it didn't portray sufficient evidence of translation symmetry. There were three main causes for this phenomena.

In the first case, only a single figure was drawn. This did not occur very often. In this case, the instructions don't appear to have been followed, although the intention of a drawer can not be stated with any degree of certainty unless questioned directly. The underlying symmetry group of the single figures was not usually C_1, but some other group such as a strip-filling C_2. Despite the lack of translation symmetry, the design usually had some other symmetry present in it.

In the second case, a sequence of figures was drawn with little or no evidence of repetition. In this case, the word 'repeating' was interpreted in some other way besides translation. The figures were most often examples of the groups D_1, D_2, D_3, D_4, D_5, D_6, D_∞ and occasionally C_2, or C_4. Instances of these included isosceles triangles (with vertical symmetry), people's faces, green peace signs, cars, trucks, houses, trees, equilateral triangles, rectangles, squares, stars, asterisks, circles and 2- or 4- armed swastikas. Some C_1 figures often showed approximate vertical reflection symmetry. Other C_1 figures, such as spirals, 'suggested' rotation.

Figure 4.1

Many students may consider the filled in strip, without any extension, to be 'the pattern'. For example,

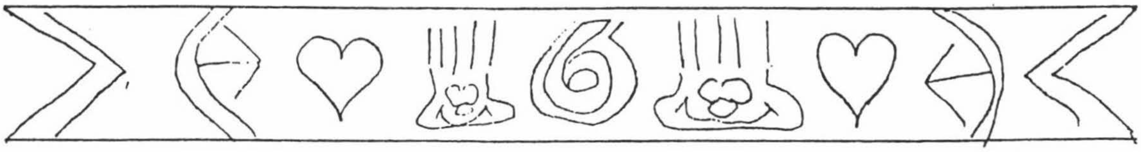


Figure 4.2

In the third case, not enough of the design was drawn in the strip for the classifier to be confident that translation symmetry existed. (Two translation units were required for a pattern to avoid this category). This is not to say that translation symmetry wasn't *intended*, but that this intention wasn't sufficiently clear to the classifier from the drawing. It is conceivable that such a pattern may have been intended to be a symmetry group. Therefore it could be classified as 'FG Unclear', since the symmetry group was often ambiguous, but this isn't done because priority in classification is given to translation symmetry (see subsection 3.2.2). Of course, the very fact that a pattern suggested one or more symmetry groups implies that it contained some symmetry.

In short, a good deal of intuitive transformation geometry was often present in designs with no apparent translation symmetry. Very few patterns which fell into this category contained no symmetry. Even the exceptions often had some non-rigid transformations in them, such as shear or enlargement/dilation. For example,

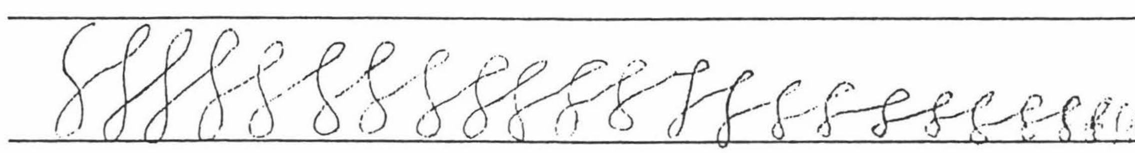


Figure 4.3.1

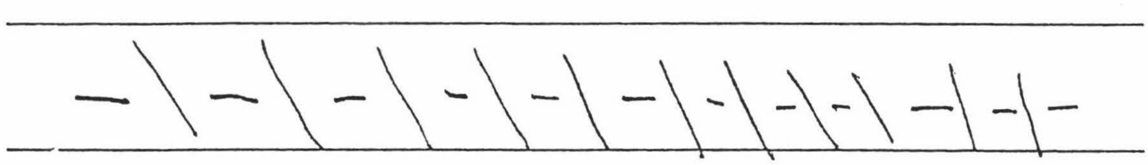


Figure 4.3.2

In the first case (fig. 4.3.1), the dilation transformation seems to have been deliberate. The second example (fig. 4.3.2) is more difficult to assess. The shear, or progressive slant, may have been an accidental result of the creator keeping his elbow in one fixed place on the table as he drew each stroke from left to right.

Other 'topological' transformations occasionally occurred.

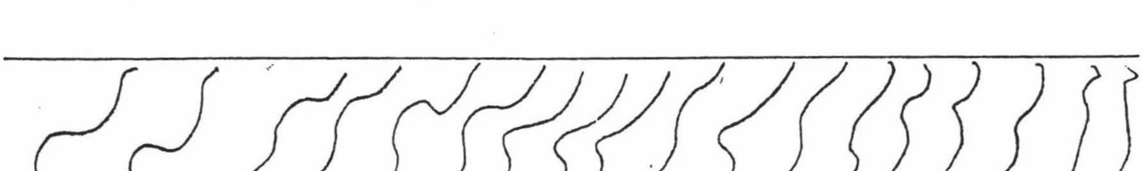


Figure 4.4

It is not clear what intentions the creator had in her design (fig. 4.4).

4.3.2. Frieze Group Unclear - Patterns With Ambiguous Symmetry Groups

In Chapter 3, a strip pattern with translation symmetry was defined as 'FG Unclear' if it had some ambiguity in the pattern which meant it could be conceived of as more than one symmetry group. The reasons why patterns were classified in this way were numerous. The following examples are an attempt to sketch some of the ways frieze group ambiguity could be introduced into a pattern.

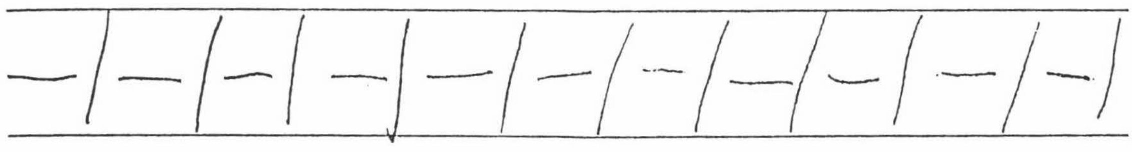


Figure 4.5

The *slant* was ambiguous in this case. Were the 'vertical' line segments actually intended to be sloped? "No" implies pmm2; "Yes" implies p112.

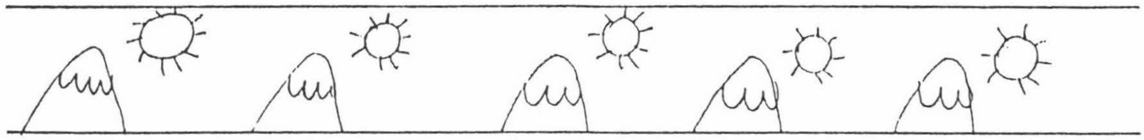


Figure 4.6.1

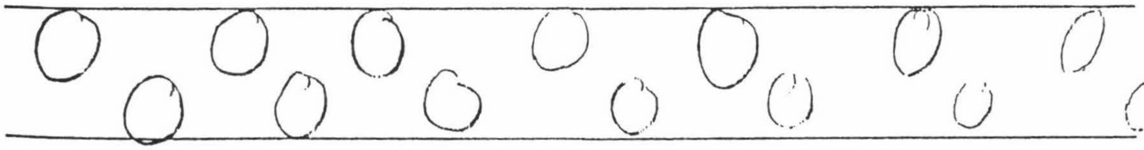


Figure 4.6.2

The *spacing* was ambiguous in both patterns (figs. 4.6.1 and 4.6.2). In the first case, is the underlying symmetry group p111 or pm11? In the second case, is it an example of a pma2 or a p112?

The following patterns (figs. 4.7.1 and 4.7.2) display an ambiguity of *position*. The creator of the first pattern was not clear about her intentions as to the position of the objects with respect to the border lines. Should the pattern be classified as a pmm2 or a pm11?

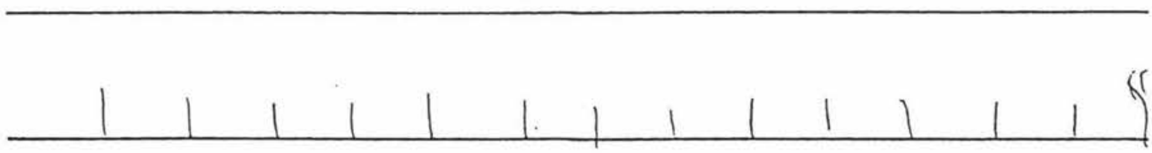


Figure 4.7.1

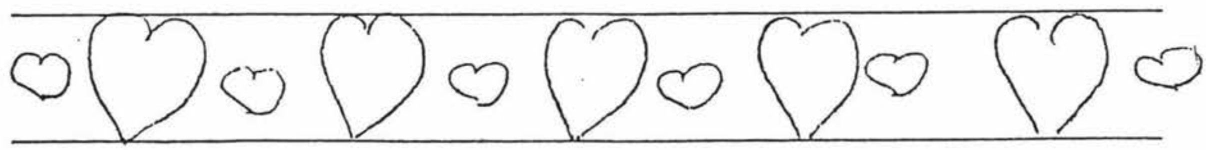


Figure 4.7.2

Is the second pattern (fig. 4.7.2) a p111 or a pm11? The ambiguity results because the relative position of the big and small hearts is not clear. Frieze group ambiguity could also result from a creator using *'topological'* shapes, figures or curves. Four examples follow:

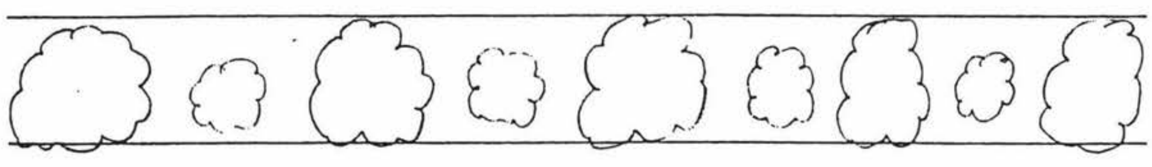


Figure 4.8.1 (Clouds)

Figure 4.8.2 (Scribble)

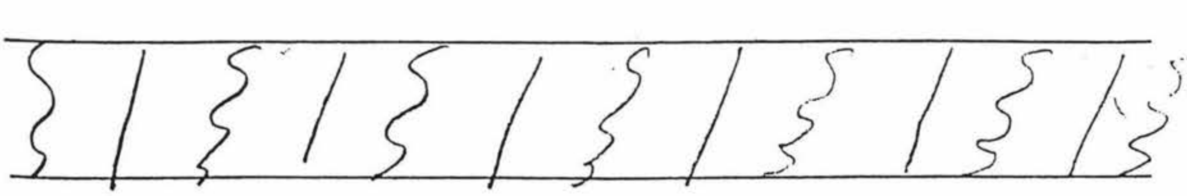


Figure 4.8.3 (Squiggles)

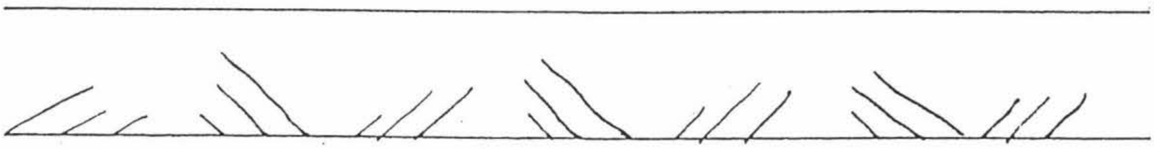


Figure 4.8.4 (Other)

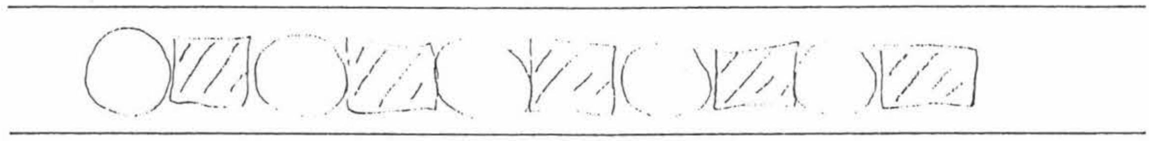


Figure 4.9

The pattern above (fig. 4.9) has an ambiguous frieze group because of the colouring. Finally, the last type of frieze group ambiguity arose from the 'perspective' drawing of 3D objects, or overlapping, as the following examples illustrate.

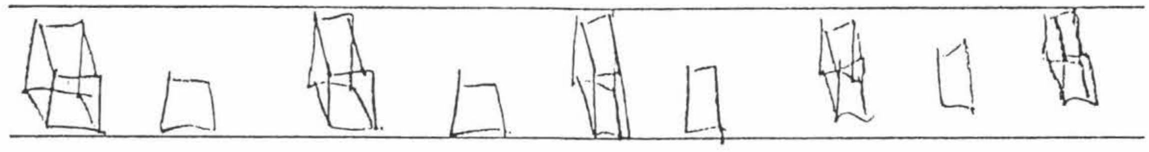


Figure 4.10.1 (Solids)

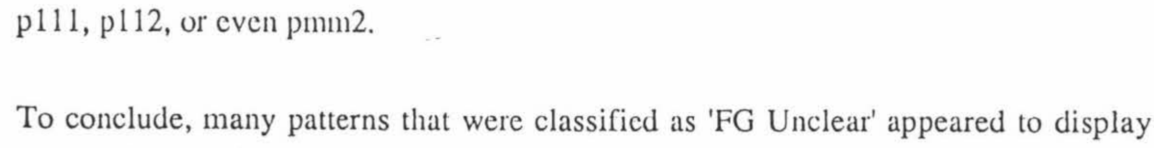


Figure 4.10.2 (Overlap)

The possible 'overlap' (fig. 4.10.2) suggests a weaving pattern. The symmetry could be p111, p112, or even pmm2.

To conclude, many patterns that were classified as 'FG Unclear' appeared to display evidence of transformation geometry in them, including translation symmetry. Unfortunately, it wasn't always clear *which* non-translation transformations were present or intended.

4.3.3. Symmetries Not Indicated By Frieze Group Classification

One of the shortcomings of the frieze group classification is that symmetries of figures within a particular pattern, and the transformations that exist between figures, can easily be 'ignored' by the symmetry group assigned to that pattern. In order to overcome this inadequacy, we can either refine the classification, or invent a new one. In the former case, ambiguities can easily arise; a practical problem which can make interpretation difficult. On the other hand, inventing a new system may reflect the 'internal symmetries' well but miss the overview of the symmetries of the whole pattern that the frieze group analysis provides; devising such a system is a theoretical problem yet to be explored according to Grünbaum *et al.* (1986).

The fourth section of chapter two discusses some of the possibilities of classification at length, and most of these are used in various parts of this chapter. It is beyond the limits of this thesis' time constraints to give more than a brief, descriptive analysis of the disjoint activity (a) patterns using the 'alternative' incidence classification. Instead, we will examine some examples of the seven frieze groups that have more transformation geometry in them than revealed by the crystallographic group, and try to make some reasonable conclusions about the nature of the use of intuitive transformation geometry.

It would be easy to conclude that a person having most of her, or his, 16 patterns classified as p111 did not have a strong intuitive sense of transformations besides translation. After observing many p111 patterns drawn by Primary school children, that conclusion now seems false. Consider the following patterns.

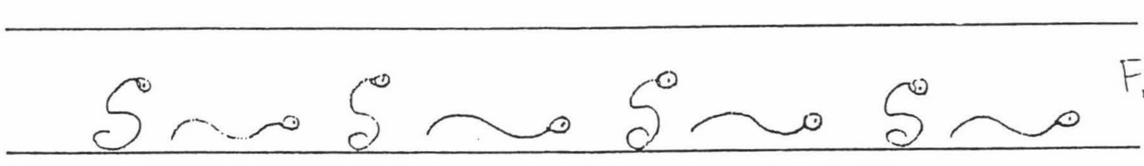


Figure 4.11.1

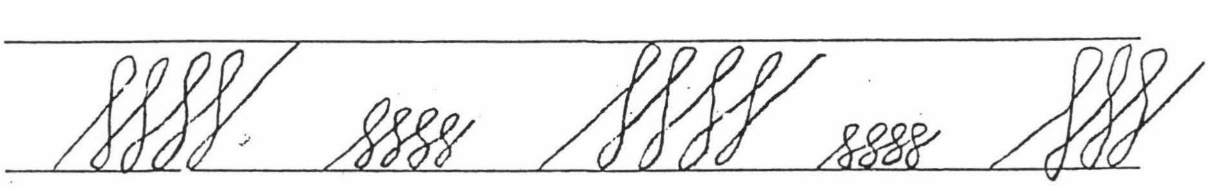


Figure 4.11.2

The first example (fig. 4.11.1) contains figures that suggest, or approximate, the C_2 symmetry group. Because of the slight asymmetry (i.e., the head of the 'snake'), the frieze group classification is p111. In the second case (4.11.2), the C_2 figures are clear, but the positioning of both base pattern elements on the bottom line instead of 'in the middle' resulted in the whole pattern (including the border lines) having only translation symmetry.

The occurrence of *approximate* vertical reflection was also quite high. Many $D_1(v)$ figures appeared to have been altered or ornamented deliberately. In one interview, a child said that she did this to make them "more interesting". Let's consider the following pattern (fig. 4.12) and extract from an interview:

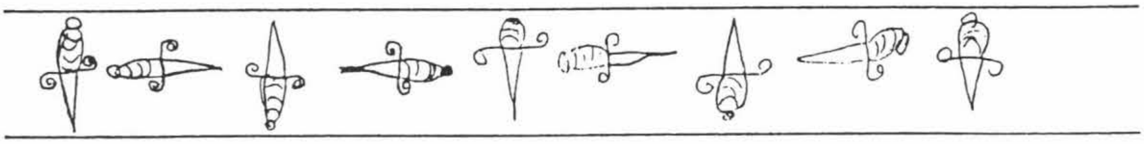


Figure 4.12

I: Pattern number 4 ... Are these daggers or knives or something?

I: Yeah!

I: When you did the curls on them, what did you do ... what were you thinking of when you did them?

M: I dunno.

I: 'Cause you've got one going like that ... and one going like that.

M: Mmm ...

I: But you didn't draw this, you didn't go [draws a dagger with both curls down: a D_1 figure] ... you didn't go like this.

M: I dunno ... I went ... instead of going two down, I put one down and one up.

I: Okay, did it give it a different look when you do it like that ... in any way?

M: Mmmm ... I think so. I dunno ... 'cause when I do them I do them at home ... I start drawing like that [points to interviewer's sketch] ... but then I had an idea to do them like that ... and they look better ... more interesting.

Mark was aware that he had changed the look of the dagger, although he mentioned nothing that explicitly identified the vertical reflection symmetry of the dagger. The asymmetry therefore appears to be implicit, since it is probably intentional.

The use of deliberate asymmetry is not peculiar to the Primary children's patterns. Indeed, many western art works display this phenomenon according to Weyl (1952, p13) He went on to say:

> "But seldom is asymmetry merely the absence of symmetry. Even in asymmetrical designs one feels symmetry as the norm from which one deviates under the influence of forces of non-formal character."

The presence of intentional asymmetries certainly suggests that, in some way, the symmetry that has been removed is intuitively perceived by the creator of the pattern. In some circumstances it seems natural that such asymmetry is pursued. In their article on symmetry-making and -breaking in visual art, Molnar and Molnar (1986) state:

> "... the experts are in agreement with the majority of people in stating that the strictly symmetrical figures are less satisfying aesthetically. In another context (for example in ornamental art), symmetry would be, on the contrary, judged as attractive."

Other patterns also contained highly symmetrical figures or a considerable degree of incidence. This clear presence of intuitive transformation geometry was not recognised by the general frieze group classification. For the sake of brevity, only one example of each symmetry group will be shown to illustrate this point.

Figure 4.13

In the p1m1 case (fig. 4.13), a base pattern comprising a $D_4(v)$ figure and a $D_1(h)$ figure yielded a pattern with translation and horizontal symmetry only.

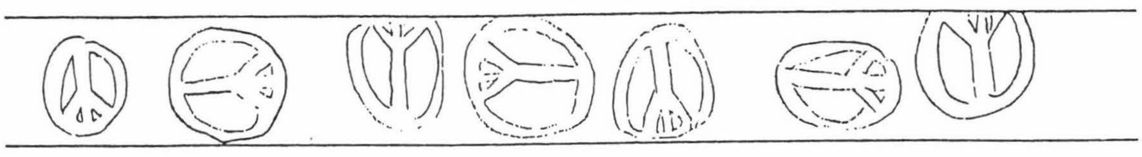


Figure 4.14

The above pm11 pattern (fig. 4.14) contains a D_1 motif and quarter-turn incidence between each motif. Between consecutive $D_1(v)$ figures, there is half-turn symmetry. Yet the overall symmetry group classification of the pattern only includes translation and vertical reflection symmetries. Not only did the drawer of this pattern have a non-symmetry group motivation, but he also used a higher degree of transformation geometry than the common mathematical classification of such a pattern would suggest.

No example was found for *p1a1*.

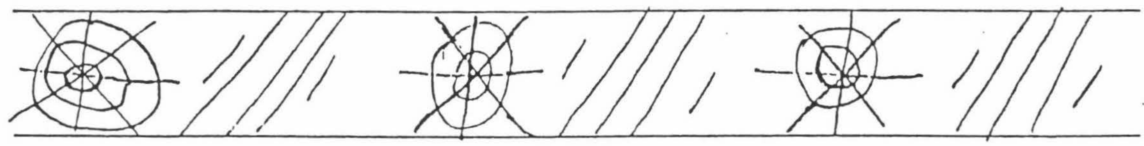


Figure 4.15

The base pattern of the p112 design illustrated above (fig. 4.15) contains one D_4 figure, and a set of disjoint straight lines arranged in such a way as to have half-turn symmetry. The repetition of the base pattern results in the 'whole' pattern having the lesser 'amount' of symmetry than the components, that is, half-turn symmetry.

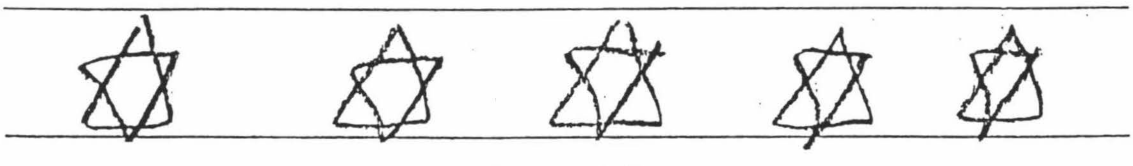


Figure 4.16

The 'Star of David' figures are examples of the D_6 symmetry group. However, the frieze group classification, pmm2, only recognises the resulting half-turn, vertical and horizontal reflection symmetries of the whole pattern.

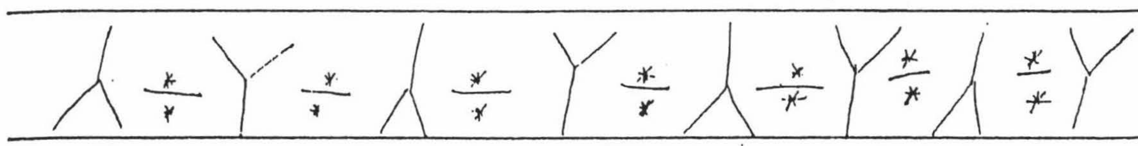


Figure 4.17

Finally, the pma2 pattern (fig. 4.17) contains elements such as $D_1(v)$ and $D_2(v)$ figures. In fact, the asterisks could be interpreted as $D_4(v)$ figures; the whole pattern certainly does not have 'diagonal' reflection.

This discussion has attempted to reinforce a salient point. In some patterns, the symmetry of the figures drawn, and the high order incidence relationships between them, suggests that an intuitive use of transformations may have been employed by some students, a use which is far beyond any formal geometry instruction given to children at this stage of their education.

4.3.4. Accidental Symmetries

Full sections of interview material in this chapter and the next are presented to indicate the construction methods associated with a pattern's frieze group, kind, style and type. However, in this present discussion, a few interview extracts that illustrate accidental symmetries are included to reinforce the idea that not all symmetries found in patterns are intuitive to the creator. This is supplemented by a more theoretical discussion of other potential symmetry accidents.

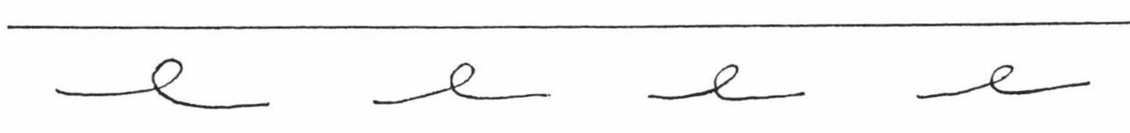


Figure 4.18 (Mary)

I: Okay, what about this one, what were you thinking of there?

M: Ummm ... it's sort of like an 'e'.

I: Sort of like an 'e' ... anything else? ... can you sort of describe that pattern to me?

M: Um ... it's a line with a loop in the middle ...

I: Uh-huh ... tell me about the loop ... is the loop meant to be straight up and down or is it meant to be sort of on a slant?

M: It's sort of like on a slant.

I: Are you sure?

M: Uh-huh.

Any possibility of intuitive vertical reflection can be excluded in the construction of this pattern (fig. 4.18).

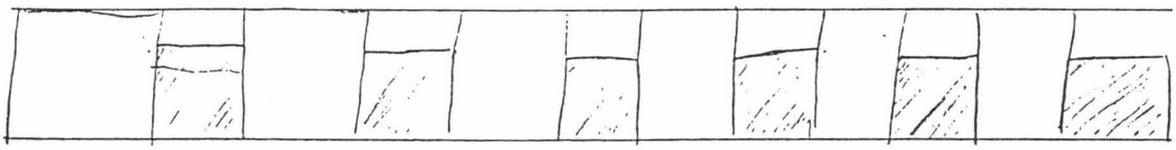


Figure 4.19 (Alice)

I: See this one here ... did you deliberately shadow it in this direction, or did you just mean for it to be shaded?

A: Just shaded.

.... I: And is that line meant to be right in the middle, or is it meant to be more ... higher than the middle?

A: Umm? ...

I: Do you see what I mean? Is that line actually meant to be there? (points halfway between border lines)

A: No.

I: It's not. It's meant to be higher?

A: Yep!

The shading was non-directional (fig. 4.19), implying that the overall symmetry group has been correctly identified as pm11, since the side length of the smaller square is more than half of that of the larger square. However this frieze group identification seems meaningless in this instance since the vertical symmetry of the whole is probably an accidental result of the symmetries of the two squares in the base pattern, and the spacing of the base patterns (see Alice's construction methods in interview section 4.4). Any symmetry that may be intuitive in this pattern is most likely to be implicit in the drawing of each of the base pattern components, namely, the squares. However, it seems reasonable to conjecture that the rotation symmetries, which are the composition of the reflection symmetries, could easily be incidental.

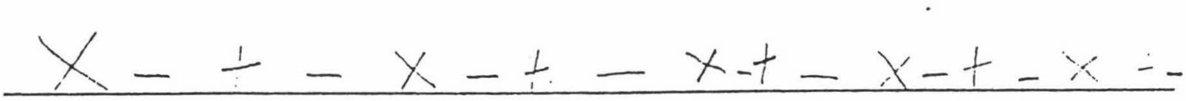


Figure 4.20 (Kate)

I: See this pattern here ... what were you thinking of there? What did you do?

K: Well I couldn't think ... I just thought of that one because it was things to do with [maths] ... it was something different that I thought of... I used that as well ... but I decided to do all the maths signs in them.

I: Uh-huh.

K: Which made quite an interesting pattern.

Kate also said she "noticed them on the blackboard .... so I decided to make a pattern out of them ... times, minus, plus, minus, ... like that (points at each figure, left-to-right)." This seems to be a base pattern construction, especially given the spacing of the design (fig. 4.20), and the use of everyday mathematics symbols (mono-oriented objects). This suggest that the intuitive transformation geometry present is implicit in each figure, but accidental in terms of the pattern as a whole.

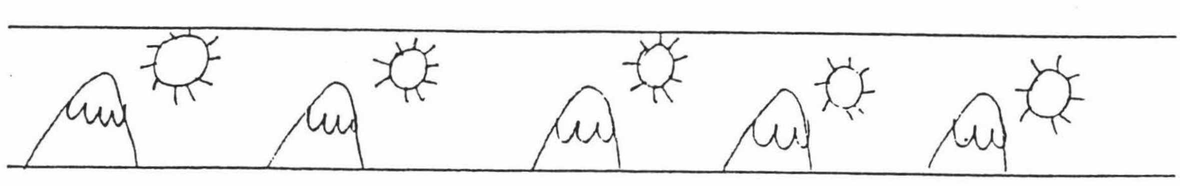


Figure 4.21 (Amy)

I: Number 12 [fig. 4.21]. What was the order you did things in there?

A: I did the triangle [mountain]... and then I did those little bits there ... they look like snow ... and then ... I think I put the sun on it.

I: Okay ... so you didn't do all the mountains first and then go back and do the suns?

A: No. I don't think so.

This pattern looks like a pm11, but according to Amy (later), the sun and mountain were intended to be grouped together, not evenly spaced. She constructed the pattern strictly from left to right which, given that the pattern is disjoint, strongly intimates a base pattern construction. Again, the intuitive transformations present are likely to be implicit in the mountain (vertical reflection) and possibly the sun as well. Translation has been used explicitly as a repeating operation.

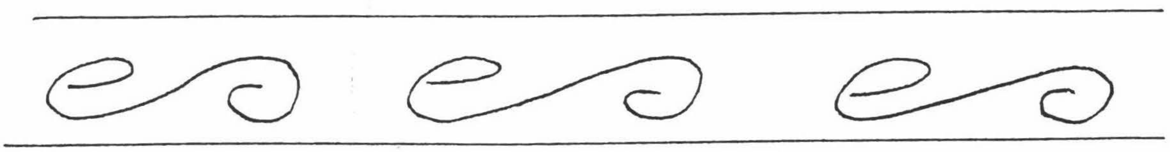


Figure 4.22 (Richard)

Each figure in this pattern (fig. 4.22) appeared to have half-turn symmetry, but in fact, this was not the case:

I: It that curl supposed to look like that curl in some way?

R: Nup. It's supposed to look like an 'e' but it just flicks ... It's not really supposed to be the same ... that one's kind of a 'e' and that one's just half.

The only transformation intended in the pattern above (fig. 4.22), therefore, was translation.

Perhaps another potential occurrence of unintentional symmetries are those resulting from the composition of other symmetries present in a pattern. To illustrate a simple example, consider the following design:

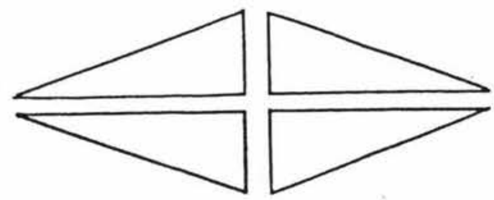


Figure 4.23

Besides the identity transformation, the underlying symmetry group has three elements: half-turn, vertical reflection and horizontal reflection symmetries. Imagine a person making such a pattern; she or he might intentionally make the 'top and bottom', and 'left and right' look the 'same' (i.e., the horizontal and vertical reflection symmetries are to some degree intentional, although not particularly well-defined). The combination of the two reflection symmetries gives the pattern an extra symmetry (a half-turn). If the half-turn is not intended and not perceived, we want to be able to say that the presence of horizontal and vertical symmetries is intuitive, whereas the half-turn symmetry is an accidental spin-off of the construction; that is, an *incidental symmetry*.

As discussed in chapter 1, the word *intuitive* can have slightly different meanings depending on the context in which it is used. In this discussion, it is defined as the non-accidental use of transformation geometry independent of any formal learning of the subject. The degree of intention behind the use of transformations could be considered to vary on a continuum from *explicit* (conscious, though not formal) to *implicit* (subconscious) to *accidental* (an incidental occurrence of symmetry). For the sake of ease, only these three classes are considered.

In the case of the seven frieze patterns, the same principle will easily apply. To do this methodically, it is helpful to think of each group in terms of its minimal generating set(s). The minimal generating sets represent different ways of generating the patterns, without necessarily using all the transformation types present in the pattern as a finished product. The anomaly, p111, has only one type of symmetry-of-the-whole, so it is impossible for any composition of that symmetry to yield another accidental symmetry. In this sense it is unique.

The other frieze groups *can* be produced without using all their generators. If we can identify those symmetries a person has intended, these sets allow us to systematically raise questions about incidental symmetries (extra symmetries which were present but not necessarily used to generate the group). Theoretically, even the translation symmetry in p1a1 could be an accidental by-product of the repeated use of the glide reflection transformation. For a symmetry group such as pmm2, a whole myriad of different uses of transformations will generate it. So, mathematically, there is more than one way of using transformations to generate six of the seven frieze patterns. With all these possibilities, it would be rather surprising if a drawer did intend all the symmetries present in a frieze pattern such as pma2 or pmm2.

Table 4.1

Frieze Group	Types of Symmetries Present	Minimal Generating Sets
p111	$\{t\}$	$\{t\}$
p1m1	$\{t, h, g_t\}$	$\{t, h\}, \{t, g_t\}, \{h, g_t\}$
pm11	$\{t, v\}$	$\{t, v\}, \{v_1, v_2\}$
p1a1	$\{t, g\}$	$\{g\}$
p112	$\{t, 1/2\}$	$\{t, 1/2\}, \{1/2_1, 1/2_2\}$
pmm2	$\{t, v, h, g_t, 1/2\}$	$\{t, v, h\}, \{t, v, 1/2\}, \{t, v, g_t\}, \{t, h, 1/2\}, \{t, 1/2, g_t\}, \{v, h, 1/2\}, \{v, h, g_t\}, \{v, 1/2, g_t\}, \{h, 1/2, g_t\}, \{v_1, v_2, h\}, \{v_1, v_2, 1/2\},\{v_1, v_2, g_t\}, \{1/2_1, 1/2_2, v\}, \{1/2_1, 1/2_2, h\}, \{1/2_1, 1/2_2, g_t\}$
pma2	$\{t, v, g, 1/2\}$	$\{v, 1/2\}, \{v, g\}, \{g, 1/2\}$

t = translation symmetry, h = horizontal reflection symmetry, v = vertical reflection symmetry, $1/2$ = half-turn symmetry, g = (non-trivial) glide reflection symmetry, g_t = trivial glide reflection symmetry. Numerical subscripts denote independent symmetries of the same type. A similar table featured in Schattschneider's (1986) article, although she appeared to omit 6 of the possible minimal generating sets for the symmetry group pmm2.

An implication of this table is that it is not always possible to confidently state, even theoretically, which symmetries were intended or used in a design, and which were not. Of course, in particular cases, when the number of different symmetries is two or less, that is, translation and possibly one other sort kind of symmetry, we can make some conclusions, at least tentatively.

p111, for instance, requires some use of translation. The frieze group p1m1 almost certainly requires the use of translation and horizontal reflection; in the two cases where one of these transformations isn't a generator, it seems very likely that some use of horizontal reflection and translation is used in the trivial glide which can be thought of (and often is by geometry students) as the *composition* of translation and horizontal reflection. pm11 requires the use of vertical reflection (but not necessarily translation since it may be combined with another vertical reflection). p1a1 requires the use of glide reflection, and p112 requires the use of half turn. The ordering of symmetries from the results of these five patterns which appears in subsections 4.1.2 and 4.2.2 seems justified. From table 4.1 above, however, it is clear that the other two symmetry groups, pmm2 and pma2, introduce ambiguity.

We can obtain more information by refining the classification system so as to reflect something of the drawers' methods. For instance, categorising patterns into kinds, styles or types. However, the best approach is probably not theoretical. Watching the drawers construct their patterns, and discussing the motivations, intentions and understandings the creators have of their own patterns seems to be a more revealing investigation method.

4.3.5. Unsure - Patterns With Ambiguous Kind

The most common practical problem that introduced ambiguity between disjoint and connected patterns was confusion between tilings and a set of disjoint figures or line segments that touched the strip borders. This was a direct result of the border lines of the strip being already provided for the drawer (see appendix E1). Typical examples of the disjoint/connected ambiguity follow for each of the 7 frieze groups and for the 'FG Unclear' and 'No Translation' categories.

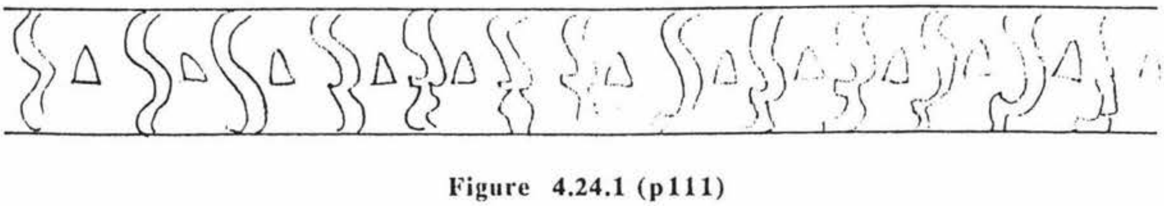


Figure 4.24.1 (p111)

Figure 4.24.2 (p1m1)

Figure 4.24.3 (pm11)

No example was found for the p1a1 symmetry group.

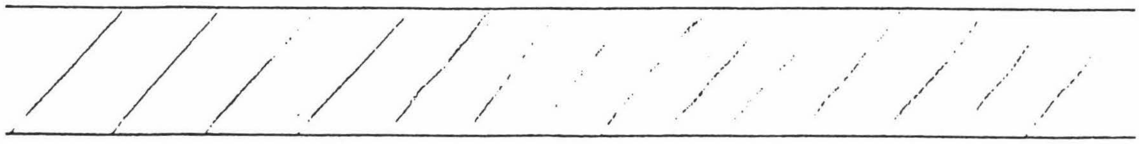


Figure 4.24.4 (p112)

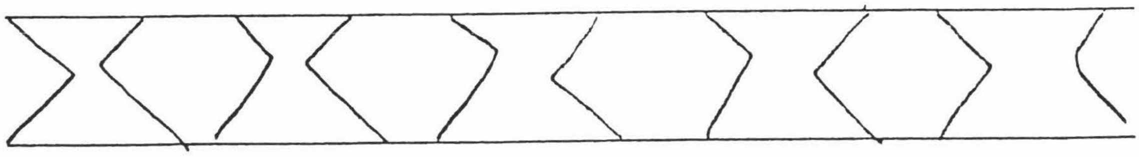


Figure 4.24.5 (pmm2)

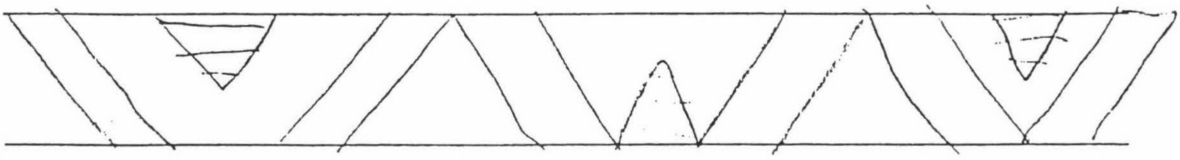


Figure 4.24.6 (pma2)

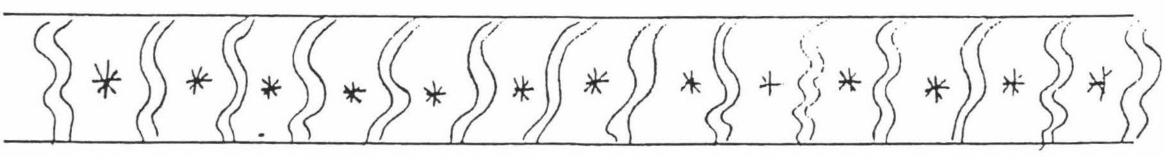


Figure 4.24.7 (FG Unclear)

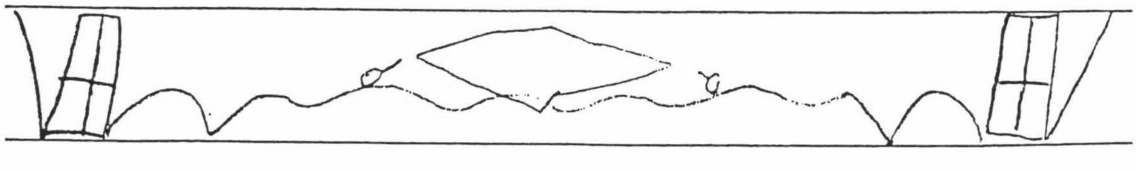


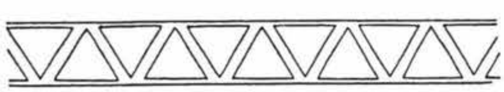
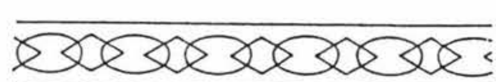
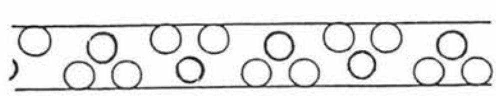
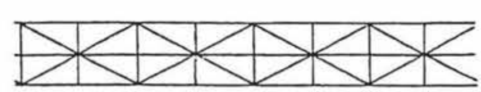
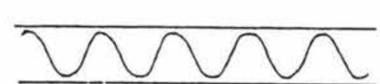
Figure 4.24.8 (No Translation)

Besides outlining the kind-ambiguity present in the patterns, some further points can be made. The results of the pattern kind analysis (disjoint and connected) for the Primary children were clear for all of the frieze groups except for p1a1. The disjoint patterns were by far the most frequent and common kind produced by this pre-formal-transformation-geometry framework group. The construction of such designs suggests that a 'repeating parts' conception was the predominant one for this age group.

However, the results for the older age group seemed to split the connected and disjoint categories fairly evenly. This shift in construction method is then due to (a) the students having had some exposure to the formal transformation framework, and/or (b) some cognitive development having taken place; for example, most of the Primary students are at the concrete operations stage, whereas the majority of Secondary students are at the formal operations stage. See Wadsworth (1979) for a synopsis of Piaget's theory of cognitive development.

While the disjoint patterns suggest 'repeating parts', the construction of a connected pattern doesn't always imply a 'connected whole' conception by the drawer. The reason for this can be seen by considering a further refinement of the two categories, as defined in section 3.4.

Table 4.2

Disjoint	**Connected**
Discrete 	*Touchings* 
Non-Discrete 	*Tilings* 
	Continuous 

Only the patterns in the *filamentary* connected category strongly suggested a 'whole' construction. The touchings, for example, actually suggest a 'repeating parts' construction in many instances, whilst the tilings may be made by either construction method (depending on the drawer).

The pie diagrams in section 4.1, therefore, underestimate the proportion of patterns drawn as 'repeating parts'. Unfortunately, it is difficult, if not impossible, to present results of a further breakdown of the disjoint/categories since the practical problem of distinguishing these refinements as listed above is too great for activity sheet (a)'s frieze groups (as mentioned in subsection 3.2.2). To illustrate this difficulty, consider the following commonly drawn pattern:

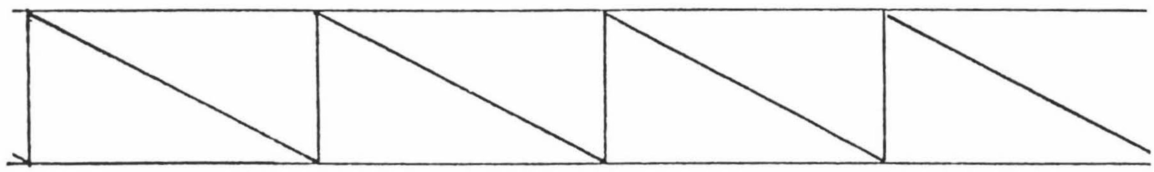


Figure 4.25

Was this intended to be a touching (fig 4.26.1), a tiling (fig. 4.26.2) or a filamentary pattern (fig. 4.26.3)?

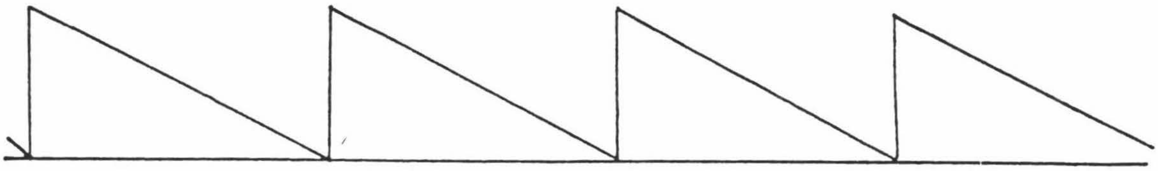


Figure 4.26.1

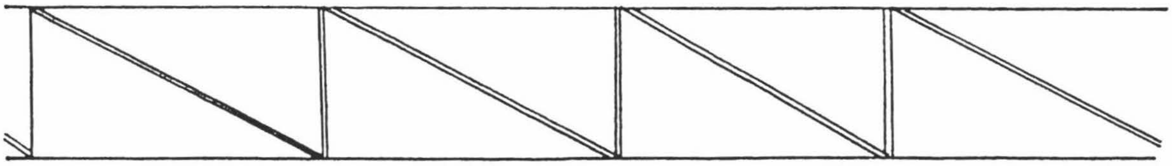


Figure 4.26.2

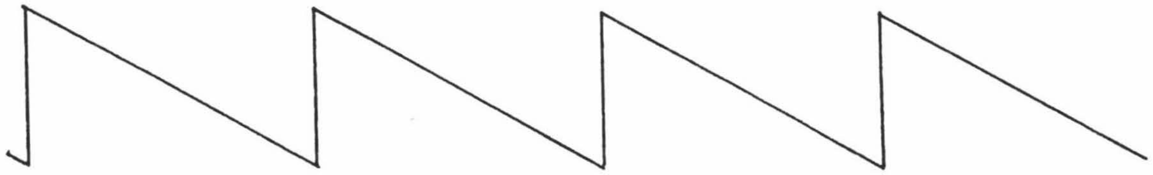


Figure 4.26.3

The number of these ambiguous cases was too great to be able to draw any conclusions about the use of each *style*. As a consequence, it was necessary to watch, and indeed question, the creators of these patterns to discern the actual construction methods and intentions behind many patterns. Again, this supports the belief that it is often *not sufficient* to use mathematical structures to examine the final written results of a child's work to be able to describe the internal processes used to obtain such results. This conclusion is one which is a theme of much mathematics education literature in general. For example, Moyer and Johnson (1978) wrote:

> " ... young children and older students alike have their own way of structuring mathematical concepts, which does not necessarily conform to the way the finished mathematical product is structured." (p 285)

Similarly, Hans Freudenthal (1983) introduced his book by noting that:

> "No mathematical idea has ever been published in the way it was discovered. ... Rather than behaving antididactically, one should recognise that the young learner is entitled to recapitulate in a fashion the learning process of [human]kind." (p ix)

4.3.6. Vertical Patterns

Table 4.3

Frieze Group	p111	p1m1	pm11	p1a1	p112	pmm2	pma2
No. of Students Producing	2	1	2	0	1	3	1

There was no difference suggested between the distribution of the frieze groups in activity (a) and the corresponding distribution of *vertical* patterns drawn by the 10 interviewees. However, the D_1 figures used to produce the two pm11 patterns were the same ones used by the interviewees to produce the rare horizontal p1m1, aligned differently with respect to the strip but the same with respect to the drawer. This suggests that the pm11, when drawn vertically, may not be as intuitive as it may seem here. Conversely, the frieze group p1m1 may be more intuitive when drawn vertically.

Since the symmetry of the figures used in the horizontal or vertical patterns often determined the overall symmetry group, the above frieze group conjecture would be consistent with the opinion of Howard (1982):

> "It is difficult to find an object in an ordinary room that does not have at least one plane of symmetry and, apart from pieces of modern sculpture, I have never seen a symmetrical mono-oriented object which does not have one of its planes of symmetry vertical when it is in its upright position." (p 522)

4.4 Interviews - Case Studies

The analysis in subsections 4.1.1 and 4.1.2 is limited in what it can reveal about the intuitive use of transformation geometry. Without doubt, the inquiry into the way the Primary students were thinking about their patterns has proved to be the most important aspect of this investigation. The children's own explanations of the processes they used to make their designs presented a more lucid, albeit less well-defined, description than the mathematical classifications of their patterns.

The following modes of pattern construction, therefore, not only represent a summary of the information discerned from observations of the designs but, more importantly, are *based* on the interviews conducted with ten Primary children. Where applicable, mathematical terminology used is from section 2.4. and subsection 2.3.1. While these methods have been related in a seemingly logical way below, several patterns discussed in the interview seem to have combined some of these methods in a somewhat ad hoc manner. The families of pattern processes listed, therefore, represent the best attempt by the researcher so *far* to describe the construction paths taken by the Primary school pupils. Indeed, in all the attempts this thesis has made to classify pattern construction, several examples of designs have existed that fell into more than one category. It seems inevitable that some ambiguity must be present when discussing types of pattern construction, since the motivation of the drawers is often multi-faceted or unclear.

Not all the methods listed here were used by each of the subjects. The list is designed to give an overview of most of the construction methods employed by the Primary school children, and what intuitive transformation geometry is implied by each construction method. Each construction method has at least one interview example to illustrate it. Naturally, the examples cannot cover the entire range of resulting patterns that emerge from each of the processes. It suffices to state that most methods generated more than one kind, style and type of pattern, which indicates that the quantitative analysis above is suggestive, but not conclusive.

4.4.1. Pattern 'Parts' Construction

A 'parts' construction is one in which a pattern is constructed (in a left to right fashion), and some transformation relationship between adjacent elements is engaged. The first kind, with simple 'translation incidence', involves the construction of a base pattern which is then repeated along the strip. This is labelled a *base pattern construction.* It appears to have been the most common of all the construction methods. While each base pattern may have any number of elements in it, the Primary children usually included one or two *(alternation type 1)*.

The second type of 'parts' construction, termed an *incidence construction*, involves other transformations, or more than one type of translation; the relationship between adjacent motifs often shows reflection or rotation incidence. This was not as common and often the presence of undifferentiated transformations seemed likely in patterns with 'inversion' (See subsection 2.3.1).

The third type of 'parts' construction has little implication to transformation geometry between figures. It is called *alternation type 2* and merely alternates between two non-congruent elements in a strictly left-to-right fashion. Examples of these three 'parts' construction methods are illustrated with interview extracts.

Base Patterns

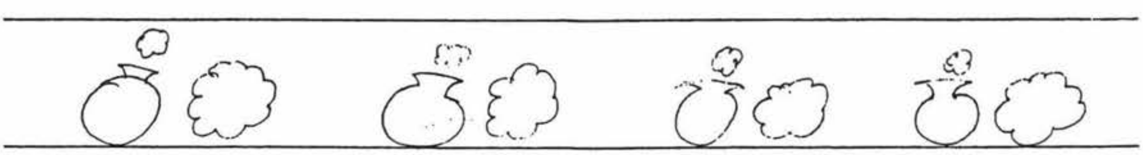


Figure 4.27 (Amy)

I: The space between that one and that one ... is that meant to be the same as the space between that one and that one, ... or are you grouping these two together?

A: Yeah. They're meant to be all one pattern (points to the 'vase' and the 'cloud' beside it).

... I: When you drew that, what order did you do it in? What did you do first, second and so on?

A: Well, I drew those ['vases']... whatever they are ... first ... then I drew the little clouds ... then I drew the big ones.

I: What did you do next?

A: Um ... went on to the next one [points at second base pattern].

Any intuitive transformation geometry was thus explicitly present in the translation of the base pattern, and implicitly used in the base pattern figures (fig. 4.27). The 'vase' had vertical reflection symmetry, but it is difficult to say if any implicit symmetry was used in the drawing of the clouds due to their topological character.

The fact that the following pattern (fig. 4.28) was made using the base pattern process demonstrates that the mere observation of a pattern doesn't always reveal its construction method. The need for interviews to discern the pattern construction method was thus apparent.

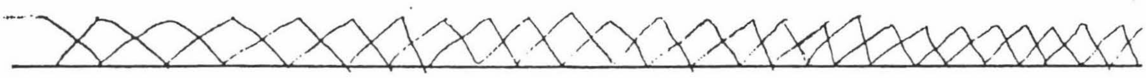


Figure 4.28 (Toni)

I: Tell me ... how did you draw that ...what did you start with, what did you do next and so on?

T: I just did ... cross ... and then went down from the top part and up from the bottom part ... and down from the top and up from the bottom.

I: So you did a whole lot of crosses?

T: Yeah ... just joining.

I: So you didn't do one 'zigzag' and then go back and do another 'zigzag'?

T: No.

This pattern would have been classified as connected in the survey. But from Toni's explanation, it hasn't been made with a sense of the whole, but of repeating parts, where each part has a relationship to the former. Toni also thought of her pattern as crosses joined together. The latter description represented a base pattern construction. The earlier discussion also suggested an incidence method.

The intuitive transformation geometry was therefore in the explicit translation of the base pattern (cross), and the implicit symmetry of the base pattern figure (cross). An ambiguity was raised here: if each line segment was drawn in relation to the previous adjacent one, some reflection or rotation incidence may have been explicitly used.

Incidence Construction

This construction method was not nearly as common, and was most often found in pma2 (PS11) designs with "up down up down ..." explanations. The presence of undifferentiated transformations and positional or directional judgements are both feasible possibilities in this case.

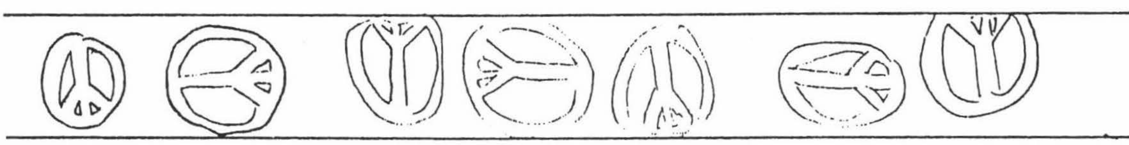


Figure 4.29 (Mark)

I: This second one [pattern 2]. Tell me what you did first.

M: I did that one first ... and then I didn't know what else to do so I did another one pointing a different way ... and then I thought turn it up ... so that's going that way ... and ... like they were rolling.

I: It's rolling ... okay, now which way is that rolling?

M: (pause) ... backwards ... cause if that was there and you turned it there it'd be going like that [gestures an anticlockwise rotation with his hand as he moves it from left to right].

I: Okay, so it was rolling that way (mimics Mark's gesture), which was the opposite way to which a clock goes?

M: Anti-clockwise.

I: And then what happened when you went over there? (points to right of second figure).

M: Um.

I: After you drew the second one, what did you do then? Did you go over here? (points to fifth figure in pattern).

M: Oh ... I did that there (points to the third figure).

I: The third one?

M: Yeah.

I: And did it roll again ... like last time? Were you thinking about it rolling?

M: Oh yeah ...'cause it sort of goes like that ... like um ... points of a clock ... there ... there ... there ... like it's going backwards on the points of a clock.

This interview extract suggests that an explicit rotation incidence method (quarter-turn) has been employed with a D_1 figure. The symmetry of the D_1 figure was probably implicit, particularly since it was a well known motif. The overall symmetry classification of the pattern (fig. 4.29) is p112, clearly not a good indicator of the complexity use of transformation geometry in this example. (See fig. 2.24.2 for a similar construction).

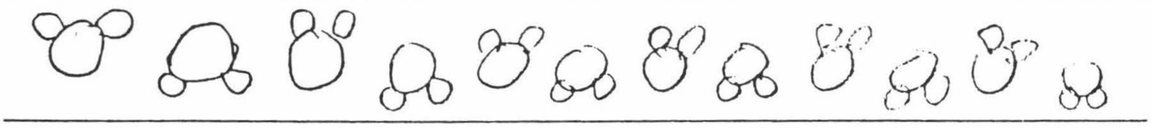


Figure 4.30 (Rachel)

I: This one here. Tell me how you drew this pattern. What did you draw first, what did you draw second, and so on ?

R: I did that one and then I did that one.

I: So you did the one at the far left ...?

R: Yeah.

I: And then you did this one [figure adjacent to far left]?

R: Yeah.

I: So what were you thinking of when you drew the pattern?

R: Bears.

I: Okay. What did you think of when you went from there to there ... from the first to the second one?

R: ... (long pause) ... not sure.

I: Alright. So you've said that you went that one, that one, that one, and so on.

R: Yeah.

I: You didn't draw all of those ones first and then go back and do those ones, did you?

R: No.

I: So ...um ... if you were going to describe this pattern, what would you say?

R: Just bears ... going up down up down ... I dunno.

Initially, Rachel mentioned only two elements; a base pattern construction seemed likely. However, her expression "up down up down" strongly supported the possibility of half-turn or possibly horizontal reflection incidence. But while this is explicit, the explanation may have been a directional judgement, or the transformation may have been undifferentiated. The latter seemed likely, since Rachel didn't distinguish the horizontal reflection and the half-turn of a C_1 figure further on in the interview. Therefore, this example was probably a classic example of inversion. Unfortunately, it was difficult to draw any further information from Rachel to confirm this; furthermore, her description didn't necessarily coincide with her construction method.

Base pattern and incidence constructions certainly display some similarities in the use of transformation geometry. A somewhat different 'pattern parts' method is alternation type 2, in which the use of transformation geometry does not appear to be particularly rich.

Alternation Type 2

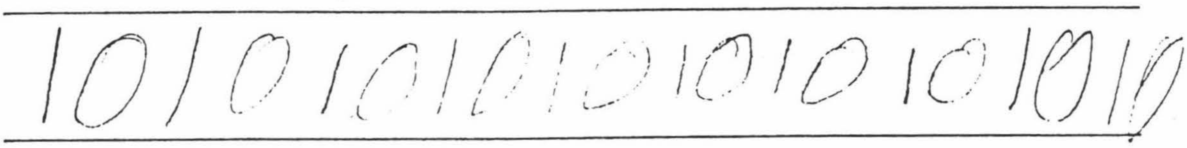


Figure 4.31 (Alan)

I: What order did you do this pattern in? What did you do first ... and what did you do second?

A: I did the line first and then the 'O' ... back and forth like that.

Clearly, any intuitive transformation geometry was only present in terms of the implicit symmetries of the figures, and the implicit translation of the translation unit. This contrasts with the alternation type 1 where the use of translation is more explicit.

4.4.2. 'Whole' Pattern Construction

This method of frieze design was characterised by a sense of the whole pattern. It occurred in two main forms: tilings and filamentary patterns. In the first case, a 'whole pattern' was divided into congruent parts, that is, a monohedral tiling of the strip. The second instance was the drawing of a curve extending the length of the given strip.

Division of the Strip into a Tiling

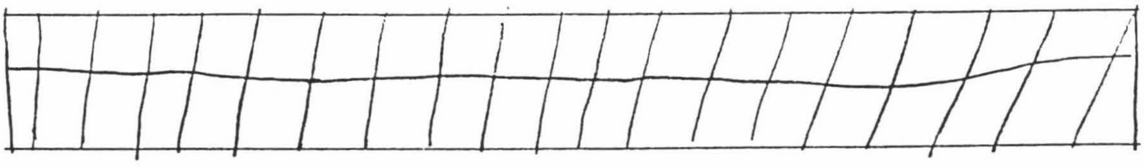


Figure 4.32 (Alice)

I: Tell me how you made this pattern.

A: Well ... I just made a line down ... along there ... and then I went back and did the downwards lines to make ... um ... squares.

A sense of the whole pattern is clear from the first phase of the drawing. This is clear example of a tiling. The translation symmetry of the whole was probably explicit (division of the whole), while the other symmetries of the whole (e.g., vertical or horizontal reflection, half-turn) were more likely to have been implicit. The symmetries of the parts, namely the squares, were also implicit.

Filamentary Patterns

A *filament* is defined by Grünbaum and Shephard (1987) as the image of a strip's midline under a homeomorphism of the strip onto itself. But for the purposes of describing human behaviour, it seems more appropriate for the expression 'filamentary pattern' to mean that the creator has drawn a curve (not necessarily simple or smooth) extending the length of the given strip without lifting her or his pencil. Such a pattern may have other markings or superimposed patterns on it as well.

Patterns with smooth curves are clearly filamentary in construction method. For example:

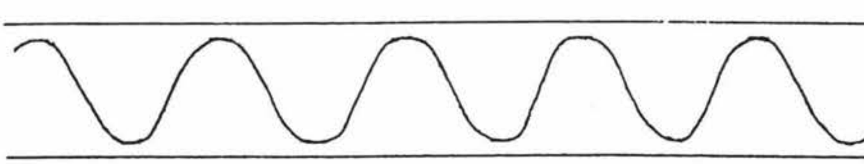


Figure 4.33

Since the whole pattern was an infinite figure (curve), that is, a complete entity in itself, the symmetries or transformations used were probably implicit, including translation. Certainly, no incidence method has been used to produce this. On the other hand, what appeared to be filamentary patterns at first glance, were later revealed in the interviews to be base pattern constructions. See Toni's pattern above in the 'Base Pattern' discussion.

There may be more Primary school students who don't imagine the patterns extending to infinity than indicated by the number who did not produce No Translational Symmetry. Many may have simply been filling in the empty strip not imagining it extending any further; the design that could actually be seen *was* the pattern. This possibility seems quite likely given the regular occurrence of 'finite' descriptions in activity (c); see chapter 6 for details.

4.4.3. Superposition Constructions

Superposition involves the construction of a pattern in two, or sometimes more, distinct phases. The first phase usually used either a form of incidence (base pattern) or occasionally, a filamentary pattern. This was done in a strictly left to right fashion. The second phase comprised the altering of the original pattern in some way, which often changed the frieze group classification or the symmetry of the figures in the pattern. This second phase was done in a variety of ways, from the light ornamentation of a pattern's figures to the placement of another row of figures alternating with the previously drawn figures in the original to the overlaying of another filamentary pattern. Superposition then, could involve either, or both, 'repeating parts' construction and 'whole' construction.

Ornamentation

It is not easy to define the process of ornamentation. For our purposes, it will be considered to be a marking within a pattern's figures or tiles, or motifs proximate to a filament. The distinction between this method and the overlapping of rows is psychological, not mathematical: essentially, it is the difference between embellishing a pattern and superimposing another pattern over the top of a first.

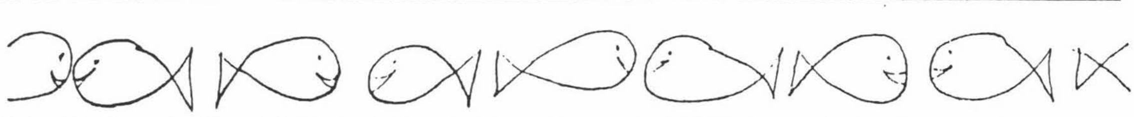


Figure 4.34 (Mary)

In the first phase of this disjoint example, this pattern constructed was a pmm2. Mary indicated that she drew the fish from left to right. She "made them go the opposite way each time". In this case, a vertical incidence construction has generated the 'phase 1 pattern'. Of course, what appears to have been an indication of vertical reflection transformation may simply have been a directional judgement or an undifferentiated transformation. This wasn't established; but in any case, it was explicit.

No questioning was undertaken regarding the horizontal reflection symmetry of the fish. Nevertheless, the fact that it was a mono-oriented object strongly suggested that this symmetry was implicit. Furthermore, Mary didn't consciously notice any asymmetry effect from ornamenting the fish (in phase 2) with facial features other than 'filling in' each one:

I: Did it change the look of the pattern after you did that?

M: Yes ... it didn't look empty.

I: Anything else?

M: Nup.

After ornamentation, the pattern became a pm11. This final frieze classification conveyed Mary's explicit use of transformations quite well. Two generators of the pm11 pattern, v_1 and v_2, seemed most appropriate to describe the process of constructing the pattern. In this case, the symmetry group element not in the generating set was translation and it was therefore at least implicit. For contrast, let us now consider the following 'connected example.

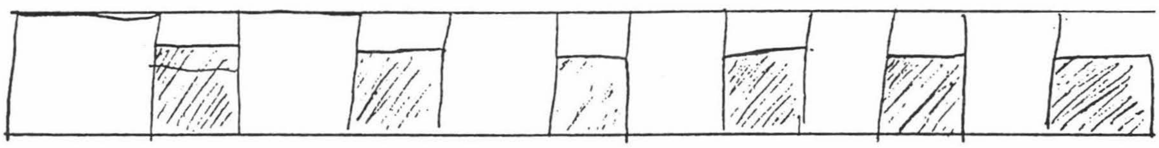


Figure 4.35 (Alice)

I: ... how did you make this pattern here... number 3?

A: Um ... I just did a line ... and then ... and then ... that was supposed to be a big box ... and then that a smaller one ... and just going like that. (with two fingers pointing to each of the squares on the far left of the pattern, Alice moves her hand in discrete movements from left to right).

I: Yeah.

A: I just did a line there and then a line there and left a space and then I just did another line along there.

I: Okay, ... then did you do the shading at the end, after you had drawn all the squares, or did you do it as you went (points left to right along pattern)?

A: I did it at the end.

The intention here was initially the repetition of , spacing the base pattern so the whole pattern would be connected. This base pattern construction was clearly not a tiling, despite appearances. The shading was an example of ornamentation, since it was done after the whole pattern was drawn and within a figure. Alice probably didn't have a sense of the symmetries of the whole of this pattern, but rather implicitly used some symmetry in the finite figures, that is, the squares. Which of these were intentional, and which were accidental was not explored; in general it was difficult to explore implicit symmetry.

Superimposing Rows

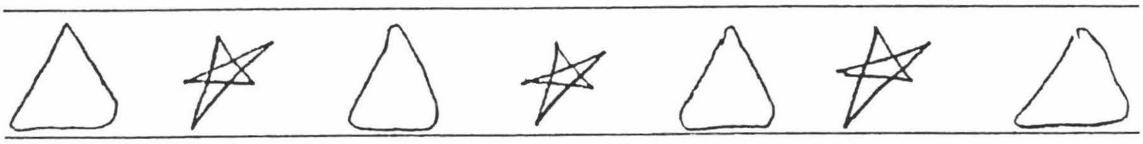


Figure 4.36 (Richard)

I: What order did you do this pattern in?

R: I can't remember.

I: For instance ... did you go 'triangle star triangle star triangle star', or did you put all the triangles in first and then go back and do the stars?

R: Oh ... I did all the triangles in first. And it looked funny so I put the stars in ...I was going to put triangles up the top but I thought "na!" and put stars in the middle instead.

Richard has shown a clear example of the superimposition of two rows of figures, namely triangles and stars. There seems to have been a slightly stronger sense of the whole in this pattern; Richard looked at his design at the end of phase 1 and evaluated "it". This perception involved a sense of translation symmetry (the 'whole') combined with the implicit symmetry of the mono-oriented figures that Richard used.

Overlaying Filamentary Patterns

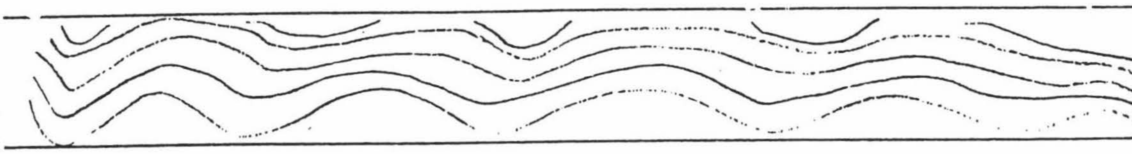


Figure 4.37 (Kate)

I: Tell me about this pattern.

K: Um ... I don't if that was meant to be a pattern or whether that was a mistake ... I think it was a mistake.

I: Oh ... well ... whereabouts in the pattern did you start?

K: I started with this bottom one ...

I: With the bottom one ...

K: I did that middle one ... and I went up and up ...and then I decided that I couldn't go over [the strip border] ... so then I just did those small pieces there.

This example reinforces the view that the intuitive symmetry in continuous curves was probably implicit, and almost certainly not explicit. Indeed, Kate seemed puzzled about how it could be a pattern, which suggested that the translation or repeating facet of the pattern was not explicitly used in the sketching of each curve. However, there seems to have been an explicit use of a vertical translation of the 'sine curve' at each phase succeeding the first.

Besides the methods given above, more elaborate superposition also ensued. This appeared in some of the patterns drawn in more than two phases. For example,

Figure 4.38 (Alice)

This ornamentation occurred in three phases. The first consisted of drawing a horizontal line midway between the strip borders. This could be classified as a pmm2 filamentary pattern. The second stage involved drawing a continuous curve beneath it which rendered the new pattern as a pm11. Any use of symmetry here was thus implicit. The final part of the design was the ornamentation with 'stripes'; the frieze group became p111. (Not surprisingly, Alice was one of the few Primary children who made nearly all of her 16 patterns as connected. These included all three styles of connected patterns, an overlapping pattern, and six of the seven symmetry groups, including two examples of p1m1 and excluding only p1a1).

One pattern, by Mary, appears to display an integration of both a 'parts' and a 'whole' conception.

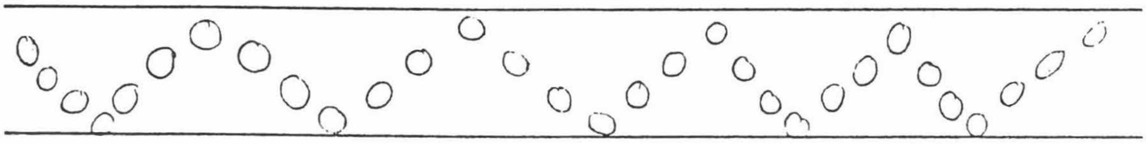


Figure 4.39 (Mary)

4.5 Summary of Activity (a)

In both the Primary and Secondary age groups, the most commonly drawn frieze groups were p111, pm11 and pmm2. Both p1m1 and p1a1 were very rare. The frequency distribution of the five frieze groups with translation and (at most) one other kind of symmetry allowed a rough ordering to be made of 'how intuitive' each of the four rigid transformations of the plane was in the construction of these patterns. This list is shown below.

1. Translation (common and frequent)
2. Vertical Reflection.
3. Half-Turn
4. Horizontal Reflection.
5. Glide Reflection (uncommon and infrequent)

However, the implications of these results to frieze patterns with more than one kind of symmetry (besides translation) was argued to be dubious.

The classification of the patterns was further refined and it was found that all the symmetry groups, apart from pma2, were most commonly disjoint for the Primary group. Indeed 75% of all the Primary students' patterns were disjoint. The Form Four group differed in that the proportion of disjoint to connected patterns was almost equal.

A qualitative discussion followed, highlighting the common classification ambiguities, and the intuitive transformations or symmetries present in those designs without translation symmetry. As predicted by Moyer and Johnson (1978), and Lesh (1976), the frieze group classification had the benefit of being methodical and quantifiable but it had shortcomings. Three of these are listed below:

Firstly, there was no reason to suppose that any symmetries of the whole pattern were even implicitly intended by the drawer, apart from some continuous examples. In fact, there is evidence to suggest many Primary children conceive their patterns to be finite.

Secondly, several accidental symmetries of the whole seemed to result from the positioning and spacing of figures. Furthermore, it seems quite likely that some symmetries could easily have been accidental spin-offs from the composition of other intended ones. This motivated the design of activity (b).

Thirdly, some patterns contained more complex symmetries than indicated by the frieze group classification.

Therefore, while the quantitative results gave *clues* to the construction methods used by the students, a more qualitative exploration involving interviews was undertaken to find out the motivations of the Primary school subjects. Most importantly, the determination of the order in which a pattern was constructed gave the strongest indications of the character of the intuitive use of transformation geometry. Several different construction methods were identified, each having its own particular use of symmetry operations and understandings of those operations.

For example, a base pattern construction implied an explicit use of translation and usually an implicit use of other symmetries, most often the vertical reflection symmetry of mono-oriented figures. This appeared, from observations, to have been the most common design mode. Alternation type 1 was a special case where the base pattern comprised of two disjoint, non-congruent elements. Other construction methods are outlined below.

A 'non-trivial' incidence construction implied the use of an explicit transformation to 'generate' the pattern. This occurred in several different ways, the most common being the 'turn upside down' undifferentiated transformation where a half-turn or a horizontal reflection, combined with translation, weren't distinguishable to the creator. Directional or orientational judgements may have been integral to the meaning of this construction.

Alternation type two implied a strict left to right alternation, each figure being drawn in reference to its immediate predecessor. Therefore, the intuitive transformation used was the implicit translation of two figures. The translation symmetry of the whole was not established by interview.

A division of the whole pattern into equal parts implied a sense of the whole pattern The intuitive use of translation was thus explicit and, quite possibly, of the whole. Internal, implicit symmetries of the tiles were usually present. The relationship of the tiles to one another wasn't explored.

Filamentary patterns provided the best examples of a sense of the 'whole'. The intuitive transformation geometry used was clearly implicit however.

The superposition of patterns occurred in three main forms: ornamentation, superimposing of two rows (alternation type 3) and the overlaying of filamentary patterns. The feature of these designs is that they were drawn in two or more phases. A sense of the whole is suggested to some extent by this. However, the character of the transformation geometry used was chiefly determined by the method used at each phase of the pattern's construction.

In general, it was not possible to determine the construction method used to make a pattern from the end result, since more than one method existed for most patterns, particularly disjoint ones. From the arguments in the literature review (subsection 2.2.1), it was not surprising to find that the children's descriptions of their methods, rather than the frieze group classification, provided the most accurate reflection of the actual intuitive use of transformation geometry in the unrestricted construction of frieze patterns.

5 *Results of the Restricted Pattern Construction Activity*

In this chapter we examine some of the trends in the *restricted* construction of frieze patterns involving the same set of students from the previous chapter (as well as additional Secondary classes and a Tertiary group), and attempt to link these trends to intuitive transformation geometry. The students were required to fill in as many of the empty strips as they could using only right-angled, scalene triangles (in any way they wished) to make different 'repeating patterns'.

As before, a quantitative analysis of the patterns drawn in activity (b) is made and is supplemented by a descriptive analysis of the patterns (although this is not as detailed as in the previous chapter). In addition, some case study material is included. Tables of the results can be found in the appendix B; the related column charts are in sections 5.1 and 5.2. A summary of the activity (a) and activity (b) pattern construction methods and the intuitive transformation geometry associated with them is given in subsection 5.4.5.

5.1. Survey Results from the Primary Schools

5.1.1. Frieze Group Analysis

The frieze group analysis of the restricted pattern construction is of the same form as for the unrestricted pattern construction. Graphs 12, 13, 14, and 15 respectively indicate the distribution of the designs amongst the frieze groups, the number of Primary students that made each group, the average number of each pattern produced*, and the frequency distribution of the first three patterns produced. Again, strong similarities between the profiles of these four column charts indicate a fairly clear ordering of the 7 frieze groups:

1. p111	$\{t\}$	(common and frequent)
2. pm11	$\{t, v\}$	
3. p112	$\{t, {}^1/_2\}$	
4. pma2	$\{t, v, {}^1/_2, g\}$	
5. pmm2	$\{t, v, {}^1/_2, g_t, h\}$	
6. p1a1 $\geq$ p1m1	$\{t, g\} \geq \{t, g_t, h\}$	(uncommon and infrequent)

The generators for each frieze group have been listed beside them for consideration in the transformation and symmetry analysis portion of this subsection.

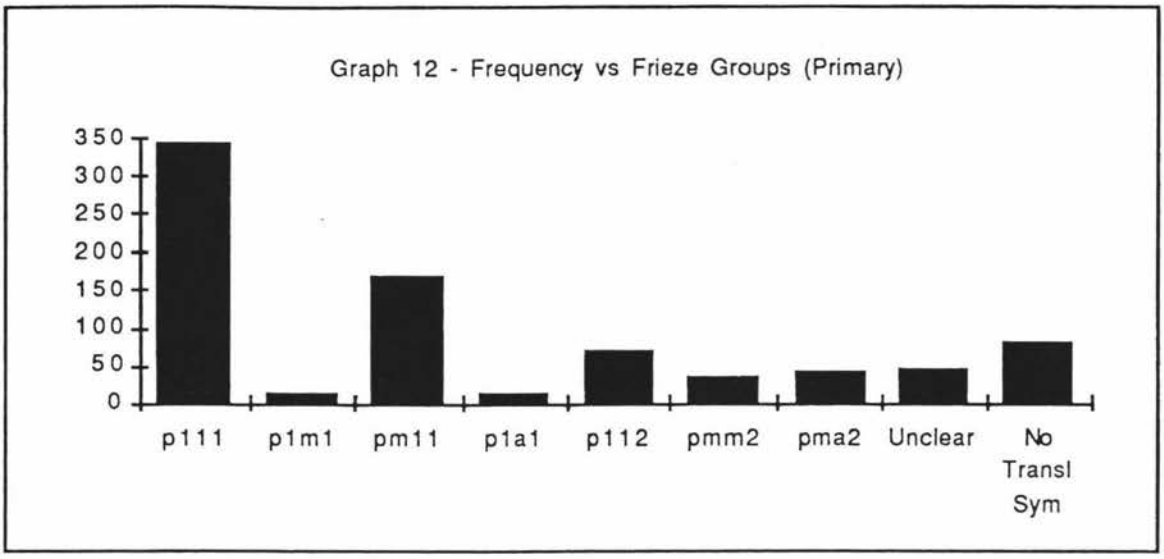
The relative 'intuitive-ness' of the frieze patterns suggested by the results of the restricted activity seem to be more clear than the corresponding inferences made from the unrestricted activity. Note also, that pma2 has been itemised above pmm2 since it is higher on three of the four measures (with the average-number* graph presenting a slight disparity). In addition, p1a1 and p1m1 have been placed on the same level since they are equal on three of the four measures, with p1a1 marginally higher in graph 15.

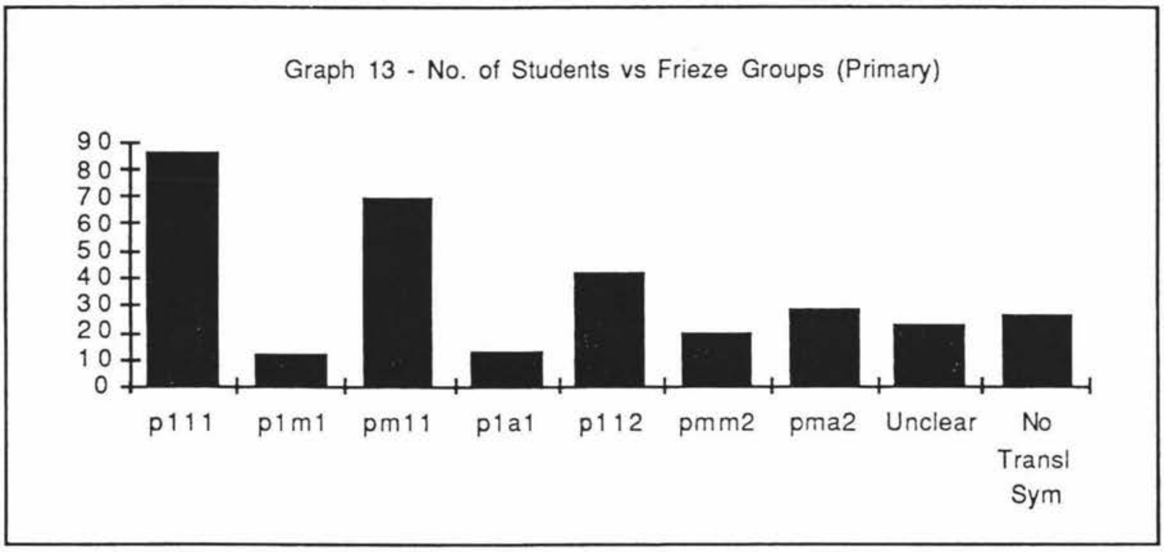
In all four graphs, p111 stands out very clearly above the other categories. For instance, 87% of the children made at least one p111 pattern and it comprises 42% of all patterns produced. Half that number of patterns were made as the next most popular frieze group, pm11, and 7 out of every 10 students made a pattern with this underlying symmetry group. For each of the other categories, less than half the students made a design of those types. The percentage of patterns that were p1a1 or p1m1 was particularly low (1.9% and 1.7% respectively).

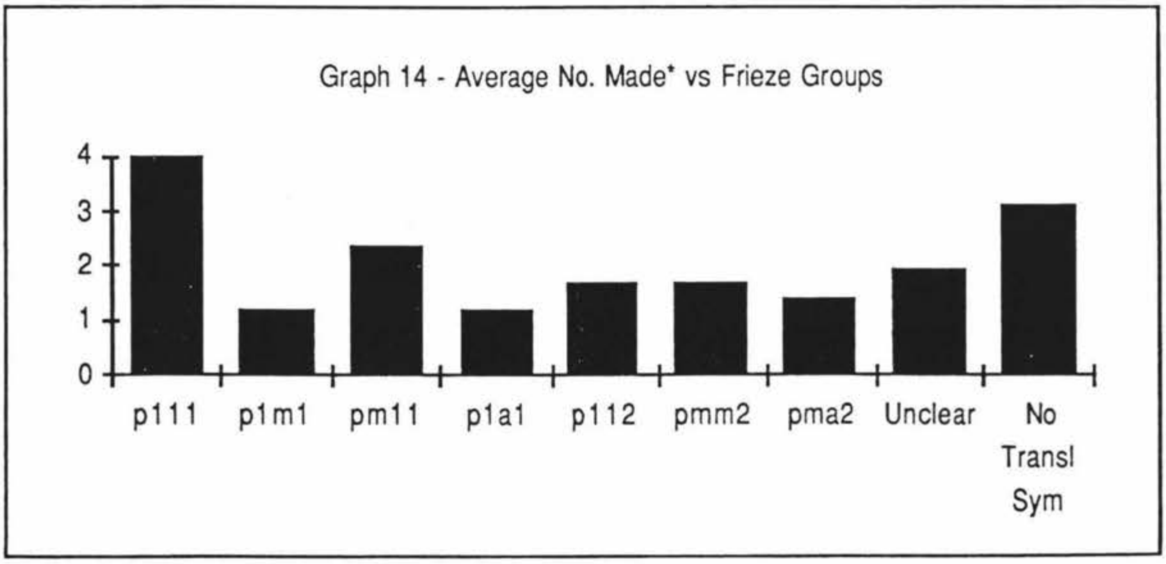
There are several important differences between the frieze group results for activity (a) and activity (b) that should be noted. Firstly, the proportion of patterns that were p111's in activity (b) was double that of activity (a). Secondly, the frequency and commonality of the pm11 was clearly less than p111, but greater than all the other frieze groups. Similarly, the position of pmm2 dropped considerably in the 'intuition order', and the p112 became a clear third, above the previously more commonly drawn frieze groups, pma2 and pmm2.

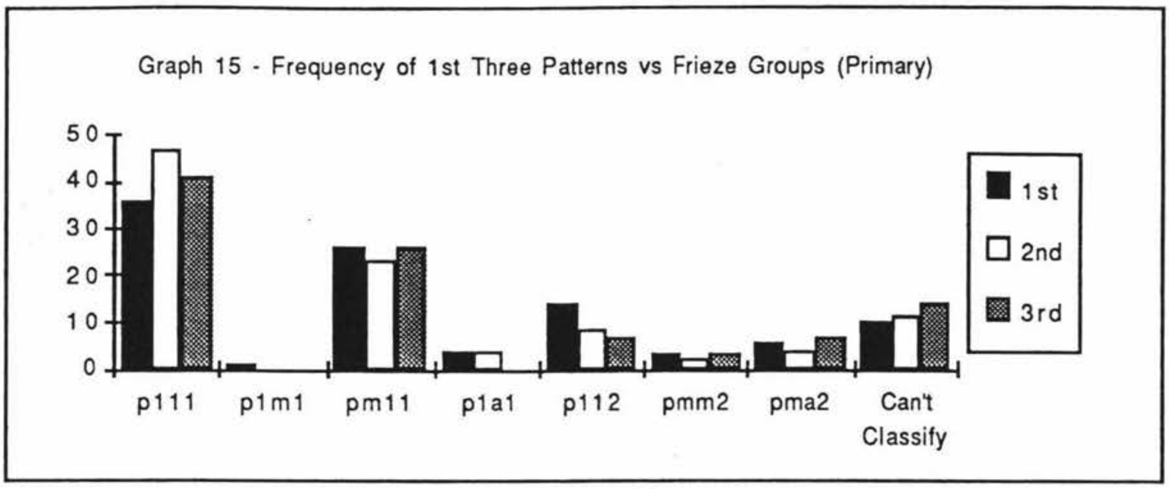
The activity (b) results suggest that there may have been more p112's resulting from the unrestricted pattern construction activity than indicated by the graphs in chapter 4, because many ambiguities between p111(touchings) and p112 (tilings) existed. With the decrease of this ambiguity in activity (b), the increase in the p112 proportion of the 'restricted' total was not unexpected.

Finally, the number of 'No Translation' cases comprised a greater proportion of the patterns than the corresponding designs in activity (a). No explanation is given for this other than the fact that most of the Primary students reported that they found activity (b) harder than activity (a); the difficulty of drawing right-angle scalene triangles may have distracted them from the task of displaying repetition in their designs.


Graph 12 - Frequency vs Frieze Groups (Primary)
350
300
250
200
150
100
50
0
p111
p1m1
pm11
p1a1
p112
pmm2
pma2
Unclear
No Transl Sym


Graph 13 - No. of Students vs Frieze Groups (Primary)
90
80
70
60
50
40
30
20
10
0
p111
p1m1
pm11
p1a1
p112
pmm2
pma2
Unclear
No Transl Sym


Graph 14 - Average No. Made* vs Frieze Groups
4
3
2
1
0
p111
p1m1
pm11
p1a1
p112
pmm2
pma2
Unclear
No Transl Sym



5.1.2. Transformation and Symmetry Analysis

From the ordering given above, we are now in a position to make some inferences about the presence of transformations (intentional or not) in these patterns. Clearly, translation was common to almost all the designs. This was not surprising, since the instructions explicitly asked for translation via the phrase "repeating patterns". Indeed, translation is a mathematical requisite of the 7 frieze groups. The evidence of translation's intuitive use is therefore suggested by the relative ease in which the children understood the concept. Expressed quantitatively, the percentage of designs which didn't show enough evidence of translation symmetry to be classified as a frieze group (11.4%) was not great.

For the non-translation transformations, we can make stronger inferences from the symmetry elements of each frieze group. In particular, upon considering the ordering of the groups with translation and one other kind of symmetry in them, we obtain the following list (using a similar argument to that of section 4.1.2):

1. Translation (common and frequent)
2. Vertical Reflection
3. Half-Turn
4. Glide Reflection $\geq$ Horizontal Reflection (uncommon and infrequent)

If this ordering of the rigid transformations' relative 'intuitive-ness' is correct, then it seems reasonable that the frieze groups pmm2 and pma2, which have *both* rare and common symmetries, are in the middle of the list. For instance, pmm2 may have been made from vertical reflection and horizontal reflection as well as translation. In a similar vein, pma2 may have been produced from vertical reflection and non-trivial glide reflection as well as translation. Furthermore, since the proportion of generating sets for pma2 containing only the commonly drawn symmetries is higher than pmm2, it also seems natural that pma2 has occurred higher up the list than pmm2.

But these conjectures assume independence in the combination of the symmetries used. It may be that the combination of vertical reflection and half-turn symmetries with translation is more difficult for the Primary children to manage than either symmetry is with translation by itself. This difficulty would *also* result in pmm2 and pma2 being below p111, pm11 and p112 on the list. This question of independence is pursued further in the activity (b) interviews and in chapter 6.

Another matter to be considered is the comparison of the activity (a) and (b) results. For instance, the symmetry group pmm2 occurred far less frequently than expected from the activity (a) results. This suggests that some of the symmetries present in pmm2 in activity (a) may have been incidental or even accidental. In particular, *if* the symmetries were used independently then (from the list above) it seems likely that the horizontal and trivial glide symmetries may have been accidental in some instances of drawing pmm2 in activity (a). Similar questions are raised with respect to accidental symmetries in other symmetry groups, especially pma2.

But an even deeper question evolves. The discussion above has proceeded as if the differences between activity (a) and (b)'s results are attributable solely to a reduction in accidental symmetries. But, as pointed out in chapter 3, the differences may also lie in the nature of the figure that was used to make up the activity (b) patterns with. After all, many of the figures drawn in activity (a) were representational; and the representations were frequently of mono-oriented, $D_1(v)$ symmetry groups. Could it be that some symmetries or transformations are more difficult, or easier, than others to perform when using a right-angled scalene triangle than with some other C_1 figure? Given the literature (Lesh, 1976) on the effect that an object's *form* can have on the use of various transformations, the answer may be "yes."

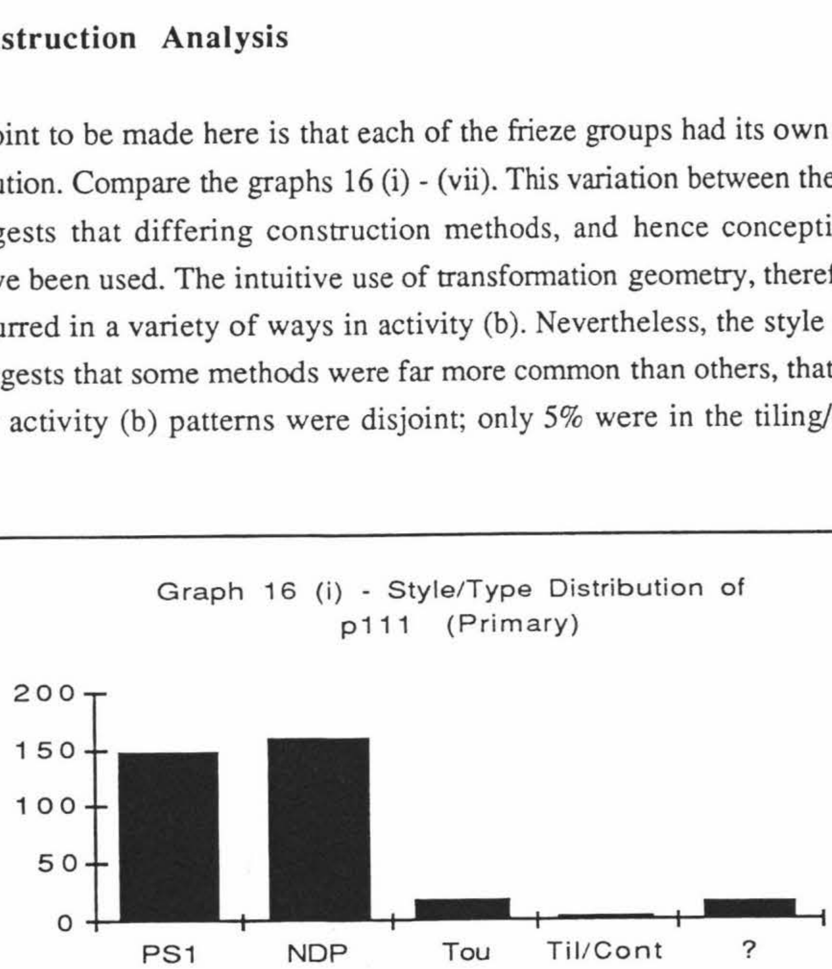
Furthermore, connected frieze groups were not as likely to be constructed with triangles. It may have been that connected patterns were not suggested by the instructions: to make repeating patterns with right-angled scalene triangles (i.e, a 'repeating motif' framework may have been set up). Given the strong relationship between the method employed to produce a pattern and intuitive transformation geometry concepts, it does not seem surprising that the restricted activity, by reducing the possibilities for making a connected frieze pattern, displays some differences to activity (a) in the frieze group distribution.

In this study then, one main reason for the differences between the two drawing activities was probably a reduction in accidental symmetries; but this is coupled with other considerations such as the differing construction methods associated with each activity, and the effect of the figure used to construct the activity (b) patterns. The implications of the restricted pattern construction results to the intuitive use of transformation geometry should be understood in that light.

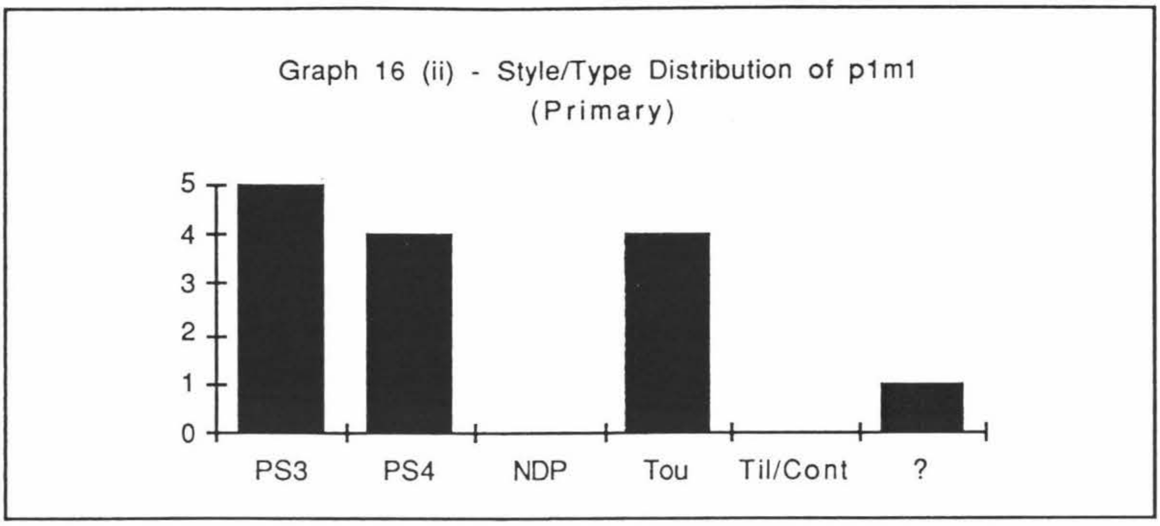
5.1.3. Construction Analysis

The main point to be made here is that each of the frieze groups had its own distinctive style distribution. Compare the graphs 16 (i) - (vii). This variation between the symmetry groups suggests that differing construction methods, and hence conceptions of the patterns, have been used. The intuitive use of transformation geometry, therefore, seems to have occurred in a variety of ways in activity (b). Nevertheless, the style analysis of the total suggests that some methods were far more common than others, that is, 78% of the Primary activity (b) patterns were disjoint; only 5% were in the tiling/continuous category.

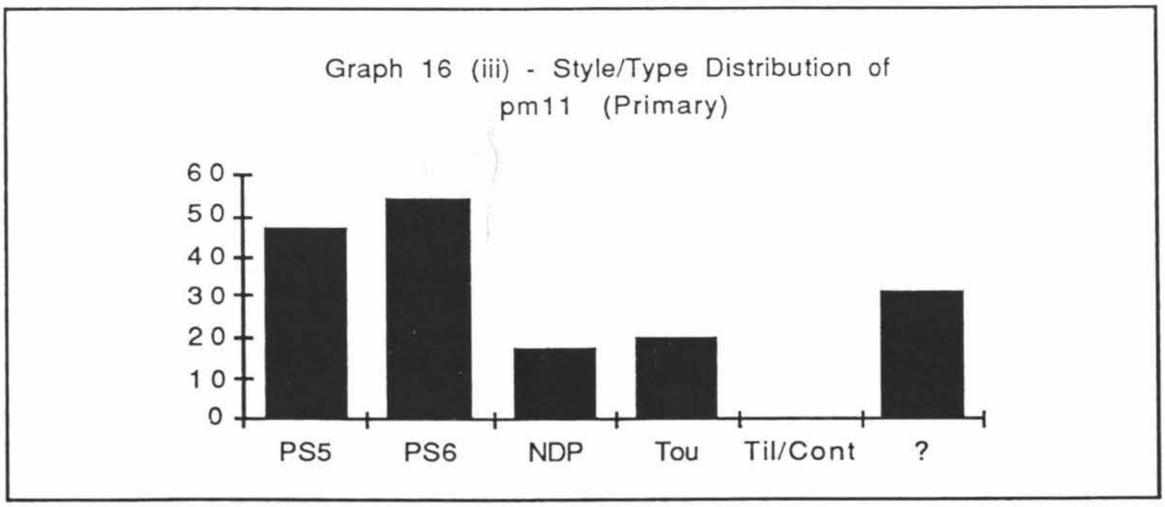
Graph 16 (i) - Style/Type Distribution of p111 (Primary)



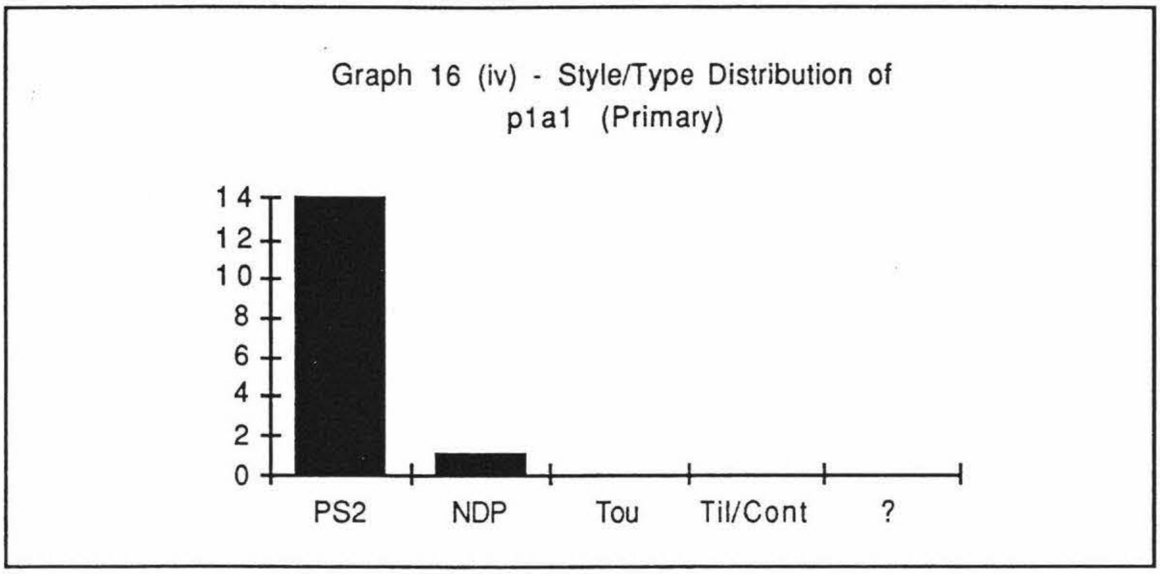
The pattern p111 was predominantly disjoint. Interestingly, there were as many p111 examples which are non-discrete, mono-motif patterns (NDP's) as there are discrete patterns. The most common reason that disjoint patterns were non-discrete was that the motif was not connected. If the disjoint motif was comprised of two connected components, then some form of alternation was probably present. If more than two components made up the motif then a base pattern construction has probably taken place. This information suggests that plenty of implicit symmetry of figures could be present in these patterns, and that translation has been used explicitly.



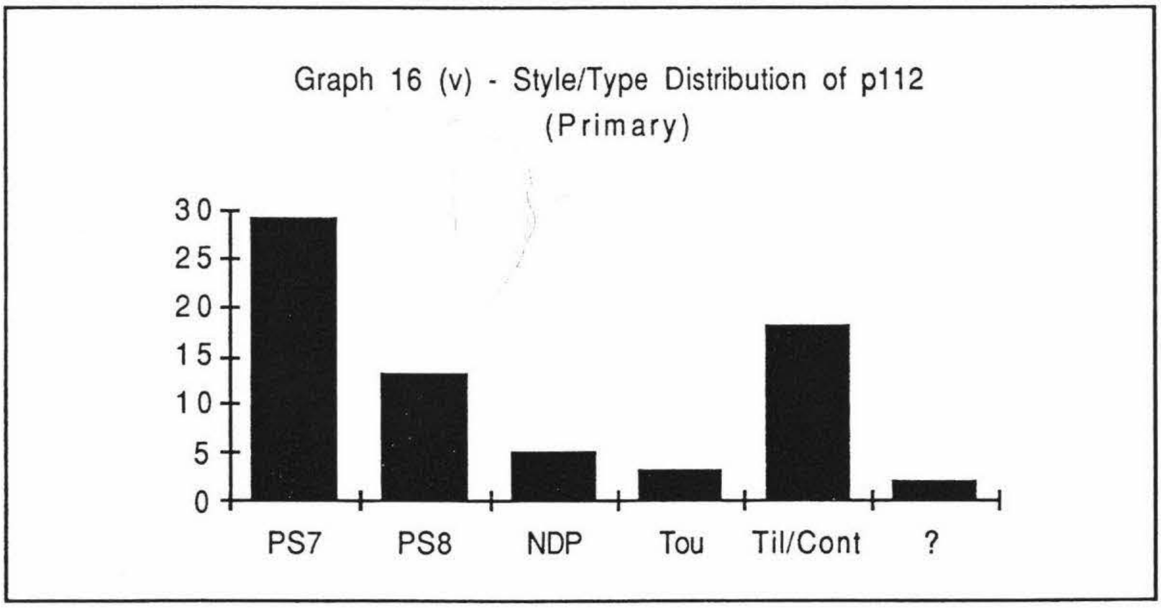
As expected, p1m1 occurred with very low frequency, so our conclusions are tentative in this case: it seems that it was unusual to draw non-discrete and tiling p1m1's. The number of PS3's and PS4's was roughly equal. In the first case, the horizontal symmetry was explicit (although not necessarily differentiated from half-turn, i.e., both are upside down). In the second case, because the participants were using C_1 figures, the horizontal symmetry may also be explicit but possibly less so since the motif must be a connected $D_1(h)$ figure. No examples of NDP's were produced, indicating that the horizontal reflection of a complicated or disjoint motif was not intuitive. Some touchings were also observed, and suggested a repeating $D_1(h)$ figure, as for PS4.



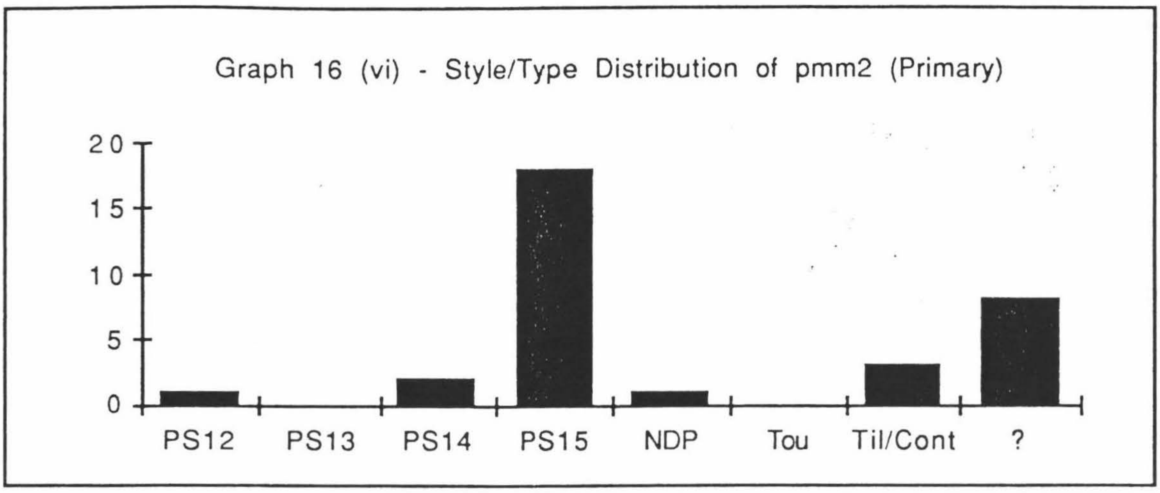
The graph of pm11's style breakdown indicates that the disjoint styles were also the most natural way to construct this frieze group with the right-angled scalene triangles, with the connected $D_1(v)$ figure patterns (PS6) occurring slightly more often than the PS5. This suggests that both the 0-order and high order incidence associated with each of these types may have been used. The number of pm11 patterns made using shapes that weren't right-angled scalene triangles was relatively higher than most other groups. These patterns were generally made up of vertically aligned isosceles triangles; a version of PS4. The vertical reflection used in this case was probably implicit.



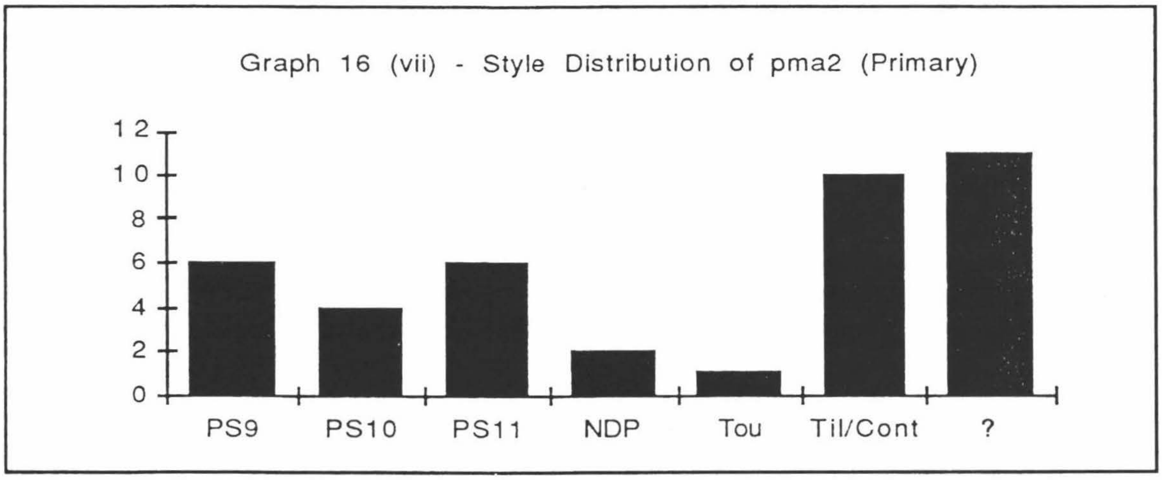
The graph of p1a1, although based on small numbers, suggests that the discrete PS2 is the predominant style. It does not appear to be 'natural' for the children interviewed to draw connected or even non-discrete, mono-motif patterns of this symmetry group.



The distribution of the p112 styles is different from the previous symmetry groups considered in that the tilings feature prominently in the graph's profile. This seems to be one of the two main ways that right-angled triangles are fitted together to fill out the strip.



The pmm2 profile is very clear. The most common way of producing such a pattern is by translating a $D_2(v)$ motif (a base pattern construction)



The graph of the pma2 frieze group is characterized by tilings and the three discrete types. It seems that pma2 is another natural result of fitting the right-angled triangles together to fill an empty strip. PS9 and PS11 occurred more often than PS10, suggesting that using figures with half-turn symmetry in them was not as 'natural' for the subjects as using figures with vertical reflection symmetry in them. All three discrete types were probably made by an incidence construction. Also, a high proportion of pma2's were made from inappropriate triangles. Similarly, most of the pm11 patterns consisted of vertically aligned isosceles triangles, suggesting an incidence construction.

Comparing the results and interpretations of the seven frieze groups for each activity reveals other more general conclusions. For example, while the activity (a) patterns often featured familiar figures, suggesting that the use of symmetry was usually implicit, many activity (b) patterns showed a more explicit use of transformation geometry. In the restricted exercise, the most likely examples of the implicit use of symmetry were the PS4, PS6, PS11 and PS15 patterns made with isosceles triangles.

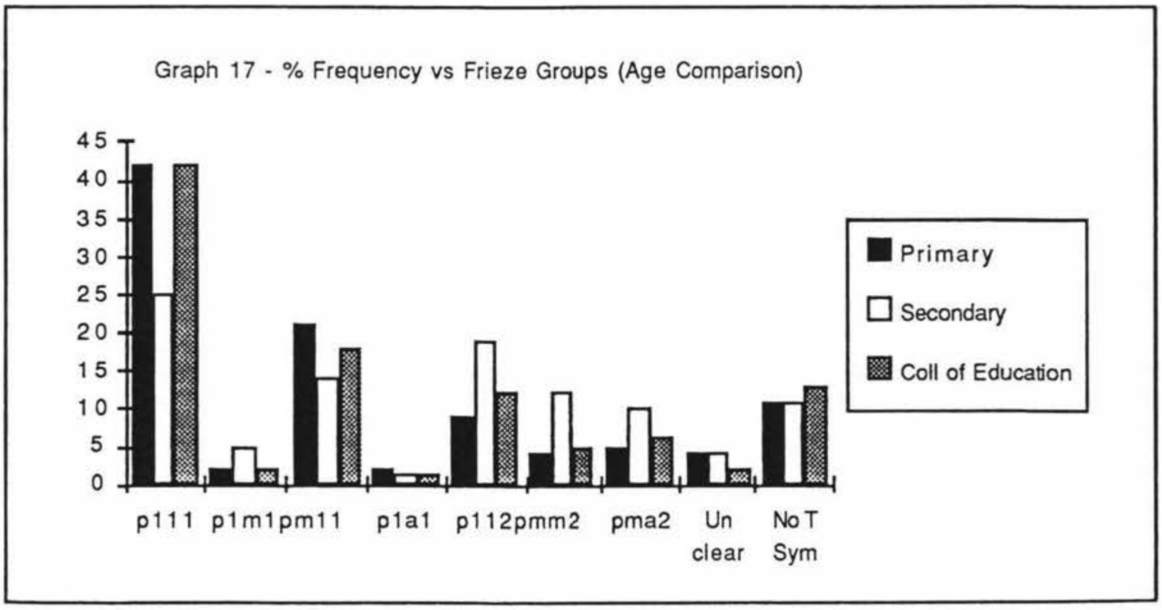
However, while it may be reasonable to suppose that base pattern or incidence constructions were used to produce the disjoint patterns, it is more difficult to assess the way in which transformation geometry has been used to produce the activity (b) tilings. This information is important to find out since p112 and pma2 were the most frequent results of a tiling by the Primary school children. Interviewing is needed to find out more detail about how the tilings were constructed.

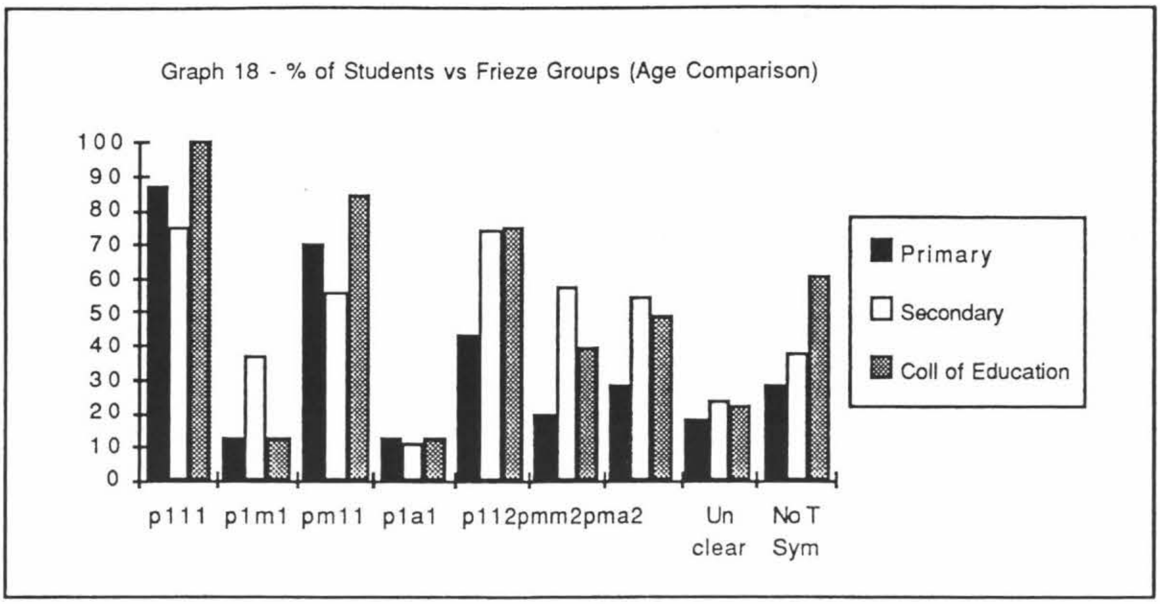
Finally, the use of C_2 figures (e.g. PS8 and PS10) was relatively low, and was much less frequent than the use of $D_1(v)$ figures, although more frequent than the use of $D_1(h)$ figures. This ordering of the finite symmetry groups shows a close parallel to the ordering of the frieze groups and the rigid transformations given above.

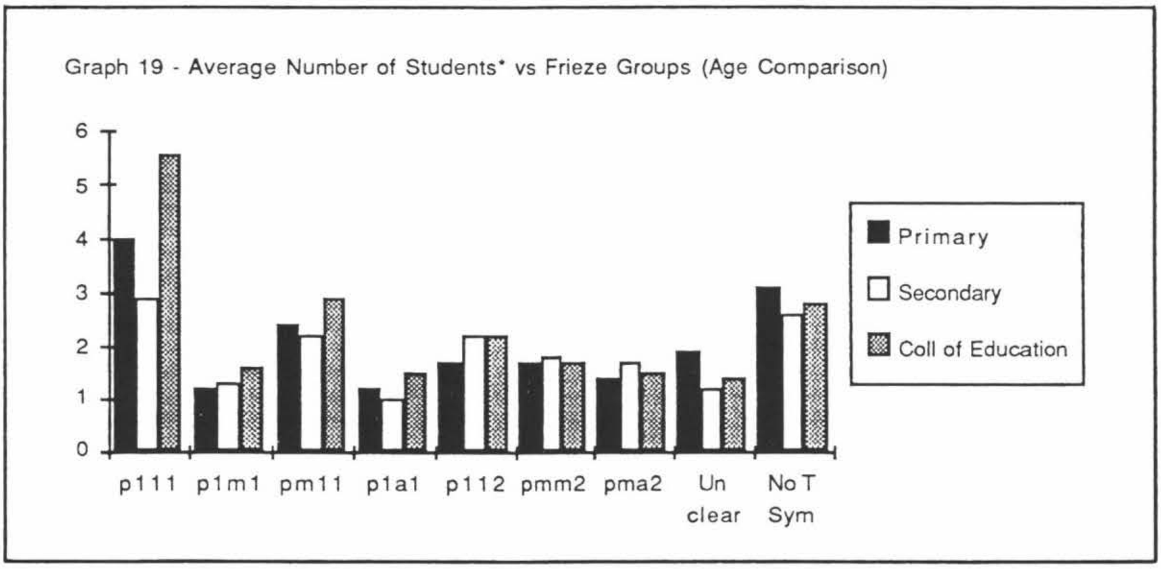
5.2 Age Group Comparison

5.2.1. Frieze Group Analysis

By considering graphs 17, 18, and 19, we can construct an ordering of how 'natural' the various frieze groups were for each age group (as argued in subsections 4.1.1, 4.1.2 and 4.2.1).




Graph 18 - % of Students vs Frieze Groups (Age Comparison)
100
90
80
70
60
50
40
30
20
10
0
p111 p1m1 pm11 p1a1 p112pmm2pma2 Un clear No T Sym
Primary
Secondary
Coll of Education


Graph 19 - Average Number of Students* vs Frieze Groups (Age Comparison)
6
5
4
3
2
1
0
p111 p1m1 pm11 p1a1 p112 pmm2 pma2 Un clear No T Sym
Primary
Secondary
Coll of Education

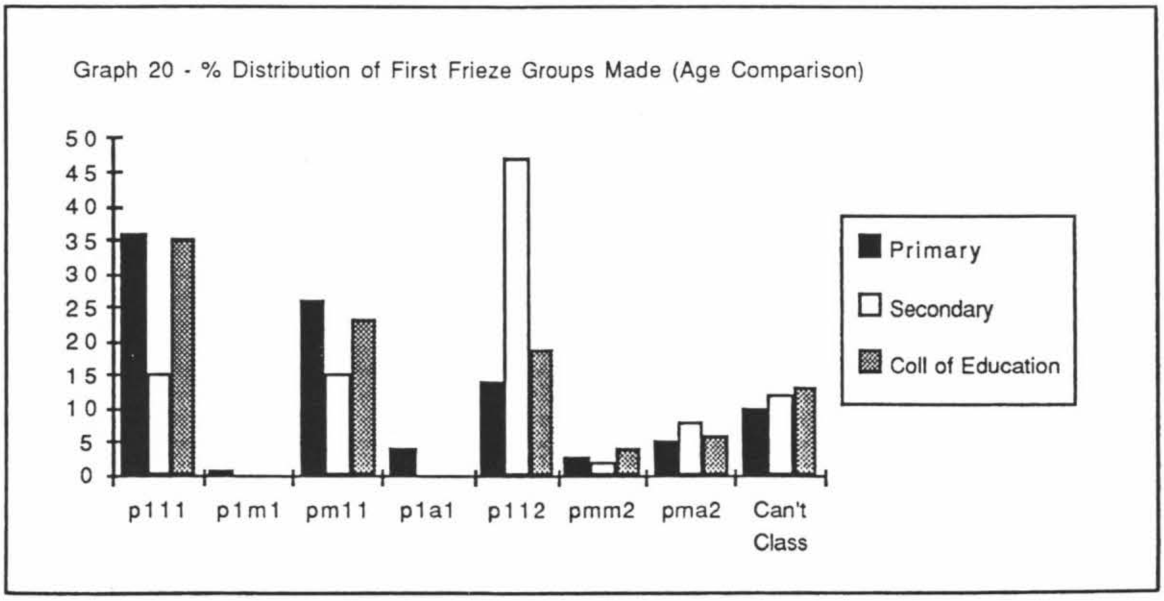

Graph 20 - % Distribution of First Frieze Groups Made (Age Comparison)
50
45
40
35
30
25
20
15
10
5
0
p111 p1m1 pm11 p1a1 p112 pmm2 pma2 Can't Class
Primary
Secondary
Coll of Education

Table 5.1

	Primary Order	Secondary Order	Tertiary Order
1	p111	p111	p111
2	pm11	p112	pm11
3	p112	pm11 ≥ pmm2 ≥ pma2	p112
4	pma2	p1m1	pma2 ≥ pmm2
5	pmm2	p1a1	p1m1 ≥ p1a1
6	p1a1 ≥ p1m1	-	-

The most striking features in this comparison of the ordering of the frieze groups drawn by each age group are:

1. p111 always appears at the top.
2. p1a1 and p1m1 are at the bottom.
3. pm11 and p112 are listed above all the other frieze groups, apart from p111.
4. pmm2 and pma2 always appear between the commonly drawn frieze groups (p111, pm11, p112) and the uncommonly drawn frieze groups (p1a1, p1m1).

Also, between 10 and 13% of the patterns contained insufficient evidence of translation symmetry for all three groups, and the number of patterns for whom the frieze group classification was ambiguous was also small in every age group.

Another interesting point to note is that the results of the pre-formal (Primary) and post-formal (Tertiary) age groups appear very similar in graphs 17, 18, 19 and 20, as well as in the ordering of the list of frieze groups above. The during-formal group (Form Fours) stands out in a variety of ways, the most obvious being the proportion of Secondary students that made p112 as their first pattern (47%); for the Primary and Tertiary students, 14% and 19% respectively. The most important differences, however, are the distributions of the frieze patterns and the number of students that made them.

The method of construction also appears to have varied between all the groups. For example, almost 80% of the Primary patterns were disjoint, for Tertiary 60% and Secondary 30%. This suggests that base pattern construction, or possibly incidence construction, appears to have been the most popular mode of design for the pre-formal and post-formal transformation geometry groups. When we consider that touchings were also probably the result of incidence, that conclusion seems even stronger. In contrast, 41% of the Secondary patterns were tilings (or of the continuous style) indicating that as a group, the Fourth Form subjects probably used a greater variety of methods than the other two groups, and hence a more rich use of transformation geometry.

The fact that the Primary and Tertiary groups did produce similar frieze group profiles on four 'measures' of 'intuitive-ness', and seemed to use similar construction methods, indicates that the Tertiary students' use of transformation geometry may be more intuitive than that of the Form Four students'. In other words, the formal transformation geometry framework seems to have had an effect on the Secondary students' patterns, but this effect may diminish substantially thereafter.

The four main observations related to the 'ordering' of the frieze groups (with respect to 'how intuitive' they were) showed strong similarities between the three different subject groups. We can summarise them as follows:

1. p111 (common and frequent)
2. pm11, p112
3. pma2 ≥ pmm2
4. p1m1, p1a1 (uncommon and infrequent)

Again, the patterns with translation and more than one other kind of symmetry occurred below the p111, pm11 and p112 frieze groups and above p1a1 and p1m1. This fact raises the same question as outlined for the Primary children section above: is the position of pmm2 and pma2 in the list a result of a *dependent combination of intuitive symmetries*, or is it due to an *independent combination of both intuitive and non-intuitive symmetries*?

5.2.2. Transformation Analysis

We might be tempted to conclude that vertical reflection symmetry and half-turn symmetry are equally intuitive from the above results, but the following consideration shows that this is probably not the case. The Primary and Tertiary results both suggest that vertical reflection (pm11) is more intuitive than half turn (p112), whereas the Secondary results imply the converse (this is why pm11 and p112 are placed together). When we consider that the Primary and Tertiary groups are not actively engaged in the learning of the formal transformation geometry framework, it seems fair to conclude that the Fourth Form results reflect the influence of this framework, and are therefore not as intuitive according to our working definition. For this reason, vertical reflection is placed above half-turn in the intuitive ordering.

We can now state with some confidence, the ordering of intuitive transformations used in those patterns with translation and one other kind of symmetry:

1. Translation (common and frequent)
2. Vertical Reflection
3. Half Turn
4. Horizontal Reflection and Glide Reflection. (uncommon and infrequent)

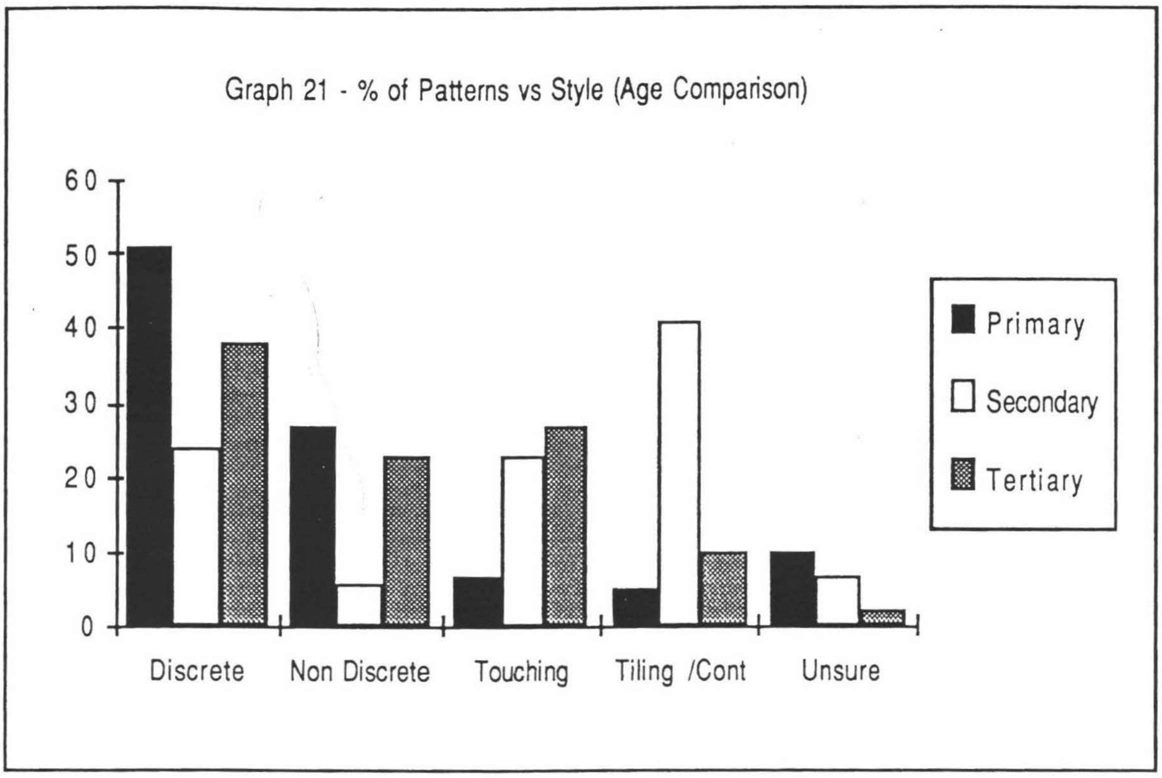
The fact that the ordering of the frieze groups, and hence the transformation ordering, was roughly the same for all three age groups suggests that, while the formal transformation geometry framework has some effect (on the Tertiary and especially the Secondary students), a strong *intuitive* transformation geometry influence exists in the construction of frieze patterns for all three age groups surveyed.

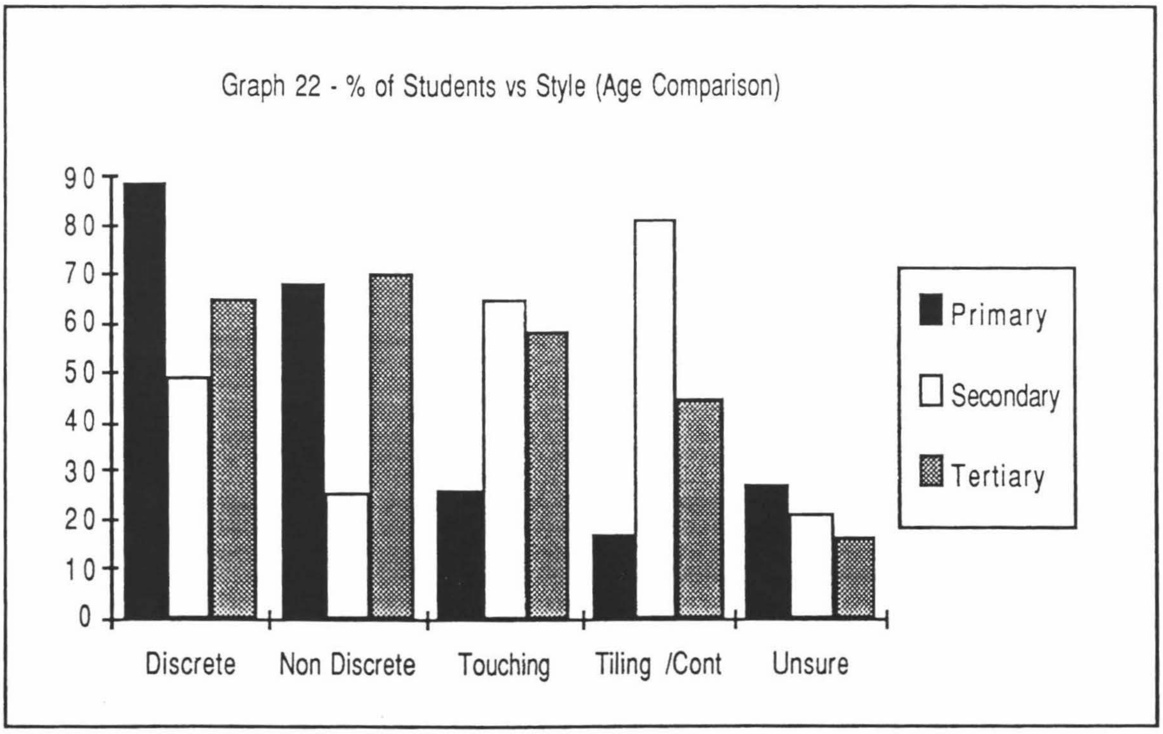
5.2.3. Style Analysis

From graphs 21 and 22, it is quite clear that the Primary and Tertiary students produced similar style of patterns; the distribution of the Secondary students' pattern styles contrasts markedly with the other age groups in three categories: discrete, non-discrete mono-motif patterns, and tilings. Since the styles were designed to provide helpful clues to the construction methods used to make them, we can conclude tentatively that the Primary and Tertiary students were probably more alike in their construction methods than either group was to the Secondary methods.

The Form 4 subjects, having been recently exposed to the transformation geometry framework, displayed a different style distribution that suggested a greater emphasis on the methods associated with tilings. Of course, the proportion of Secondary and Tertiary touchings is roughly equal. Why the distributions of these two agree in some styles, and not in others, is difficult to say. It might be the effect of the exposure that some of the Secondary pupils have had to tessellations and the like recently; of course this is merely conjecture.

More telling, is the distribution of styles within various frieze groups. However, rather than make detailed comparisons between the age groups for every style in each frieze group, a few summarising observations will be made. For the relevant data see appendices B1.3, B2.3 and B3.3.





p111: The Primary students focussed on the disjoint types, whereas the older students made touchings as well. This suggests that more subjects from the older age groups may have had a 'sense of the whole' when constructing p111 patterns.

p1m1: The Primary and Tertiary subjects made very few of these, with most examples either discrete or touching styles. The tilings and touchings formed the highest proportion of Secondary students' p1m1 patterns.

pm11: All three age groups produced a similar style distribution for this frieze group.

p1a1: Interestingly, the Primary group produced a far higher proportion of p1a1's than the other groups, although numeracy was still small. The predominant style for the Primary group was PS2; for the post-Primary students, both PS2 and touchings.

p112: This frieze group displays a sizable contrast between the Secondary and the other two groups. 80% of all the Secondary p112 patterns were tilings, compared with 51% for Tertiary and 26% for Primary. In fact, 60% of the Secondary students made a p112 tiling. Also, only 15% of the Secondary patterns were disjoint, compared with 47% for Tertiary and 60% for Primary. The NDP was rare for all three groups.

pmm2: The frequency of patterns, for each of the age groups, was low for PS12, PS13 and PS14. Most of the Primary patterns, however, were the discrete type PS15; the majority of the Secondary patterns were tilings and few touchings; and the bulk of the Tertiary pmm2's were touchings.

pma2: The distribution of the Primary and Tertiary patterns was quite similar for this frieze group, with the NDP and touchings occurring infrequently, and a fairly even distribution amongst the other styles. The Form 4 students focussed strongly on the tilings for the pma2 pattern (76%). Indeed, almost half of the Secondary students produced a pma2 tiling.

The style analysis of the frieze groups has revealed that the Primary and Tertiary groups are more alike than either is to the Secondary group in their pattern styles. However, for particular frieze groups (p111, p1a1, pmm2) the Primary and Tertiary results do differ somewhat in the style distribution, and therefore probably in the emphasis they gave to the various construction methods that they used. Having said that, the Primary/Tertiary differences in these three frieze groups may not be as pronounced as the above analysis implies, since the interview data shows that the base pattern construction is commonly associated with all the disjoint and touching styles. See the interview subsection for details.

5.3 Further Observations

5.3.1. Frieze Group Ambiguities

The occurrence of frieze group ambiguities dropped considerably from activity (a) to activity (b). The two main causes of frieze group ambiguity was the use of the strip border lines as part of the pattern, and the nature of the triangles drawn.

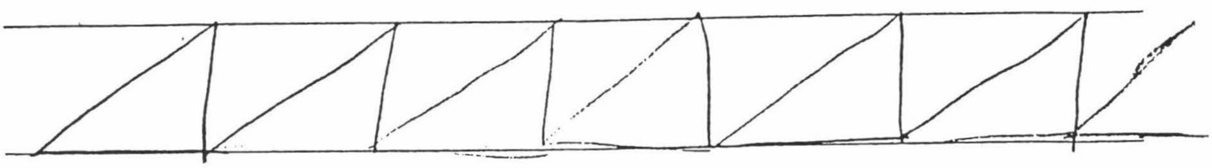


Figure 5.1 (Alice)

Alice explained that she started by drawing a single "right angle triangle", and then repeated it along the strip. Until pointed out in the interview, she hadn't noticed the other set of triangles 'pointing' the other way. While this pattern looks like a p112, from Alice's explanation of her intentions, it is clearly a p111. The ambiguity has arisen from two aspects of the pattern: the presence of the border lines, and Alice's spacing of the triangles. This example was the most common form of frieze group ambiguity (from the point of view of the classifier).

The use of non-right angle scalene triangles could also bring about categorisation difficulty, as the following pattern (fig. 5.2) shows:

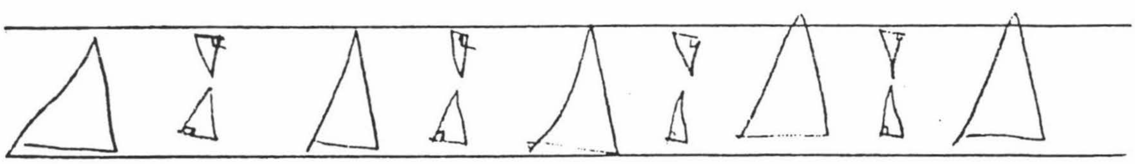


Figure 5.2 (Carla)

The right-angle marks shown in this pattern were made by Carla during the interview. Before this was done, the frieze group was difficult to identify. It may have been classified as either p111(NDP), p112(PS7), or pmm2(PS13). After her graphic explanation, it seems clear that this was a p112 (PS7). Other interviewees did use isosceles triangles in some instances. This confusion between right angled scalene triangles and isosceles triangles may be due to similar side lengths or perhaps an affinity for D_1 figures.

5.3.2. Style Ambiguities

Alice's example (fig. 5.1) illustrated not only a frieze group ambiguity, but a style ambiguity as well. Given the way she made the pattern, it is best described as a touching, not a tiling. Similarly, Carla's pattern (fig. 5.2) above also had a style ambiguity as well as an ambiguity in the symmetry group. It could easily have been classified as a non-discrete mono-motif pattern (P111) if the triangles were intended to be isosceles.

The reasons for ambiguity in the patterns were the same whether they were between frieze groups or styles. The properties responsible for ambiguities were:

1. Spacing of the triangles in a way that tilings and touchings can't be distinguished.
2. Use of the border lines already provided.
3. Ambiguously drawn figures.

5.3.3. Accidental Symmetries

Some symmetries recognised by the classification weren't intended by the drawer. The character of the accidental symmetries in this activity sheet was different from that of activity (a). In activity (a), many D_n figures contained symmetries which weren't intended by the drawers. In contrast, most of the activity (b) symmetry accidents arose from the confusion between symmetry groups due to the positioning of the triangles in the strip. In the above example, Alice had not even seen the 'other set of triangles', much less intended the half-turn transformation between adjacent triangles or of the whole pattern.

It should also be noted that those patterns with properties 1 and 2 (above) often had extra symmetries which were not intended by the drawer. However, overall, far fewer examples of extra symmetries were found for activity (b) than for activity (a). Similarly, no examples of accidental symmetry were identified in the activity (b) interviews either.

5.3.4. Vertical Patterns

Table 5.2

Frieze Group	p111	p1m1	pm11	p1a1	p112	pmm2	pma2
No. of Students Producing	1	1	1	5	2	0	0

The rarest types of symmetry groups for the horizontally drawn patterns were p1m1 and p1a1. Yet, in this small interview survey of vertically drawn patterns, p1a1 accounted for 5 of the 10 patterns! It seems that the drawers' orientation relative to the empty strip has effected the nature of the frieze groups produced. A number of explanations could be given to support this view. For example, in the real world, the most common example of glide reflection is that of footprints (semi-egocentric) and walking (propriocentric), both of which are usually perceived in the direction of the median plane (x-axis).

Another possibility is that the prevalence of (vertical) p1a1's may be due to a human's natural affinity for uprightness of the triangles (Fisher, 1978). Furthermore, vertical reflection incidence was not uncommon for the horizontally aligned patterns. From observation and questioning, the drawers appeared to use a vertical reflection (relative to themselves) as part of an incidence construction (or base pattern construction) in the vertical patterns; a 'glide reflection' pattern naturally arises.

5.4. Interviews - Case Studies

Introduction

The most important information gained from the interviews with respect to the restricted pattern construction was the *methodology* behind the patterns made. The subjects' explanations of their construction methods pinpointed the intuitive use of transformation geometry far better than the pattern classification system.

In activity (a), the (unrestricted) construction methods identified were 'base pattern', 'incidence', 'alternation (type 2)', 'division of the strip into a tiling' and 'filamentary'. Combinations of these in phases yielded various forms of superposition, such as 'ornamentation', 'superimposing of rows' and the 'overlaying of filaments'. Several of these methods were used to make the activity (b) patterns too. But rather than present another full explanation of those methods here, this discussion will give one interview example of the same methods, leaving the more detailed discussion for the different construction methods that arose in activity (b). In addition, the activity (a) methods which didn't seem to have appeared in the activity (b) interviews will also be noted, and a table summarising the activity (b) construction methods will be displayed. Finally, a brief outline of the contrast between the interview responses to activity (a) and activity (b) will be made.

5.4.1. Similar 'Restricted' and 'Unrestricted' Construction Methods

Figure 5.3 (Mary)

Mary began this pattern (fig. 5.3) by drawing the triangle on the far left. Then

M: ... I put one back to it.

I: ... ah ... what do you mean?

M: That's back to back.

I: So is this triangle [second] meant to be the same as that one [first]?

M: Mmm ... but ...

I: And how is it different?

M: But it's sort of mirrored.

I: I see ... after that, did you repeat the first pair or did you sort of mirror that one [second]?

M: I did more back to back triangles (pointing to the first pair drawn).

Reflection incidence can be ruled out; Mary's method was a clear case of base pattern construction.

In fact, the most common example of discrepancy between the survey interpretations and the children's explanations was the method of constructing discrete patterns with C_1 motifs. The survey analysis argued as if these were most likely to have been made with an incidence method. The children who were interviewed, however, often made these by a base pattern construction.

High order incidence explanations were made for some discrete patterns where the motif was not a C_1 figure *and* was not the translation unit. This suggests that the discrete types probably drawn with an incidence construction were PS10, PS11, PS13, and PS14. Whilst high order incidence may have been used to produce NDP's or touchings, it seems that it was rarely used since only 26 of the 808 patterns drawn were one of the 4 types listed. One example follows:

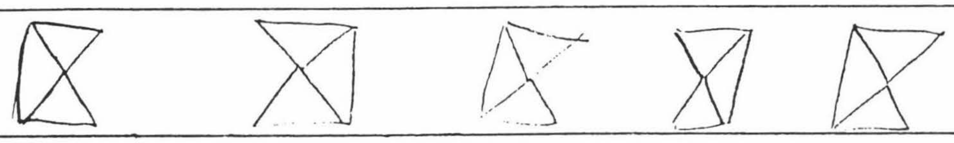


Figure 5.4 (Mary)

Firstly, Mary explained the construction of the figure on the far left:

> I: What were you thinking of there?
>
> M: Um ...sort of [___?___], something like that ... sort of triangle there and a triangle there ... and just thought of putting a line down there to make another one.

The horizontal or half-turn symmetry present have not been mentioned, indicating that the use of such symmetry may be implicit and possibly undifferentiated in this pattern. Also, she has made the first triangle isosceles which suggests implicit vertical symmetry. But Mary continued to explain the rest of the construction, a clear example of the explicit use of vertical reflection incidence:

> I: ... And then ...?
>
> M: Did the opposite ... so it'd mirror.
>
> I: Okay, what did you do then?
>
> M: Well that one [second motif drawn] mirrored that one [first motif] ... and that one [third motif] mirrored that one [second motif] ... and so on.

Richard's explanation of his pattern (fig. 5.5) displayed a clear example of superposition by ornamentation.

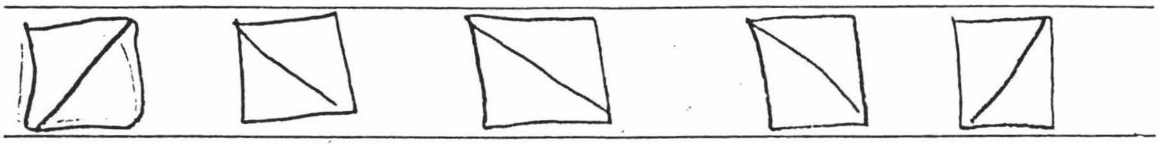


Figure 5.5 (Richard)

R: I just drew the squares, and then I went back and put the lines in.

Amy also employed superposition (fig. 5.6). Her method differed in that two rows were constructed separately. Also notable is her explicit, although possibly undifferentiated, explanation of a transformation of the whole bottom row.

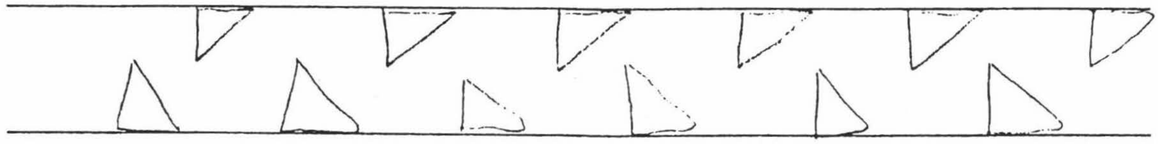


Figure 5.6 (Amy)

A: First I went along the bottom ... and then I went and did the top ... I turned it upside down and did the top the same as the bottom.

I: ... um ...so are they [top and bottom rows] meant to 'line up' then?

A: No ... it's shifted along a bit.

5.4.2. Additional Construction Methods

The extra methods that appeared in the interviews for this activity were both tilings. The first (fig. 5.7) was the building up of a tiling from the triangles. That is, instead of dividing the whole strip into congruent tiles, as occasionally occurred in activity (a), each tile was drawn one at a time directly adjacent to the previous one so as to fill the given strip without any overlaps. The construction was very similar to the base pattern method.

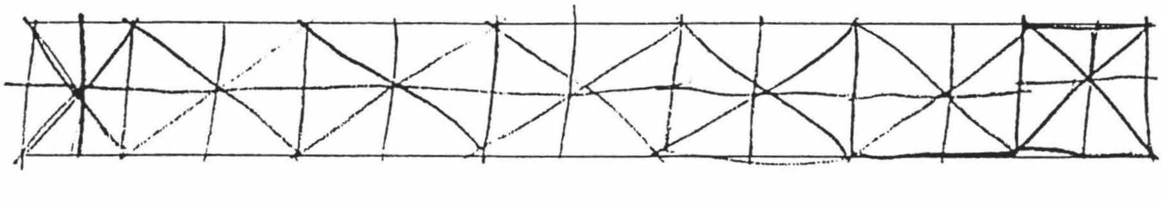


Figure 5.7 (Alice)

A more puzzling tiling method is displayed below:

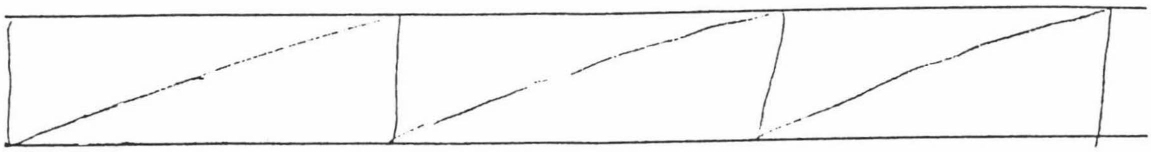


Figure 5.8 (Toni)

I: How did you make this pattern, Toni?

T: I just drew ... a straight line down, and a sloping line up, a straight line down and a sloping line up.

I: Okay ...can you tell me where the triangles you drew are?

T: There ..and there ...and there and there ...

I: So you deliberately made these and these [both sets of triangles]?

T: Yeah. I just did them as I went along.

This could be interpreted as a base pattern construction, with the translation units together. But the adjacency of each triangle made in relation to the former triangle suggests the construction was a little different. The sense of continuity associated with Toni's use of one curve and the strip borders to make the triangles *also* includes a somewhat filamentary approach not usual in a base pattern construction.

5.4.3. Different Restricted and Unrestricted Construction Methods

It was harder to identify those activity (a) methods which were not used by the Primary students in activity (b), since it was quite possible that some of the students used pattern constructions not discussed in the interviews.

Clearly, a single filamentary curve that didn't touch the border lines couldn't be drawn because triangles would not have been formed. The corollary of this observation is that the overlaying of such filaments probably did not occur. The interview example closest to that of a filamentary construction was Toni's (this is displayed above in the 'extra methods' discussion).

The 'alternation type 2' construction was also not identified in the interviews, although it is quite possible that it was used by some of the students who were not interviewed. Similarly, none of the interviewees seemed to have used the division-of-the-strip-into-a-tiling method. Toni's pattern above could have been produced by this method, but it appears to have been the result of a slightly different approach.

The interview data then, shows us that we must be cautious in making inferences about intuitive transformation geometry from the symmetry group, and even style, classification. Several different methods can produce exactly the same frieze group and style. On the other hand, it appears from the interviews that it was *most common* for one method to be used to produce a particular style of frieze group; recall, too, that most of the Primary patterns were disjoint so the base pattern construction was probably used for the majority of patterns. Therefore, the style analysis gives a methodical framework in which to make suggestive, but not conclusive, inferences about the intuitive use of transformation geometry. Given the interview data, those inferences can be modified or refined in a similar manner to the discussion of type analysis above.

5.4.4. A Comparison of the Unrestricted and Restricted Activities

In every Primary school, the unanimous opinion was that activity (b) was more difficult than activity (a). The two main reasons given for this were:

1. It's harder to think of patterns with only triangles in them.
2. Confusion about the concept of a right-angled (scalene) triangle.

Perhaps the most noticeable feature of the Primary interviewees' activity (b) explanations was the increase in the use of words that were explicit in describing transformations or symmetries. For example, differentiated terms like "mirrored", "flipped over" and "turned around" were used a lot; undifferentiated terms or directional judgements such as "pointing the other way", "turn upside down" were also common. In patterns of the same symmetry group, identifying the intuitive use of transformation geometry was easier in activity (b) than in activity (a).

The best example of this contrast between the two activities was in the explanation of the symmetries within a base pattern. In activity (a), many of the figures within the patterns were representational in form, and little explanation of the symmetry of these figures was made. The symmetry of these figures then was implicit or, in some D_n cases ($n \geq 2$), probably accidental. In contrast, constructing the connected figures within the activity (b) patterns required a more direct use of transformations and, correspondingly, the children would explicitly mention the transformation used as part of their explanation of how they made the base pattern.

We can summarize the Primary children's methods associated with each activity sheet, noting any differences between them, if any, with respect to transformation geometry.

Table 5.3

Construction Method	Explicit Transformation Geometry	Implicit Transformation Geometry	Accidental or Incidental Transformation Geometry	Extra Notes
Base Pattern, activity (a)	Translation of base pattern.	Reflection symmetry of D_n and rotation symmetry of C_{n+1} figures ($n \geq 1$).	All symmetries of whole pattern. Rotation symmetry in D_n figures ($n \geq 2$).	
Base Pattern, activity (b)	Translation of base pattern. Reflection symmetry of D_1 and D_2 and rotation symmetry of C_2 figures.		All symmetries of whole pattern. Rotation symmetry in D_2 figures*.	* C_n and D_n figures ($n > 2$) did not feature in any interview patterns from activity (b).
Incidence, activity (a)	Characteristic incidence transformation plus translation.	Symmetry of D_1 and C_2 figures. Translation of translation unit.	All symmetries of whole pattern. Rotation symmetry in D_n figures ($n \geq 2$).	Characteristic incidence transformation may be undifferentiated
Incidence, activity (b)	Symmetry of D_1 and C_2 figures. Characteristic incidence transformation plus translation.	Translation of translation unit.	All symmetries of whole pattern.	Characteristic incidence transformation may be undifferentiated.
Tiling by Division of the Strip		Translation symmetry of whole. Symmetry of resulting figures	Non-translation symmetries of whole pattern	Not identified in activity (b) interviews.
'Filamentary' Tiling	Translation of translation unit. Transformations between adjacent tiles.	Translation symmetry of whole pattern	Non-translation symmetries of whole pattern	Not identified in activity (a) interviews.
Tiling by Building (step by step)	Translation of 'base pattern'. Transformations of the tiles.	Translation symmetry of whole pattern. Symmetry of adjacent tiles.	Non-translation symmetries of whole pattern.	Not identified in activity (a) interviews.
Filamentary Pattern		Translation plus one other symmetry of whole.	Remaining symmetries of whole pattern.	Not identified in activity (b) interviews.

Ornamentation activity (a)	Translation of markings from one figure to another.	Translation symmetry of whole pattern.	Non-translation symmetries of whole pattern.	For first phase, see one of the above activity (a) methods.
Ornamentation activity (b)	Translation of markings from one figure to another.	Translation symmetry of whole pattern.	Non-translation symmetries of whole pattern.	For first phase, see one of the above activity (b) methods.
Superimposing Rows activity (a)	Translation of motifs in each row.	Translation symmetry of each row and of whole pattern.	Non-translation symmetries of whole pattern.	For each phase, see activity (a) base pattern method. A stronger sense of the whole is present.
Superimposing Rows activity (b)	Translation of motifs in each row.	Translation symmetry of each row and of whole pattern.	Non-translation symmetries of whole pattern.	For each phase, see activity (b) base pattern method. A stronger sense of the whole is present.
Overlaying Filaments	Translation of filament.	Translation plus one other symmetry of whole.	Remaining symmetries of whole pattern.	Not identified in activity (b) interviews. Both phases are filamentary (see above).

5.5 Summary of Activity (b)

For activity (b), the most frequently and commonly arising frieze patterns were p111, pm11 and p112 for the three age groups. The unusual patterns were p1m1 and p1a1.

Based on the ordering of the 5 frieze groups that include translation and one other kind of symmetry, the ordering of the rigid transformations with respect to intuition is similar to activity (a). In spite of the instructions explicitly prompting translation, this transformation is placed at the top of the list because it was used easily by the Primary children:

1. Translation (most intuitive)
2. Vertical Reflection
3. Half-Turn
4. Glide Reflection, Horizontal Reflection. (least intuitive)

The style analysis of the total number of patterns drawn, and of the separate frieze groups, revealed that the Primary and Tertiary groups were quite similar in the patterns they drew, and therefore in the construction methods use to produce them. The Fourth Form students have probably been affected by the formal transformation geometry framework they were recently exposed to, and thus their use of transformation geometry is not as intuitive as the other age groups.

Two characteristic features of the Secondary activity (b) patterns was the extensive use of half turn and also of tilings. Indeed, many p112 tilings occurred which have both features.

While several types of pattern construction existed, by far the most common method for the Primary children was the base pattern construction. Furthermore, while a variety of methods were suggested by individual pattern *styles*, most styles could have been (and probably were) made using the base pattern construction. Hence, the most common way that transformation geometry was used intuitively was via the symmetry within the base pattern and the translation of the base pattern. Both uses of translation were generally understood explicitly by the Primary school students.

A discussion of the methods used in activity (a) and (b) was made, and a summary table made to compare the intuitive use of transformation geometry in each activity implied the methods. This can be found in subsection 5.4.4.

In general, the Primary school students' use of transformation geometry in activity (b) was more explicit than in activity (a). For example, in the base pattern construction, the use of symmetry within the base pattern appeared to be more intentional in activity (b) than in activity (a).

6 Results of the Pattern Description Activity

In this activity, the subjects were given discrete examples of the seven frieze groups[1] and asked to write a description of each pattern in turn. Correspondingly, the results presented are chiefly qualitative. The general classification of the phrases is a construction of the researcher's based on the trends observed in the subjects' written responses. It includes some terminology from Piaget and Inhelder (1971) and Howard (1982). The categories are outlined in subsection 3.4.2. A detailed presentation of the *Primary* school children's descriptions is made to illustrate the classification, since the children surveyed provided the most intuitive descriptions of the three age groups (see chapter 1). In a later section, interview extracts are given to qualify the findings. This is supplemented by a more thematic comparison of the age groups which includes a brief quantitative discussion of the commonality of the type of responses given for each frieze group.

Besides the description analysis, a small survey was conducted in the interviews, which required the subjects to compare or match various examples of frieze groups. Tables summarising those matches are given and pertinent interview extracts are presented to highlight the character of the children's verbal explanations; the explanations are discussed using a slightly modified version of the criteria found in Piaget and Inhelder's (1971) chapter on *The Spatial Image and 'Geometrical Intuition'*. For further explanation of the terminology in this chapter, see subsection 3.5.2 above.

6.1 Results from the Primary Schools

For each of the seven frieze groups, the range of responses outlined in subsection 3.4.3 is illustrated by using the Primary children's explanations as examples. In the following list of examples, an asterisk (*) denotes those phrases which occurred several times; two asterisks (**) denote a very common expression. A comma (,) between two entries indicates that they are associated or similar.

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[1] In this chapter, the patterns from activity (c) are reduced to 80% of their original size.

6.1.1 p111 [PS1]

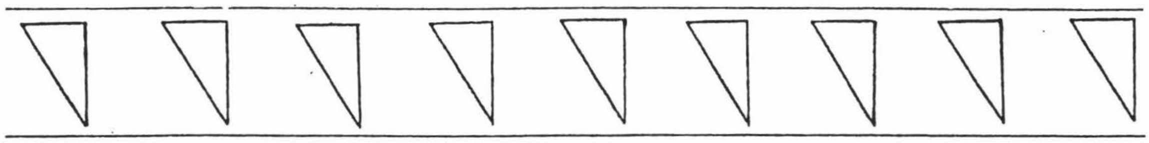


Figure 6.1

A1. Explicit: Differentiated Transformation Geometry

"triangles going across" *
"all in the line of a pattern"
"repeated pattern", "repeating down the line"

A2. Explicit: Undifferentiated Transformation Geometry

"triangles all the same shape and size"

B1. Implicit: Comparison of Whole Pattern with Another Object

"shark's teeth"
"a chain saw blade"

B2. Implicit: Comparison of Pattern's Parts with Another Object

"tiger's tooth"
"dart"
"a wing of a 'tom-cat'"
"half an arrow"
"bird's beak (top)"
"fins of a surf board"
"wing of a plane"
"half a triangle"
"a wonky ice-cream cone (with no ice-cream in it)"
"half rectangles"

C. Orientation or Direction Judgements

"upside down triangles"
"point down"
"right-angle in top corner", "right-angle facing upwards"
"standing on biggest point"
"all facing the same way"
"skinny point at bottom, fat end at top" **

D. Positional Judgements

"spaced out evenly", "spacing 2cm apart" *
"triangles in a row"

E. Miscellaneous Evaluations

"right-angle triangles" **
"sides not the same length" *

"nine in the set" *
"don't touch the top" [border line]
"heaps of them in a line"
"a 3 shape"
"sides not equal"
"very pointy"
"bare", "quite plain"
"I don't like it ...they look ugly"
"white"
"it's not a triangle"

6.1.2. p1m1 [PS3]

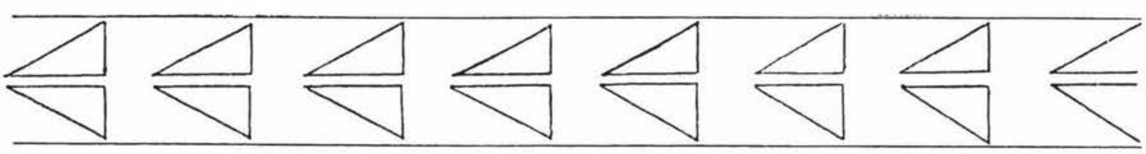


Figure 6.2

A1. Explicit: Differentiated Transformation Geometry

"pattern goes across the page"
"repeat down the line"
"both sides are metric [symmetric]"
"a triangle with a reflection"
"if you folded it they would be equivalent"

A2. Explicit: Undifferentiated Transformation Geometry

"two triangles are facing each other"
"bottom triangle just like the top"
"repeats itself upside down"
"parallel triangles"

B1. Implicit: Comparison of Whole Pattern with Another Object

no example

B2. Implicit: Comparison of Pattern's Parts with Another Object

"ramp with its reflection in a mirror"
"arrows", "arrow split down the middle", "arrows pointing one way", "arrowheads" *
"draw a backward L shape and join diagonal"
"ship sails on their sides"
"hang glider"
"triangle with line through", "form in the shape of normal triangle", "triangle cut in two", "together would make an equilateral triangle" *
"paper dart", "dart not joined up", "dart folded apart, lying flat" *
"formed in the shape of a bird's beak"
"mountains with straight sides and line through it"
"aeroplanes", "paper aeroplanes flying in a row", "front of jets", "rocket sharp thing", "like a plane's wings", "front of a plane's window" *

C. Orientation or Direction Judgements

"longest line points left"
"one upside down, the other the right way up", "one facing up, the other facing down" **
"they point left" *
"the short sides to the right, two long lines in the middle"
"the right-angles are in the middle" *

D. Positional Judgements

"spacing about 2 cms apart"
"triangle on top of the other triangle"

E. Miscellaneous Evaluations

"right-angle triangle" **
"sides not the same length" *
"The sides just about touch", "the points just about touch", "not quite touching"
"8 shapes", "16 triangles"
"white with black outline"
"a 6 shape"
"nothing in the middle"

6.1.3. pm11 [PS5]

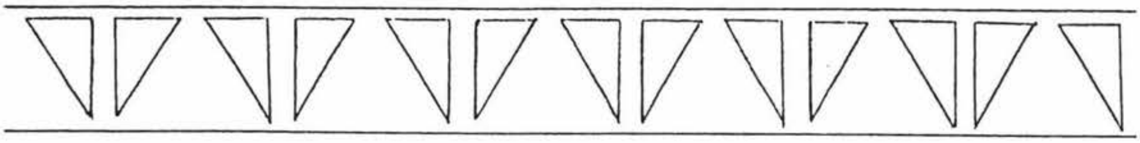


Figure 6.3

A1. Explicit: Differentiated Transformation Geometry

"in a straight line across"
"in a row"
"both sides are <u>metric</u> [symmetric]"
"the sides mirror each other"
"reflection of a triangle"

A2. Explicit: Undifferentiated Transformation Geometry

"it has been alternating having a new change of direction"
"they are side by side by side"
"straight bits pointing at each other"
"side to side"
"triangles facing each other"
"they are looking opposite each other", "two together going opposite ways" *

B1. Implicit: Comparison of Whole Pattern with Another Object

"a wall with a gap"
"a monster mouth because it has got lots of spikey teeth"

B2. Implicit: Comparison of Pattern's Parts with Another Object

"sail upside down with line through it"
"arrow with no stick in it", "arrowheads pointing down" *
"shape of hang glider"
"bird's beak"
"put together makes one big triangle"
"a rocket", "an aeroplane front" *
"upside down pyramid"
"upside down triangle cut in half"
"if turned around looks like darts going up"
"upside down ship sails with mast yanked out"
"$6^1/_2$ jets"
"a house if you put it together"

C. Orientation or Direction Judgements

"big point pointing to bottom ... short line on top ... straight line in middle"
"triangle on left has right-angle pointing right ...triangle on right has right-angle pointing left"
"two triangles standing on end"
"right-angle in middle at top"
"upside down triangles"
"pointing down"
"fat at top, skinny at bottom"
"direction is south"

D. Positional Judgement

"one beside the other"

E. Miscellaneous Evaluations

"two nearly joined right-angles"
"sides just about touch", "points just about touch"
"6 pairs ... sets of two" *
"13 triangles"
"a very good pattern"
"it is a triangle even though it doesn't look like one"

6.1.4. p1a1 [PS2]

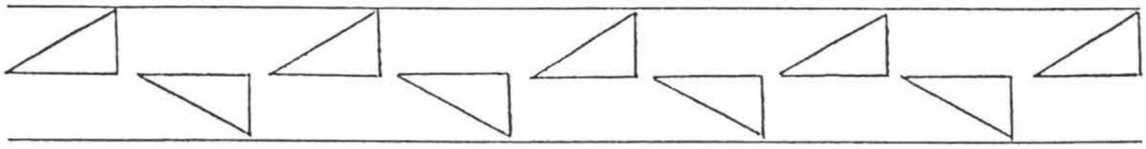


Figure 6.4

A1. Explicit: Differentiated Transformation Geometry

"like a mirror but triangles on the bottom one are a wee bit before the top one"
"you have to slide it to match it"
"it goes across the page"

"Δ's in a zig-zagging row"
"If you folded the bottom they would be the same but opposite"
"two rows of triangles"
"repeats the pattern four times"

A2. Explicit: Undifferentiated Transformation Geometry

"they go down up down up from the left", "triangles up and down facing left", "right-angle triangles going top,.bottom, top, bottom ..." *
"every triangle goes diagonally to the next", "they are diagonal to one another", "one triangle on top and one diagonally to that one" *
"one is up, one is down and they keep on going"
"every second pattern they are the other way up"
"the bottom in a line of right angles [triangles] exactly like top line except turned upside down"
"a pattern with one triangle"
"two right-angle triangles reversing in turn"
"face different ways"
"they are anglewise from each other"
"and so on", "and it continues like that all the way down the line"
"two triangles pointing left but they aren't positioned in the same place"
"one is on top, another is upside down below it"
"instead of being opposite they have a space in between"

B1. Implicit: Comparison of Whole Pattern with Another Object

"looks like a ladder
"like a draught board but with triangles"
"chain-saw blade"
"checkered"
"they look like footprints"
"could draw an imaginary line in the middle"
"a path zig-zagging"
"a motorway of cars"

B2. Implicit: Comparison of Pattern's Parts with Another Object

"draw a sail and draw a line through it and then slide it"
"looks like a dart spinning around and around"
"it looks like a face if you put some of it together"
"a man's evil eyes" [drawn]
"a sad bird"
"join them to make a triangle" [an arrow pointing to adjacent triangles is drawn]
"skateboard ramp upside down and right way up"
"fins"
"it looks like bumps"

C. Orientation or Direction Judgements

"short side (or right-angle or point) of first triangle faces right and points up ...short side (or right-angle or point) of second triangle faces right and points down" **

"one on top, one on bottom, and so on"

D. Positional Judgements

"5 on top, 4 on bottom"

"spaced 1cm apart" *

"one in front up the top, one behind down the bottom"

E. Miscellaneous Evaluations

"different because they aren't pushed together", "aren't joined like the other ones"

"halves of triangles in clusters"

"very uneven"

"these triangles have gone crazy!"

"I find it hard to describe"

"fancy"

"a very effective pattern"

"they are all over the place"

6.1.5. p112 [PS7]

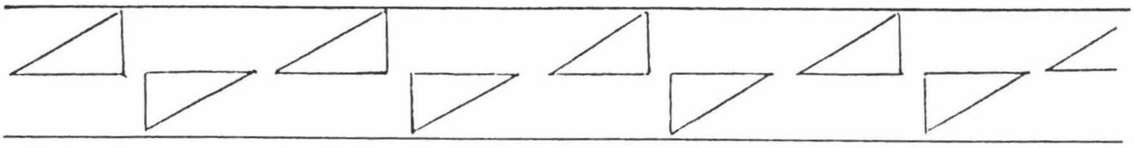


Figure 6.5

A1. Explicit: Differentiated Transformation Geometry

"every second one is up the other way and pointing the other way"

"it's repeated"

A2. Explicit: Undifferentiated Transformation Geometry

"triangles going up down up down"

"they are anglewise"

"fancy sort of pattern. It goes up down up down"

"each right angle triangle goes diagonally to the next one"

"they are point to point, pointing at each other"

"right angles together"

"facing diagonally"

B1. Implicit: Comparison of Whole Pattern with Another Object

"a spikey line going diagonally"
"traffic. five cars pointing left and another four pointing right"
"same as pattern four but bottom row is facing west"
"a zig-zagging row as well but the second triangle is a different way"
"tricked us into thinking it is the same as pattern four"
"a triangle zig-zag"

B2. Implicit: Comparison of Pattern's Parts with Another Object

"sort of looks like bows or ribbons", "a bow going diagonally but it hasn't got the knot in the middle", "bow ties going across the page" *
"traffic. five cars pointing left and another four pointing right"
"a dart going loop the loop"
"it seems like wings"
"a bird" [a sketch was made using 'first' triangle as head, 'second' triangle as body]
"two jets going different ways"

C. Orientation or Direction Judgements

"orientation of "top triangle" and "bottom" triangle using sides, angles or vertices pointing or facing up, down, left and right **
"top line faces left, bottom line faces right"
"bottom ones pointing backwards", "bottom row is facing west"
"they alternate"
"going to left in the air, going to right on the ground"

D. Positional Judgements

"spacing of a third of a cm"

E. Miscellaneous Evaluations

"very hard"
"$8^1/_2$ right-angle triangles"

6.1.6. pmm2 [PS12]

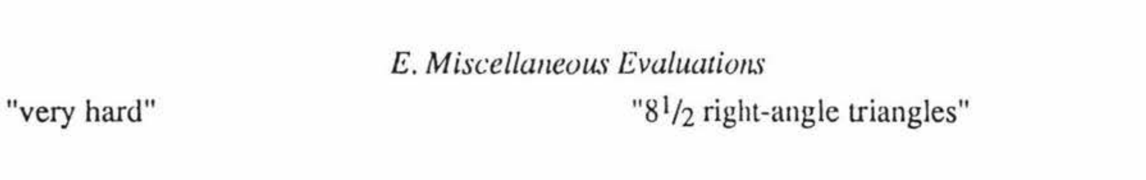


Figure 6.6

A1. Explicit: Differentiated Transformation Geometry

"it goes across the page"
"it's sort of different from the others ... it just repeats across the page"
"If you join them they will be metric [symmetric] again which is like saying they are the same on each side."
"and keeps on going down the line"

A2. Explicit: Undifferentiated Transformation Geometry

"all right angles facing"
"four triangles facing outwards"
"put two on top of each other and put two more beside it and so forth"
"same as bottom except it is upside down"

B1. Implicit: Comparison of Whole Pattern with Another Object

no example

B2. Implicit: Comparison of Pattern's Parts with Another Object

"four triangles formed so they have a cross shape in the middle", "the spacing makes a cross" *
"they are in a diamond shape", "$4^1/_2$ diamonds", "a sideways diamond" *
"a diamond split down the middle and across", "a diamond cut in fours across and up" *
"a diamond version of the red cross"
"four right-angle triangles form a kite", "a kite with no string on it"
"top half like a triangle"
"form a target that would appear on a lot of spacie games", "a target off a telescopic gun"
"put together you would get a very uneven square"
"they look like bird's beak back to back"
"looks like a flag", "NZ flag"
"if you sit on it you will soon find out about it"
"like windows all in a row but right-angle triangles instead of squares"

C. Orientation or Direction Judgements

"bottom ones have horizontal line facing upwards, top line has horizontal line facing downwards"
"two triangles point left, two point right"

D. Positional Judgements

"two triangles on the top and two on the bottom"
"two shapes on one side and two on the other"

"a gap from top to bottom and gap from side to side"

"they are all in a line"
"they are all in a row"

E. Miscellaneous Evaluations

"16 right-angle triangles"

"very very fancy"

6.1.7. pma2 [PS9]

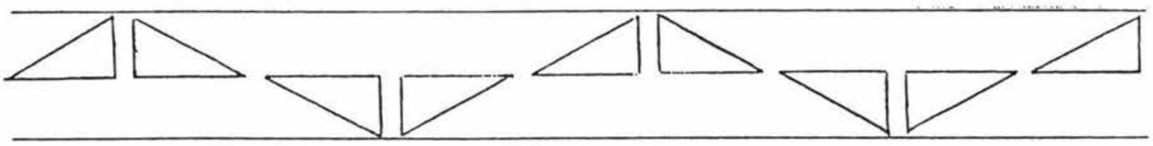


Figure 6.7

A1. Explicit: Differentiated Transformation Geometry

"and it just repeats itself over and over."
"through the middle of the whole pattern"

"the bottom has moved over to the right"
"they repeat down the line"

A2. Explicit: Undifferentiated Transformation Geometry

"one right-angle, one left angle, one right angle turned around, one left angle turned around."
"two triangles facing each other and two upside down and so on"
"they are going up and down"
"two at top and two at the bottom pointing opposite ways on a side"
"the pattern is in two and they are facing away from each other"

"two triangles back to back with flat part facing downwards"
"same on bottom except they are upside down"
"each segment is diagonal to the next"
"it is alternately going up and down"
"in each set, one triangle is pointing left, one is pointing right and the short bit is always on the inside"

B1. Implicit: Comparison of Whole Pattern with Another Object

"like pattern six but cut in half and slid across", "it's half of pattern six" *
"a pattern like this "
"like a zig-zag line"
"it's like a snake", "a snake zig-zagging"

"a bump going up and down"
"it's like an old boat"
"pattern goes up and down like a yo-yo"
"the up down pattern"

B2. Implicit: Comparison of Pattern's Parts with Another Object

"first like the top of a diamond, the next one looks like the bottom of a diamond"
"spaceships"
"a roof and a row boat joined together"
"ramps joined together and pyramids upside down"
"a bump going up and down"
"half a game targeter ..."
"a hang-glider", "a para-glide"
"a broken through the middle version of the diamond red cross"
"a Chinese helmet"
"arrowheads alternately going up and down"
"it reminds me of a fighter"
"jets going opposite ways"

C. Orientation or Direction Judgements

"has a pattern one up top and one down the bottom"
"two up, two down, two up, two down, one up"
"a slope going up and a slope going down"
"the first triangle ['pair'] is up the top and the next one is upside down and it is on the bottom"
"one facing north, one facing south"

D. Positional Judgements

"two up top in front, then two down bottom behind"

E. Miscellaneous Evaluations

"two triangles together"
"fancy"
"too hard"
"stupid"
"funny"

6.1.8 Discussion

A few concluding statements can be made. The predominant style of the Primary school children's descriptions was the use of an implicit transformation geometry phrase, likening the triangles in the base pattern to a real world object. This was very often accompanied by referring to the triangles' orientation. The number of triangles, whether they touched, and the fact that they were right-angled, were all concerns to the Primary school students, the latter probably because of the novelty of a new concept introduced from activity (b). Indeed, the descriptive domains employed by the children usually *combined* more than one of the categories (A - E) listed above. However, several students only described the orientation or 'direction' of each triangle without reference to the other triangles; that is, they used criteria C (and D or E) phrases but didn't use criteria A or B expressions. This does not imply that they did not perceive symmetry in some way; perhaps the symmetry wasn't the prominent feature of the patterns for some children; perhaps symmetry was difficult for some children to communicate.

Secondly, explicit descriptions of transformations were generally confined to translations (e.g., "repeating") and reflections (e.g., "it you fold it over it would be the same"). Glide reflection was not often spelt out explicitly but some descriptions of p1a1 included phrases such as "you have to slide it to match it" suggesting that an intuitive composition of horizontal reflection and translation was present. Rotations were barely mentioned at all, even in the descriptions of the p112 pattern.

The third point is that the expression "pattern" often referred to the base pattern and not the strip as a whole. Many explanations described only one base pattern, not even bothering to mention the repetition, as if the translation of the design was assumed or a redundant piece of information. Furthermore, a significant number of the children's descriptions indicated that the 'whole pattern' described was perceived as finite, confined to set of points drawn on the page (e.g., pmm2 [PS12] "$4^1/_2$ diamonds"). This conclusion is supported by the information from the interviews as well. It is doubtful if many of the standard three and four students surveyed did imagine the frieze patterns extending beyond the confines of the page.

Some variation of responses can be detected between the seven frieze patterns. For example, comparing the base pattern to another object (implicit) was not common for p111 and p1a1, whereas it was for the other five frieze groups, particularly pmm2. On the other hand, the explicit description of transformations was most common for p111, focussing on the translation of a triangle. For further consideration of similar trends, see the tables in the following section.

6.2 Age Group Comparison

Having sorted and identified the character of the children's description of the discrete frieze patterns shown above (figs. 6.1 - 6.6), we now proceed by comparing the differences between the Primary, Secondary and Tertiary groups. The prime reason for such a comparison is to consider the evolution of transformation geometry concepts, bearing in mind the cognitive development of each age group and the degree of exposure to the formal transformation geometry framework which each age group has had. As before, we are not interested so much in the *correctness* of the descriptions as the *type* of criterion used.

In this section a further refinement of category E is added (see subsection 3.4.3), and tables summarizing the popularity of each criteria for the respective frieze groups are given. The percentages given do not sum to 100% because some descriptions included more than one criteria.

A1 Phrases of Explicit Differentiated Transformation Geometry
A2 Phrases of Explicit Undifferentiated Transformation Geometry
B1 Comparison of Whole Pattern to Another Object
B2 Comparison of Pattern Parts to Another Object
C Orientation or Direction Judgement
D Position Judgement
E1 Topology Judgement (e.g., "not touching")
E2 Triangle Properties (e.g., "right-angled", "sides different lengths")
E3 Personal Comments (e.g., "it's too hard", "very fancy")

Table 6.1. (% of Primary Students Using Each Category)

	A1	A2	B1	B2	C	D	E1	E2	E3
p111	22	14	3	10	48	6	4	67	6
p1m1	12	34	0	58	38	11	35	29	0
pm11	9	30	4	59	42	8	25	24	3
p1a1	8	29	13	19	32	15	24	25	14
p112	1	24	27	41	47	11	16	20	9
pmm2	11	9	0	78	19	6	25	34	8
pma2	6	33	13	56	27	13	32	32	5
Av. %	10	20	8	46	36	10	23	33	6

Overall, the most popular criteria for description was the comparison of pattern parts to another object, that is, *implicit* transformation geometry. Besides p111, only one in ten students included *explicit* differentiated transformation geometry in their descriptions.

Table 6.2. (% of Secondary Students Using Each Category)

	A1	A2	B1	B2	C	D	E1	E2	E3
p111	38	19	2	5	93	53	34	71	8
p1m1	44	48	2	44	52	40	27	35	1
pm11	45	35	3	38	53	31	29	22	2
p1a1	35	31	10	16	62	48	26	34	3
p112	35	22	14	18	57	27	19	21	1
pmm2	34	36	1	62	18	22	44	26	2
pma2	40	37	11	42	36	25	23	29	1
Av. %	39	33	6	32	41	35	29	34	3

The Fourth Form students tended to describe the patterns in terms of explicit transformation geometry more frequently than the Primary students did, using terms such as "repeat", "re-occurring", "mirror image", and "reflect" and explaining the congruence of the triangles. Also, responses showed less emphasis on comparing the patterns to real world objects; the Secondary subjects seemed more concerned with the triangle's orientation and non-symmetry properties such as comparing the sizes of its angles, side lengths and so on. In short, a wider range of criteria was used. However, the comparison of the whole pattern to another object was not as common as for the Primary classes, especially in descriptions of p112.

Table 6.3. (% of Tertiary Students Using Each Category)

	A1	A2	B1	B2	C	D	E1	E2	E3
p111	52	25	0	14	80	43	45	77	6
p1m1	69	35	1	48	52	45	52	42	6
pm11	69	40	6	42	58	23	40	37	1
p1a1	55	23	15	5	65	48	46	45	3
p112	46	29	9	12	60	34	23	29	0
pmm2	66	46	0	71	25	29	52	28	0
pma2	65	43	11	28	51	37	42	28	3
Av. %	50	34	6	31	56	37	43	41	3

The College of Education students tended to use explicit differentiated transformation phrases and orientation judgements and they employed a wider range of criteria in their descriptions (on average) than the other two groups. Apart from pmm2, over half the Tertiary subjects included an orientation or direction judgement in their descriptions.

Overall, several ideas can be drawn from these results. Firstly, as the age of the subjects increased, a wider use of the range of criteria occurred. Secondly, the Primary children appeared to employ a metaphor style of description, likening a pattern part to a real world object. In contrast, it was usual for the Secondary and Tertiary groups to make orientation judgements or explicit descriptions of transformations; the use of implicit phrases was low for both groups. Furthermore, the use of explicit transformation was more often undifferentiated at the Primary level, while it was more commonly differentiated at the Tertiary level; the proportion of Secondary students using undifferentiated and differentiated descriptions was roughly equal.

Thirdly, the most common explicit transformation referred to was translation, followed by reflection. Rotation concepts were not used a great deal by any of the groups. For example, in the description of p112, Primary groups didn't use explicit criteria at all, and the Secondary and Tertiary students were just as likely to describe a half-turn in terms of a composition of two (perpendicular) reflections.

The possibility of the patterns being commonly perceived by symmetries of the whole seems remote, especially given the low percentage of students in all age groups who compared the 'whole' pattern to another object. An even stronger conclusion is that all three groups clearly showed a perception of the patterns as finite, indicated by the enumeration of triangles, or using expressions such as "repeat $4^1/_2$ times". The only obvious indications of an 'infinite pattern' conception came from a couple of students in each age group (e.g., "it keeps repeating forever").

The p112 [PS7] pattern was often compared to p1a1 [PS2] which, if not considered, would reduce the number of comparisons of this pattern considerably. Therefore, pma2 [PS9] was probably the pattern most commonly compared as a 'whole' to other real world objects. This observation corresponds nicely with the results of the continuous frieze pattern constructions described as "zig-zags" from chapter 4.

It is also interesting to note that the Primary school children were more likely to make a personal comment about the patterns than the other age groups, or tell a story about it. For instance, "... come over here, sit down, and I will tell you all about it" or "... if you sit on it you will know all about it" or even "This is an evil man's eyes. He lives in a cave by the sea and eats crabs ...". In contrast, the older subjects appeared to be more concerned with being precise.

In some respects, a direct parallel to Piaget's theory of cognitive development can be made. Children of 9 or 10 years old are usually at the concrete operations stage; a few may be nearing the beginning of the formal operations stage. It seems natural that they would compare a pattern's parts with real world objects (or other pattern parts), as Wadsworth (1979) explained:

> "while the child clearly evolves logical operations, these operations (reversibility, classification, etc.) are only useful to him [or her] in solving problems involving concrete (real observable) objects and events" (p 97)

The period of formal operations occurs roughly between the years of 11 and 15, an age range in which Fourth Form students generally lie. Wadsworth maintained that during this stage the student is better able to "organise data, reason scientifically and generate hypothesis". It comes as little surprise then, that we observe a reduction in the use of concrete comparisons and an increase in the use of explicit transformations, a criteria which Piaget and Inhelder (1971) rate as "the most advanced type of argument."

The main difference between the Secondary and Tertiary students was a slight increase in the use of explicit transformations and a noticeable increase in the number of different criteria used to describe the patterns. This also concurs with Piaget's theory of cognitive development because, according to Wadsworth, the structure of one's reasoning does not improve after the period of formal operations, but rather:

> "A major difference between adult and adolescent reasoning capabilities is the sheer number of shemata, or structures. ... the typical adult has more ... 'content' to which he [or she] can apply his [or her] reasoning powers than does the typical adolescent." (p 118)

Lastly, from the viewpoint of a perceptual framework, such as that sketched by Foster (1984), the description of *global features* increased from the Primary to the Tertiary groups. Other perceptual characteristics such as local features and local spatial relationships were common amongst all three groups.

6.3 Interviews - Case Studies

In this section we examine the meaning or intention of some of the children's written expressions. The main reason for doing this is to see if any intuitive transformation geometry was present in the verbal explanations where it was not in the written explanations. Next we consider the similarities perceived between the seven (discrete) frieze groups of activity (c) and also explore the relative difficulty of matching other discrete or filamentary patterns with them. The criteria these ten subjects used to match various patterns are also discussed in relation to transformation geometry.

6.3.1 Explanations of Intentions and Further Perceptions

The extracts here are designed to give the flavour of the children's own understandings of their written descriptions. As in section 6.1, we consider each of the frieze patterns in turn.

p111 [PS1]

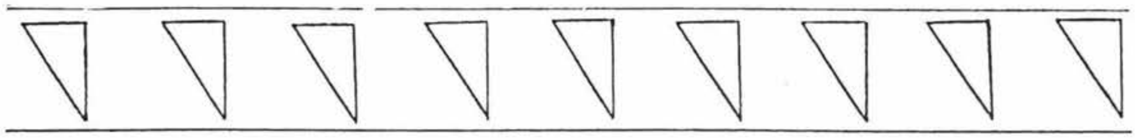


Figure 6.8

I: (Reads) "It looks like fins of a surfboard". What is "it"?

M: One of them [a right angle-triangle].

I: And you've said "it looks like a chain-saw blade"?

M: That's all of them. They look like a chain-saw blade ... the bottom.

Mark commented that this pattern was easy because "it looked like lots of things." This was typical of his written descriptions in general; his explanations were generally implicit, comparing either the whole pattern or its parts to another object. It was only when questioned that he used more explicit phrases. See pma2 below for another example.

Toni's explanation (not shown) was different. She simply described the scalene properties of one triangle and the fact that it had an interior right-angle. She didn't bother explaining the translation because it was obvious to her. Once she had described the base pattern, she deemed her written explanation sufficient. From Toni's other descriptions it seems that she considered the base pattern to be 'the pattern'. Many other children used similar written descriptions to Toni, suggesting that a common perception of a pattern was 'the thing that repeats' (i.e., a motif) in contrast to the *result* of the repetition.

The discovery of 'undifferentiated' orientations (and by implication, transformations) was made through the following extract. Amy had written "an upside down L" and was asked to draw this, which she did, matching the orientation of the triangle in pattern one (p111). The researcher then drew an alternative 'up-side down' L (reflecting hers).

> A: Oh? ... oh yeah [as if she hadn't thought of this but recognised it as upside down as well].
>
> I: So what do you mean by upside down? Can you explain a bit more about it?
>
> A: Well, upside down and <u>turned around</u>.
>
> I: Okay so are you *turning* it around like this perhaps (turns page 180°), or ...
>
> A: Mmm?
>
> I: Or are you *flipping* it over or something else perhaps?
>
> I: Mmmm ...you need to turn it over to put it backwards [horizontal reflection] ... then you need to flip it over [vertical reflection]... or the other way around.

While Amy's earlier explanation (underlined) sounded like rotation, in fact she was explaining the composition of two reflections in fairly explicit fashion. "Upside down" can have at least two specific meanings (as Amy's recognition indicated) and a broad meaning (as her surprise indicated). A highly complex use of explicit, differentiated transformations has been revealed in the interview where her written description had only appeared to be an orientation judgement.

p1m1 [PS3]

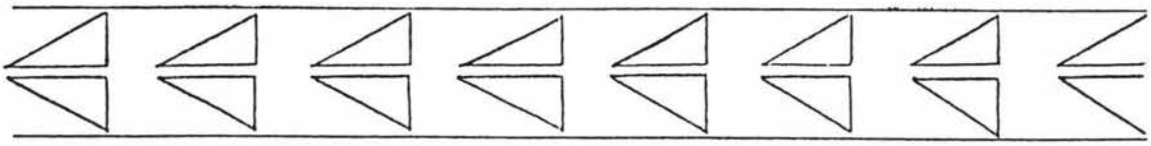


Figure 6.9

I: ...and then you said "it mirrors itself". Now what does that mean?

M: It halves itself down there ... this way [points between the two triangles on the far left of the page]

I: So do you mean each one of these ... each 'pair' ...[pointing at the two triangles on the far left of the page] or the whole pattern [uses arm and 'folds' it over to indicate the reflection of a row].

M: Um ... oh ... that one mirrors that one [first 'pair'], that one mirrors that one [second 'pair'] ...and so on ...so they all mirror the same way.

The horizontal reflection symmetry detected by Mary was in each base pattern, and not the pattern as a whole. The translation seems to have been fairly explicit also by the phrase "and so on". In another pattern description, (p1a1), she used the phrase "and it keeps on going". This suggests that she may have imagined the pattern extending beyond the confines of the page, but further questioning revealed that this was not so.

pm11 [PS5]

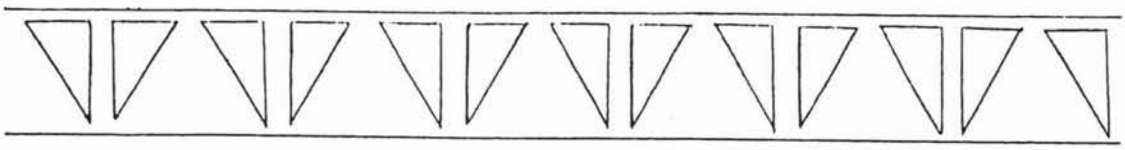


Figure 6.10

Alice wrote "the long straight bits are pointing at each other". She explained:

A: Those ones there [triangles in first 'pair'] are facing each other ... like a mirror.

Alice's written comment that "All of it looks like a bird's beak" referred to the base pattern since she explained by pointing to each 'pair', labelling them "bird's beak, bird's beak, bird's beak, ..." Her description, classified as undifferentiated transformation (relative direction), when more fully explained intended to convey a vertical reflection.

pla1 [PS2]

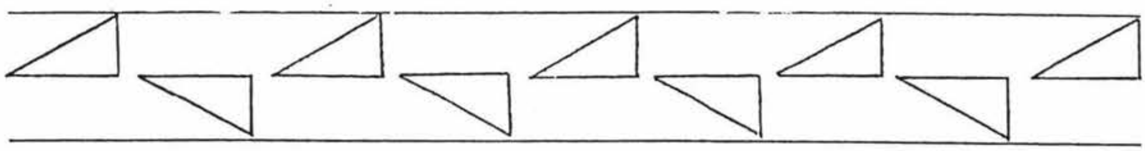


Figure 6.11

I: Why is it "fancy".

C: Well .. this is more of a ... more of a harder one to do ...

I: Right.

C: 'Cause you go up down up down up down (enthusiastically).

Some alternating property has attracted Carla to this pattern. It is still difficult to tell if the expression "go up down ..." implies a perception of the figures changing direction or an undifferentiated transformation (relative orientation). The word "fancy" seems to mean that the pattern was complex, making it hard to draw or describe.

pl12 [PS7]

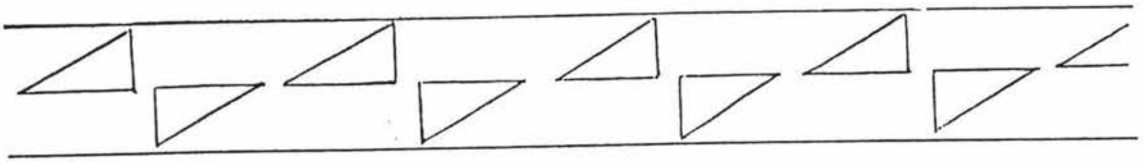


Figure 6.12

I: What do you think is repeating in this pattern?

R: Well they're all repeating each other. That one's repeating that one ... and that one's repeating that one ... and these ones are the same ... up the top ... they're repeating.

I: Oh ... so you think that the ones down the bottom are repeating and the ones up the top are repeating?

R: Yeah ... but those ones [triangles in bottom row] aren't repeating the top ones.

This 'rows' perception was common in the interviews and the written descriptions. It seems that half-turn was not often used to explain this pattern. Instead, the translation of triangles in each row was explicit. It is almost as though two separate patterns were being described (which is why the corresponding constructions are classified as the superposition of rows).

pmm2 [PS12]

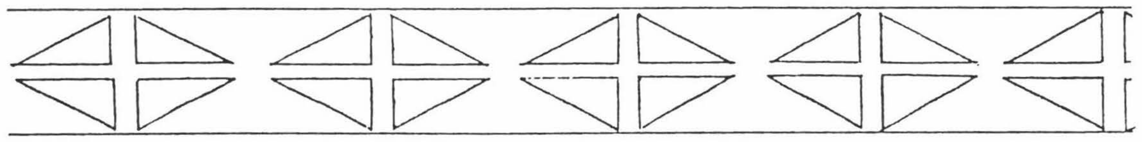


Figure 6.13

I: Anything else you can tell me about the pattern?

R: It looks like the sign off the "Dukes of Hazard"'s car ... um ... it looks like bunches of triangles bunched together.

I: How are they bunched together?

R: Equally apart ... all the same size ... and they repeat each other. That's all I can think of.

I: What if you look at it this way? (turns the pattern 90°)

R: Looks like the crafts off "Star Wars."

Richard's comparisons did not seem to have any explicit transformations in them, apart from translation. This was not surprising given the high percentage of students who wrote implicit transformation geometry descriptions.

pma2 [PS9]

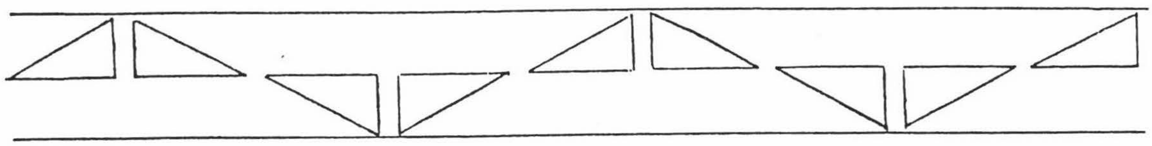


Figure 6.14

To illustrate the fact that the same expressions can be used to mean two different things, consider the phrase "turned around". For Carla it probably meant a half-turn. In her description she wrote "one right-angle[1], one left angle[2] (down), one right angle[3] turned around, one left angle[4] turned around."

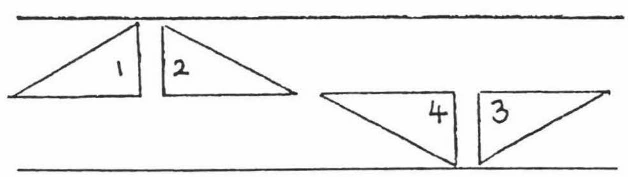


Figure 6.15

After she explained that the joining of the first two triangles (1 & 2) would make another triangle, she turned the page and showed how the pair would then "go to those ones" (3 & 4). This apparent intuitive half-turn was checked. The page was returned to its original orientation, and Carla was asked to explain it again without turning the page. Again she used the expression "turned around" and the 'mapping' was consistent with the previous explanation.

On the other hand, Mark meant a flip (as a component of a glide reflection).

> I: Is that one the same as that one? [comparing first 'pair' with second 'pair']
>
> M: Oh ... yes ...but if that one there was turned around ... it'd be the same as that one.
>
> I: Show me what "turned around" means.
>
> M: [Marks turns his hand over] You know ... flipped over.

Among other things, Mark has recognised congruence where he had not written it before. Both children have used explicit differentiated transformations in their oral explanations. In Mark's case, his written description had been implicit. In Carla's case, her written expression was undifferentiated. Carla response was particularly surprising given the rarity of any explicit half-turn in the written explanations.

Some Concluding Remarks

The use of explicit transformations sometimes became more obvious in discussion than in the written descriptions. It may well be that an intuitive sense of symmetry was a component of some of the Primary children's perception, but other aspects of the patterns strike them as more important to write down.

It also seems clear that once a translation unit has been described, it was seen as a sufficient description by most subjects interviewed. Indeed, the 'pattern' *is* the base pattern according to some subjects; the repetition was simply obvious and not worth mentioning. In this case, translation was intuitive, even though it wasn't explicitly stated. Related to this point, and equally lucid, is the conclusion that the children's perception of the patterns was *finite*, limited to the design on the page, and no more. For instance, even when a description included a comparison of the 'whole' to another object, it often had the sense of a *finite* 'whole'. This may well be the case for many of the students in the older age groups, but this consideration is not pursued in this thesis.

Let us reiterate a familiar theme: A feature of geometric intuition is that a child may have used or recognised a transformation, but not in a way that distinguished it from another transformation. Such examples are termed *undifferentiated transformations* as noted above. It appears, from the interviews, that the most common example of this phenomena was the use of the expression "turn upside down" to denote the result of a half-turn, glide or horizontal reflection alike. There are mathematical as well as psychological reasons for this occurring, as explained in subsection 4.3.4.

In some case studies, the base pattern of the pmm2 pattern was discussed. Interviewing revealed that a child who described the relationship of the lower triangle to the upper figure as "turn it upside down" often made no distinction between reflection or rotation by this expression. (e.g., Richard said: it's the same thing!") And indeed, this pattern has both horizontal reflection and half-turn symmetries, whose effects are identical (in this case). Lesh's (1976) comment that children may focus on the end product and not the 'process' of a transformation appears to be supported by this observation.

However, often the same child still thought in terms of 'upside down' when the transformed figure itself had no reflection symmetry. It then becomes difficult to know whether the lack of transformation distinction was due to the nature of the object (i.e., an orientation judgement) or a faulty generalisation from the example sketched in the previous paragraph. It *may* be that half-turns and horizontal or glide reflections are grouped together in the child's cognitive structures under the heading of 'turn upside down'. Given the large proportion of mono-orientated objects in a child's world which have vertical reflection symmetry in them (Howard, 1982), it would not be surprising if this generalisation has been made by some children.

6.3.2. Matching Frieze Patterns

Tables of matches are given below. We also examine some of the *criteria* used to match frieze patterns, as described in chapter 3. Obviously, some of the children's responses may or may not be correct with respect to symmetry groups, but as before, the criteria, rather than the correctness of perception, are what concern us here. All of the matching criteria described in chapter 3 occurred in the interviews, but in order to preserve brevity not all appear in the extracts below. Instead, a few 'typical' responses are displayed.

Comparison of the Seven Discrete Frieze Pattern Types

In this matching activity, the students were given 7 cards with a different frieze group on each card (see appendix C1). They were asked to indicate any pair of patterns which seemed similar to them and to try to explain the similarity if they could. The results in Table 6.4 give the number of affirmative responses for each possible match. Comments and interview extracts follow.

Table 6.4

	p111	p1m1	pm11	p1a1	p112	pmm2	pma2
p111	N/A	0	3	0	0	0	0
p1m1		N/A	10	1	0	2	0
pm11			N/A	0	0	0	1
p1a1				N/A	10	0	2
p112					N/A	0	0
pmm2						N/A	10
pma2							N/A

The three matches common to all ten children interviewed were p1m1 [PS3] with pm11 [PS5], p1a1 [PS2] with p112 [PS7] and pmm2 [PS12] with pma2 [PS9]. One or two typical interview extracts are given for each.

1. p1m1 [PS3] and pm11 [PS5]

> I: So you think patterns 2 and 3 look quite similar?
>
> M: Yep.
>
> I: And why's that?
>
> M: Those two there [first pair of triangles on left of pattern 2] are pointing that way, and those [first pair of triangles on left of pattern 3] are pointing downwards ...
>
> I: Uh-huh.
>
> M: But they still mirror each other.

Mary has recognised the reflection within a base pattern of triangles for each pattern (E), as did several of the other children Note also that the symmetry of the whole was not mentioned. It seems that the pattern was perceived as the translation unit - the base pattern. A direction or orientation judgement (D1) has been used to distinguish the patterns. Similar responses, combining relative orientation evaluations (D2) or comparisons with another object (e.g., an arrowhead) and orientation judgements, were common components of this match. As noted above, several students noted the similarity between the two patterns in their written descriptions using the same criteria.

2. p1a1 [PS2] and p112 [PS7]

(i) M: Um ... all of those corners are pointing that way ...

> I: Uh-huh.
>
> M: But ... um ... this one [bottom row of pattern 5]... those corners there are pointing to each other ... the ones above them.

(ii) I: Okay, why did you choose those two?

T: Because they've both got two rows of right-angle triangles ... and that ... the top one's exactly the same but the only change is the bottom line ... 'cause the slope in the bottom line is sloping down in 5 ... and up ... it's sloping down in 4 but in the other direction.

These were very common responses, often stated on paper as well. When *comparing* the patterns, the children tended to perceive them as two rows or as a series of base patterns, describing each with positional (2), orientation judgements, or evaluations based on relative direction. Very little synonymous language to transformation geometry was used.

3. pmm2 [PS12] and pma2 [PS9]

(i) M: Well they're the same up the top and then they're the same down the bottom.

(ii) Al: 'Cause if you put that one onto that one ... it will sort of look similar to that one ... 'cause ... if you moved that over there it would be like that one.

Alice was saying that by sliding the top row of pattern 6 by an appropriate amount it will look like pattern 7, or alternatively, by sliding the bottom row of pattern 7 by an appropriate amount it will look like pattern 6. Mary's comparison was less dynamic. She perceived a similarity by the regular deletion of the appropriate triangles in pattern 6. In this match, the criteria she uses for similarity is probably positional only (B3), while Alice's response actually described a translation (E) of a row. But in neither case do they verbalise any perceived common reflections, glide reflections or rotations. This was also true of those students who noted the similarity of these two patterns on the written survey.

4. p111 [PS1] with pm11 [PS5]

I: Okay ... why do you think they're similar?

T: Um ... because if you get pattern 1 here and ... put that ... that one there is exactly the same to that one there ... this is just pattern 1 with a reverse triangle beside it ... it's like a mirror image.

This time a symmetry group mismatch has been made using gaps (C2), relative orientation (D2), and quite clearly, symmetry (E).

Comparison of Two Different Sets of Discrete Pattern Types

The original set of discrete (green) patterns were used, and the new set for comparison were on red cards (see appendix C2). Each symmetry group is correspondingly prefixed by G (green) or R (red)

Table 6.5

	Gp111	Gp1m1	Gpm11	Gp1a1	Gp112	Gpmm2	Gpma2
Rp111	10	0	0	0	0	0	0
Rp1m1	0	10	0	1	0	0	0
Rpm11	0	0	8	0	1	0	2
Rp1a1	0	0	0	10	7	0	1
Rp112	0	0	1	2	8	0	0
Rpmm2	0	0	0	0	0	10	0
Rpma2	0	0	0	0	0	0	10

The predominant technique employed for the common matches, namely the corresponding frieze groups and Rp1a1 with Gp112, was to note the 'congruence' of the motifs used in the red and green patterns, viz., the "F" and the triangle. For example, Mark explained how to change an "F" into a triangle: "if you just take that one away and then draw a line to the corners it would make a right-angle [triangle]." Usually some combination of evaluations based on orientation or direction judgments, 1-1 correspondence, or proximity was used to complete the intuitive match.

Consider Kate's match of Rp1a1 with Gp1a1. She held Rp1a1 in her hand and tried to find a match. Initially she chose Gp112, a common 'mismatch' of symmetry groups.

> K: This one [p112] ... um cause of the way it's arranged ... the way it's arranged looks similar.
>
> I: Uh-huh. ... Does it look similar to anything else, or not really?
>
> K: Now it sort of looks similar to pattern 4 [p1a1] because you've turned it around.

I: Okay, so if you could turn it any way you want, which of those two patterns does it look most like?

K: Probably this one.

I: Probably "4". Why?

K: Well ... um ... for one reason this one here's sort of chopped off at the back, that one isn't ... and 'cause of the way their facing at the moment ... they're sort of spaced out the same ... I think that's about all actually.

Kate has clearly not based her match on transformations or symmetry but a combination of direction (D1), spacing (C2) and extremities (B1). Her initial 'mismatch' is mainly due to criteria C; evaluations based on the general shape or composition of parts. On this basis, it is not surprising that Kate, like all the other interviewees, found the matching with the filamentary patterns more difficult. The use of the phrase "this one's sort of chopped off at the back" to explain her matching also strongly suggests that she has not intuited the pattern extending beyond the confines of the page (let alone an infinite strip).

This is not say that all matches were made without an intuitive use of transformation geometry. Kate recognised the vertical reflection in Gpm11 and matched Bpm11 to it using that criteria.

K: Well ... they're back to back ... like a mirror (points to first 'pair' of triangles in Gpm11)

I: Uh-huh.

K: And so are they (points to first pair of "F"'s in Gpm11).

I: And what about if I turn this one this way? (turns Bpm11 180^{o})

K: No, because now they're upside down.

While Kate has clearly used an explicit transformation criteria for her match, the orientation of the individual shapes still appears to have been a more dominant feature of her perception of the patterns. However, until Kate was asked, she hadn't mentioned the orientation of the base patterns. This indicates that some perceptions were probably not verbalised because they seemed obvious to the subject.

Comparison of a Set of Discrete Pattern Types with a Set of Filamentary Patterns

The set of discrete pattern types are the originals, prefixed as before by G (green); the filamentary patterns are prefixed by B (beige). See appendix C3 for the list of 'beige' patterns.

Table 6.6

	Gp111	Gp1m1	Gpm11	Gp1a1	Gp112	Gpmm2	Gpma2
Bp111	4	2	1	2	2	2	2
Bp1m1	0	9	0	0	0	1	1
Bpm11	2	2	7	1	0	2	3
Bp1a1	0	0	0	7	3	0	1
Bp112	5	0	0	5	6	0	0
Bpmm2	0	1	1	0	0	9	0
Bpma2	0	0	0	2	2	1	8

As before, some of the common matches will be illustrated with interview extracts and complemented with brief discussion. It is clear from comparing tables 6.5 and 6.6 that frieze group mismatches were more common in the filamentary/discrete comparison activity (44) than in the discrete/discrete comparison activity (15). From the interviews above it seems that the use of transformation criteria was not usually *dominant* in the discrete/discrete case, and this conclusion is amplified by the results of the filamentary/discrete matching activity where several of the other factors (B, C, and D) such as 'pointiness', were deliberately reduced. Many subjects still tried to use these 'eliminated' criteria and, correspondingly, the children reported having more difficulty and took far longer to decide on the matches than for the previous activity.

The general style of matches for this activity was to try to find corresponding 'lines' or 'points' with the correct orientation. The children often imagined parts of the filaments as triangles to do this.

1. Bpm11 with Gpm11

I: Okay, does it look like any others?

Ad: It could be pattern 3.

I: It could be pattern 3?

Ad: Yep ... 'cause they're all pointing downwards.

I: How about pattern four?

Ad: No.

I: Any others?

Ad: It could be the bottom of pattern 6.

I: Yeah.

Ad: Or the bottom of pattern 7.

I: So if you had to choose, which one would you pick?

Ad: It looks most like pattern 3 ... I think ... when it's upside down.

The 'points' were still used to make a match. Also, the orientation (non-relative) was important to Aden. He concentrated on the triangular spaces in pattern 3 as oppose to the the actual shapes (C2).

2. Bpmm2 with Gpmm2

I: You did that quite quickly. You must be confident. Why do you think they're similar?

R: Oh ... well ... because you can imagine those four fitting into there.

I: Uh-huh ... any other reason?

R: Well they're sort of grouped together like a diamond ... and ... um ... well ... those are ... a *bit* like a diamond.

The matching criteria in this case is, in Piaget's and Inhelder's terms, based on the general shape or composition of the parts; in particular, an evaluation based on the shape or global area. In terms of the description criteria from section 6.1 and 6.2, Richard has used an implicit form of transformation geometry by comparing a pattern part to another real world object.

The two most common 'mismatches' were Bp112 with Gp111 and with Gp1a1. In the first case, the 'triangles' formed by the filament were seen along the top of the strip but not the bottom. They were then matched with pattern one's triangles because they were 'single' triangles. "In the other patterns they have twos" said Rachel. Sometimes the 'wrong' orientation was mentioned. For instance, Kate noted that "They're not the quite the right way round though." In the second case, the common 'alternating' property, that is., the 'up-downness' of both patterns was the dominating feature. Again, 'points' were often used. This mismatch was not surprising given the number of students who thought Gp1a1 and Gp112 looked very similar. (Recall the student who wrote "this one is trying to trick us into thinking they're the same).

Bp111 was often matched with a different 'Green' symmetry group. The interviewees often noted a similarity to part of another pattern. One of the mitigating reasons for this phenomena appears to have been the orientation of the 'potential triangles' in the filamentary pattern Bp111. Each 'triangle' is 90^{o} anti-clockwise to the orientation of the triangles in Gp111. Hence they " face the wrong way" or "are around a different way." All the 'Green' patterns have subsets with triangles in the same orientation (except Gpm11, which also has only one match).

A Final Comment

In most children's explanations of their matches, evidence of transformation geometry appeared to be lacking. This was most noticeable with the discrete/filamentary exercise. Perhaps a conjecture can be made: *If* (a) transformations or symmetries are detected more easily between proximate or connected objects as articles in Lesh and Mierkiewicz's

(1978) monograph indicated, and (b) the students seem to consider 'the pattern' as the base pattern as demonstrated in several ways above, then it seems likely that children making matches with *filamentary* patterns will not yield explanations with a strong transformation geometry component.

6.4 Summary

6.4.1 Primary Survey Results

The use of description criteria varied between the frieze groups. Explicit transformations were not common for any of these groups; roughly one in five children used phrases indicating translation; reflection was mentioned even less than this. Rotation and glide reflection were rare. For Primary students, it seems that *using* transformations to make patterns is more intuitive than *describing* patterns in transformation 'language'.

There were many clear signs that the children perceived the base pattern as 'the pattern', and that the design presented was restricted to the set of figures and lines actually drawn on the page.

6.4.2 Age Group Survey Comparison

Primary students commonly used implicit transformation geometry by comparing pattern parts with real world objects. The proportion of Secondary students using implicit symmetry was smaller, whilst a clear increase in explicit transformation geometry occurred. The proportion of Tertiary students using implicit transformation geometry was the same for the Secondary students, with slight increases in explicit transformations and a few other categories. In short, as the age of the subjects increased, the description of transformations became more explicit up to formal operations stage, after which a greater scope of descriptions occurred. The use of implicit transformation geometry criteria by comparing the 'whole' pattern was not particularly common for any age group.

6.4.3 Interview

The oral explanations the children gave of their written descriptions often brought out more explicit transformation geometry phrases. The children appeared to have seen more than they recorded on paper which, given their writing skills, does not seem surprising.

A common description of the pattern was to describe the base pattern only, often using a comparison to a real world object or another pattern's translation unit. The translation of the base pattern, according to the interview material, is so obvious to the students that they don't mention it.

The interviews also suggested that the students conceived the designs displayed as precisely those points etched on the page (as opposed to an infinite extension in the manner of a frieze group). This was consistent with the results of the written descriptions.

Matching Activities

From the interview results, it seems clear that transformation geometry was not extensively verbalised as a criteria for matching frieze patterns. Other characteristics such as position, orientation and direction of points and lines were dominant in the descriptive domains of the Primary subjects interviewed. One of the strongest pieces of evidence of this was the lack of recognition of a match when a 'Red' frieze group was orientated 180^{o} to a 'direct match' so that a 1-1 correspondence of points and lines and their respective positions and directions didn't exist. For example, "no it's not the same ...but it's like pattern 2 if you turn it up the other way" (Rp1m1 and Gp1m1). A few examples occurred with transformation geometry, all with reflection, and of these most were vertical reflection. Secondly, when some of these factors were eliminated in the filamentary/discrete activity, the children showed both qualitative and quantitative signs of difficulty. If transformation relationships or symmetry had been the only salient matching criteria, this difference would probably not have been as pronounced.

7 *Conclusions*

"Transformations are like putting your shoes and socks on: it makes a difference the order which you do it in." (Anonymous)

7.1 A Review

7.1.1 An Introductory Note

The task of this thesis has been to identify and explore the character of intuitive transformation geometry concepts in the construction or perception of frieze patterns (bands with translation symmetry in one direction only). In part, the motivation to explore this topic arose from the particular relevance that transformation geometry has to New Zealand: *kowhaiwhai* (Maori rafter patterns) are examples of frieze patterns and are suggested for use at the Form 3 and 4 levels by the New Zealand *Syllabus for Schools* booklet (1987) and by the more recent draft syllabus.

The importance of learning geometry and, in particular, the value of the transformation approach was highlighted by the literature review in chapter 2. The education literature suggested that particular transformations seem to be learnt in a certain order (e.g., translation, then reflection, then rotation), although the character of the conceptualization of these transformations was still unknown. From the perceptual psychology literature it was found that vertical reflection symmetry is a salient feature of the perceptual process in many cases. The perception of figures with a single axis of symmetry is affected by a number of variables such as spacing, figure complexity and direction of the mirror line. Translation, or repetition, is also detected quite easily, although it is often described in asymmetric terms by psychologists. Unfortunately, it appears that little is known about the perception of rotation or glide reflection symmetries. Furthermore, the amount of research on the perception (or use) of transformations or symmetry in *patterns* is very small indeed, providing additional impetus and motivation for this thesis.

The General Approach

The approach to this present study has been two-pronged, using both survey and case-study research methods with Standard 3 and 4 children. For comparison, surveys were also conducted with Secondary and Tertiary students. Respectively, these three age groups correspond to the times before, during and after exposure to formal transformation geometry concepts.

The construction activities were of two types: unrestricted (in which the subject could draw any frieze pattern they wished) and restricted (in which the subject could only use right-angled scalene triangles to make their frieze patterns with). In the interviews, the intentions and the order of these constructions were explored. The perception activity required the subjects to describe seven different patterns, each pattern having a different set of symmetries to the others. The interviews investigated any further unwritten understandings of the patterns which the children had, and clarifications of the meaning of phrases were sought. In addition, the interviewed subjects performed a matching task, and the criteria for matching was described.

7.1.2 Results of the Unrestricted Pattern Construction Activity

Survey

An ordering of the (relative 'intuitive-ness' of the) seven frieze groups in the unrestricted pattern construction was made from the results of four 'measures'. This list is shown below for the Primary and Secondary subjects (1 indicates a popular and frequently drawn pattern type; 6 indicates an uncommon and rare pattern type).

Table 7.1

	Primary	Secondary
1	p111, pm11	p111, pm11
2	pmm2	pmm2, pma2
3	pma2	-
4	p112	p112
5	p1m1	p1m1
6	p1a1	p1a1

By comparing the numeracy and popularity of the Primary frieze patterns with translation and one other symmetry, an order of the relative 'intuitive-ness' of the four transformations was made. Translation has been placed at the top because repetition was used and understood easily by virtually all of the students. Perhaps the most surprising observation is the low use of horizontal reflection *by itself* in frieze patterns.

1. Translation (most frequent and common)
2. Vertical Reflection.
3. Half-Turn
4. Horizontal Reflection.
5. Glide Reflection (infrequent and uncommon)

However, the implications of these results to the intuitive use of transformations in frieze patterns with more than one kind of symmetry (besides translation) was argued to be questionable; combinations of symmetries can yield other unintended symmetries.

All the symmetry groups, apart from pma2, were most commonly drawn as disjoint for the Primary group. Indeed 75% of all the Primary students' patterns were disjoint, whereas the proportion of disjoint to connected patterns made by Form 4's was almost equal. This result *suggests* that the Primary students favoured a 'pattern parts' construction over a 'whole' construction, whereas the Form 4's used both approaches equally.

Interviews

The quantitative analysis of the 'unrestricted' patterns had several shortcomings however. Some symmetries identified weren't intended by the creators; these accidents occurred due to arbitrary choices of spacing or positioning of figures within a pattern. Incidental symmetries, resulting as a spin-off from the use of other transformations, also occurred. Conversely, the use of transformations not included in the elements of frieze groups was noted in the interviews. Another complication in frieze group analysis is that it requires the pattern to be extended infinitely, an extension probably not imagined by most, if not all, of the Primary subjects.

Several different construction methods were identified in the interviews, each method having its own particular use of symmetry operations and associated understanding of those operations. The most common method of making a frieze pattern was a *base pattern* construction, in which a motif is designed and then repeated along the strip. Other approaches included the use of incidence relations (the repeated application of a transformation on a motif to generate a pattern), tilings (building, or dividing, a strip with congruent parts) and filamentary techniques (curve sketching). Sometimes, combinations of these methods were used, resulting in a superimposition of two or more patterns.

7.1.3 Results of the Restricted Pattern Construction Activity

Survey

The restriction imposed on the subjects in activity (b) yielded some interesting results. The frieze pattern which arose most naturally for *all three* age groups surveyed were p111, pm11 and p112. The most unusual frieze groups were p1m1 and p1a1. The occurrence of pmm2 and pma2 dropped from the previous exercise, thereby raising (somewhat unanswered) questions about the deliberate use of particular symmetries present in those patterns.

Based on the ordering of the 5 frieze groups with translation and one other kind of symmetry, the ordering of the rigid transformations with respect to intuition is similar to activity (a).

1. Translation (most intuitive)
2. Vertical Reflection
3. Half-Turn
4. Glide Reflection, Horizontal Reflection. (least intuitive)

The style analysis of the total number of patterns drawn, and of the separate frieze groups, revealed that the Primary and Tertiary groups were quite similar in the patterns they drew, and therefore in the construction methods use to produce them. The Form 4's patterns differed in several ways. Two characteristic features of the Secondary activity (b) patterns was the extensive use of half turn and also of tilings. Indeed, many p112 tilings occurred which have both of these features. It seems that the Fourth Form students have been affected by the formal transformation geometry framework which they have been recently exposed to, and thus their use of transformation geometry was probably not intuitive as the other age groups.

Interviews

Despite the restriction, the methods used to make the activity (b) patterns were very similar to those of activity (a). The base pattern construction, for example, was still the most commonly used. Of course, some notable omissions and additions were discovered in the interviews. Furthermore, the relative popularity of the various methods was probably a little different to that of activity (a), with the number of tilings increasing, and a reduction in the use of filamentary and incidence techniques

In general, the Primary school students' use of transformation geometry in activity (b) was more explicit than in activity (a). For example, in the base pattern construction, the use of symmetry within the base pattern appeared to be more intentional in activity (b). The explicit and implicit transformations associated with each method, and a comparison of actvity (a) and (b) methods, are described in sections 4.4, 4.5, and 5.4.

7.1.4 Results of the Pattern Description Activity

Primary Survey Results

A number of criteria for describing frieze patterns were identified. These criteria were:

A. Explicit differentiated and undifferentiated transformations.
B. Implicit 'parts' and 'whole' comparisons.
C. Orientation or Directional Judgements.
D. Position Judgements.
E. Miscellaneous.

The explicit description of transformations was not common for any of the groups; roughly one in five Primary students mentioned translation, and even less noted vertical reflections. Rotation, horizontal reflection and glide reflection were very rarely noted by the children. It seems that using transformations to make patterns is easier (to the children) than describing patterns in 'transformation' language. This does not seem surprising given the reflective nature of *verbalizing* one's apprehension of spatial relationships.

There were many clear signs that children and adults alike conceived the 'pattern' to be the base pattern (the motif which repeats to make the whole design), and that each design presented was perceived as the set of figures and lines actually drawn on the page; it was not imagined to continue indefinitely. This perception contrasts with the infinite extension assumed for frieze group classification. Secondly, the Primary school students' descriptions of a half-turn symmetry indicated a composition of reflections just as often as a rotation. More importantly, it appears that some transformations, such as horizontal reflection and half-turn, weren't easily distinguished by many Primary subjects.

Age Group Survey Comparison

Primary students commonly used implicit transformation geometry by comparing pattern parts with real world objects; they employed a form of metaphor or simile. As the age of the subjects increased, the descriptions of transformations became more explicit up to formal operations stage, after which a greater scope of descriptions occurred. The use of implicit transformation geometry by comparison of the 'whole' pattern was not particularly common for any group.

Interviews

The oral explanations the children gave of their written descriptions often brought out more explicit transformation geometry phrases. The children appeared to have seen more than they recorded on paper which, given their writing skills, does not seem surprising.

A common description of the pattern was to describe the base pattern only, often using a comparison to a real world object or another pattern's translation unit. The translation of the base pattern, according to the interview material, was so obvious to the students that they didn't mention it. The interviews also confirmed the notion that the students had conceived the designs displayed as precisely those points etched on the page; they did not appear to even consider the possibility of an infinite extension in the manner of a frieze group. This finding adds weight to the argument presented in this thesis that any symmetries perceived are usually between parts of a pattern, and not the pattern as a whole.

From the interview material, it seems clear that transformation geometry ideas were not extensively verbalised as a criteria for matching two frieze patterns, although many correct matches between frieze groups were made. Other characteristics such as position, orientation and direction of points and lines were dominant in the descriptive domains of the Primary subjects interviewed. If an example did include a transformation geometry explanation, it was always of reflection, and usually of vertical reflection.

7.2 Implications

Pedagogical Implications

Booth (1981), and Lesh (1976) have pointed to the importance of describing *intuitive* transformation geometry concepts, in an effort to provide teachers and researchers with further insight into the concepts that a child might bring with them into a classroom. By exploiting Gagatsis and Patronis' (1990) idea of using geometrical models as a way of moving from intuitive to reflective concepts, one would hope that effective and significant learning will take place in the classroom. Success with such methods has been reported by mathematics educators, as well as teachers in others areas such as art and music. For example, Booth (1985) used young children's spontaneous pattern painting as a way to link art to mathematics and found that substantial gains were made in the transformation geometry area. Indeed, with the advent of a new New Zealand mathematics curriculum, which emphasizes relevance and practical problem solving, the links between transformation geometry and other subjects (such as culture, art, and computing) seem particularly appropriate for students to explore.

If frieze pattern construction activities are used in the classroom, it *may* be worth asking students to align some of their patterns vertically and some horizontally (with respect to themselves). Although not conclusive, the results from the restricted construction activity suggested that symmetries may be used differently in each case; hopefully, this approach would result in the class having a wider variety of symmetries to explore.

One of the recurring themes of the two frieze pattern construction chapters was the noticeable difference between the *symmetry group* analysis, the *behaviours* used to produce a pattern, and the *child's intuitive understanding* of the process they used. It seems quite likely that children don't necessarily understand concepts in the same way as a teacher may deliver them. Clearly, students are not empty vessels into which knowledge can be poured. Therefore, if a teacher wishes to evaluate a child's understanding of transformations, it would seem that students should be encouraged to *talk* about their own understanding of transformations; observing either the process or the final product of a transformation activity may not be sufficient for effective assessment.

Implications for Research

In general terms, this thesis has reinforced three important points for research:

1. Mathematical structures must be used carefully when examining *intuitive* mathematical concepts. For example, while the employment of frieze group analysis was systematic and provided general clues about the extent to which various symmetries were used in the frieze pattern constructions, in several cases it was actually misleading. This also reinforces Grunbaum's (1984) warning to anthropologists about the use of symmetry classifications for analysing designs from various cultures.

2. Case studies seem to have been the most helpful research method for identifying the *character* of intuitive transformation geometry concepts. Uncovering the *process* of construction, rather than the product, was a useful focus for this aspect of investigation. For instance, determining the order of a pattern's construction provided a number of important insights. However, the most important approach was the questioning of a subject's understanding of the operations employed. For instance, base pattern and incidence constructions can result in the same product, and appear to use the same process (i.e., construction sequence), but are understood by the creator in two different ways.

3. Similarly, obtaining children's oral explanations of their written descriptions clarified the intended meaning of many phrases. It appears that children use qualitatively different criteria to adults for making mathematical judgements. Quite clearly, the type and range of criteria used for describing (discrete) frieze patterns varied between the Standard 3/4, Form 4, and Tertiary groups. For example, the idea of undifferentiated phrases and orientation judgements, common amongst the primary children in their frieze pattern descriptions, confirmed Lesh's (1976) suspicion that children do not *necessarily* conceive of transformation relationships as slides, flips and turns, or even motions.

The scope for further research of intuitive mathematical concepts appears to be large, particularly in the area of transformation geometry. Some suggestions are listed below:

1. An exploration of the character of the use intuitive transformation geometry in patterns with more than one type of symmetry, such as pmm2 and pma2 is desirable. More generally, a description of the intuitive perception and construction of 2-dimensional (wall paper) patterns would be useful.

2. The effect of a frieze pattern's alignment relative to its designer has only been briefly considered in this study. The suggestive results from the restricted activity indicate that it may be an area worth pursuing in order to establish whether students use transformations differently in various orientations of the strip.

3. In the mathematics education field, it would be helpful to find out *all* the variables affecting frieze pattern perception, since these factors probably affect how students learn transformation geometry from patterns. For instance, what role do symmetries, colour symmetries and non-rigid transformations play in this process? Are there any notable differences between cultures, genders, or personality types?

4. It may be more appropriate for students to learn and use other mathematical structures in describing patterns *before* using transformations or frieze groups. The development of such structures, for use at Primary level, is a potential area of research.

5. Standard 3 and 4 children have a tendency to describe pattern parts using simile or metaphor. Does this extend to the description of other mathematical phenomena, and if so, how can this be exploited in the classroom?

This project set out to identify some of the transformation ideas that a child may bring into a classroom before formal concepts are taught. There can be little doubt that the description of children's intuitive transformation concepts is not trivial. A variety of techniques are needed to explore this area, especially those based on the learner's conceptions of transformations rather than imposing a finished mathematical product as *the* framework to understand a child's cognitive structure. A child's intuitive understanding of transformations is often rich and apt. Rather than being disregarded, this understanding needs to be *built on* by the learning of reflective formal ideas.

Indeed, the frieze pattern construction activities appeared to be interesting to the students involved and resulted in a wide variety of uses of transformations. The pattern description exercise provided many clues about how transformations are perceived. Relevant, appropriate, revealing and motivating, there seems to be a strong case for making *frieze patterns* an integral part of the teaching and learning of transformation geometry.

Appendices

In appendices A and B, an asterisk in an 'average number made' column denotes that the average is calculated only for those who actually drew the particular category of pattern shown by the respective rows. In table A2.4 for example, the Secondary students who made a connected p112 produced about 2 of them on average, whilst the average over the whole group was only 0.8.

A blank indicates that an entry would be non-sensical, either because it would involve dividing by zero, or because the particular combination of column and row categories has no meaning.

Appendix A: Result Tables of Activity (a)

Table A1.1 - Frieze Group Results of the Primary Children's Unrestricted Pattern Construction

Frieze Group	Numeracy	% of Total Patterns	No. of Students	% of Students	Average No. Made*	Average No. Made
p111	371	27	86	87	4.3	3.7
p1m1	41	3	31	31	1.3	0.4
pm11	340	25	87	88	3.9	3.4
p1a1	3	0	3	3	1.0	0.0
p112	91	7	54	55	1.7	0.9
pmm2	254	18	75	76	3.4	2.6
pma2	110	8	62	63	1.8	1.1
FG Unclear	97	7	48	48	1.8	1.0
No T. Sym	78	6	20	20	3.9	0.8
Total	1385	100	99	100		14.0
Different FGs	398					4.0

Table A1.2 - A Frequency Breakdown of the Primary Children's First Three Unrestricted Patterns

Frieze Group	% Drawn 1st	% Drawn 2nd	% Drawn 3rd	% of Total Patterns
p111	26	30	31	27
p1m1	1	2	1	3
pm11	27	21	25	25
p1a1	0	0	1	0
p112	9	8	6	7
pmm2	15	17	18	18
pma2	10	8	6	8
Can't Classify	11	13	10	13

Table A1.3 - A 'Kind' Classification of the Primary Children's Unrestricted Pattern Construction

Kind	No. of Patterns	% of Total Patterns	No. of Students	% of Students
Disjoint	1040	75	96	97
Connected	279	20	54	55
Unsure	65	5	39	39

Table A1.4 - Frieze Group/Kind Results of the Primary Children's Unrestricted Pattern Construction.

This table displays a refinement of the frieze group analysis of the unrestricted pattern construction. Note: no pattern can be judged as displaying no evidence of translation symmetry *and also* be sensibly classified as 'connected'; the design displayed may be an element of a larger discrete pattern, or it may simply be a finite design.

Frieze Group	Kind	No. of Kind	% of FG	% of Total Patterns	No. of Students	% of Students	Av. No. Made*	Av. No. Made
p111	Disjoint	344	93	24.8	81	82	4.2	3.5
	Connected	17	5	1.2	14	14	1.2	0.2
	Unsure	10	2	0.7	8	8	1.3	0.1
p1m1	Disjoint	39	96	2.8	30	30	1.3	0.4
	Connected	1	2	0.1	1	1	1.0	0.0
	Unsure	1	2	0.1	1	1	1.0	0.0
pm11	Disjoint	245	72	17.7	80	81	3.1	2.5
	Connected	89	26	6.4	47	47	1.9	0.9
	Unsure	6	2	0.4	5	5	1.2	0.1
p1a1	Disjoint	1	33	0.1	1	1	1.0	0.0
	Connected	2	67	0.1	2	2	1.0	0.0
	Unsure	0	0	0.0	0	0		0.0
p112	Disjoint	57	63	4.1	41	41	1.4	0.6
	Connected	19	21	1.4	14	14	1.4	0.2
	Unsure	15	16	1.1	13	13	1.2	0.2
pmm2	Disjoint	179	70	12.9	67	68	2.7	1.8
	Connected	60	24	4.3	34	34	1.8	0.6
	Unsure	15	6	1.1	12	12	1.3	0.2
pma2	Disjoint	37	34	2.7	27	27	1.3	0.4
	Connected	65	59	4.7	40	40	1.7	0.7
	Unsure	8	7	0.6	6	6	1.3	0.1
FG Unclear	Disjoint	65	67	4.7	38	38	1.7	0.7
	Connected	26	27	1.9	18	18	1.4	0.3
	Unsure	6	6	0.4	6	6	1.0	0.1
No T. Sym	Disjoint	72	92	5.2	18	18	4.0	0.7
	Connected							
	Unsure	6	8	0.4	4	4	1.5	0.1
Total		1385		100.0	99	100		14.0

Table A2.1 - Frieze Group Results of the Secondary Class's Unrestricted Pattern Construction.

Frieze Group	Numeracy	% of Total Patterns	No. of Students	% of Students	Average No. Made*	Average No. Made
p111	60	20	18	95	3.3	3.2
p1m1	12	4	5	26	2.4	0.6
pm11	71	24	18	95	3.9	3.7
p1a1	0	0	0	0		
p112	33	11	13	68	2.5	1.7
pmm2	64	21	17	89	3.8	3.4
pma2	36	12	14	74	2.6	1.9
Unclear	23	8	11	58	2.1	1.2
No T. Sym	2	1	2	11	1.0	0.1
Total	301	100	19	100		15.8
Different FGs	85					4.5

Table A2.2 - A Frequency Breakdown of the Secondary Class's First Three Unrestricted Patterns

Frieze Group	% Drawn 1st	% Drawn 2nd	% Drawn 3rd	% of Total Patterns
p111	21	16	11	20
p1m1	5	0	0	4
pm11	11	26	32	24
p1a1	0	0	0	0
p112	16	11	11	11
pmm2	5	26	32	21
pma2	26	21	11	12
Can't Classify	16	0	5	9

Table A2.3 - A 'Kind' Classification of the Secondary Class's Unrestricted Pattern Construction

Kind	Numeracy	% of Total Patterns	No. of Students	% of Students
Disjoint	147	49	18	95
Connected	143	48	19	100
Unsure	11	4	6	32

Table A2.4 - Frieze Group/Kind Results of the Secondary Class's Unrestricted Pattern Construction.

Frieze Group	Kind	No. of Kind	% of FG	% of Total Patterns	No. of Students	% of Students	Av. No. Made*	Av. No. Made
p111	Disjoint	43	72	14.3	14	74	3.1	2.3
	Connected	17	28	5.6	9	47	1.9	0.9
	Unsure	0	0	0.0	0	0		0.0
p1m1	Disjoint	7	58	2.3	5	26	1.4	0.4
	Connected	4	33	1.3	2	11	2.0	0.2
	Unsure	1	8	0.3	1	5	1.0	0.1
pm11	Disjoint	41	58	13.6	15	79	2.7	2.2
	Connected	30	42	10.0	14	74	2.1	1.6
	Unsure	0	0	0.0	0	0		0.0
p1a1	Disjoint	0		0.0	0	0		0.0
	Connected	0		0.0	0	0		0.0
	Unsure	0		0.0	0	0		0.0
p112	Disjoint	14	42	4.7	10	53	1.4	0.7
	Connected	15	45	5.0	8	42	1.9	0.8
	Unsure	4	12	1.3	3	16	1.3	0.2
pmm2	Disjoint	32	50	10.6	15	79	2.1	1.7
	Connected	30	47	10.0	13	68	2.3	1.6
	Unsure	2	3	0.6	2	11	1.0	0.1
pma2	Disjoint	3	8	1.0	3	16	1.0	0.2
	Connected	33	92	11.0	14	74	2.4	1.7
	Unsure	0	0	0.0	0	0		0.0
FG Unclear	Disjoint	5	22	1.7	4	21	1.3	0.3
	Connected	14	61	4.7	3	16	2.0	0.7
	Unsure	4	17	1.3	7	37	1.3	0.2
No T. Sym	Disjoint	2	100	0.7	2	11	1.0	0.1
	Connected							
	Unsure	0	0	0.0	0	0	0.0	0.0
Total		301		100.0	19	100		15.8

Appendix B: Result Tables of Activity (b)

Tables in appendix B display the results of the restricted frieze pattern construction for each age group surveyed: Primary (B1), Secondary (B2) and Tertiary (B3).

Table B1.1 - Frieze Group Results of the Primary Children's Restricted Pattern Construction

Frieze Group	Numeracy	% of Total Patterns	No. of Students	% of Students	Av. No. Made*	Av. No. Made
p111	342	42	86	87	4.0	3.5
p1m1	14	2	12	12	1.2	0.1
pm11	169	21	69	70	2.4	1.7
p1a1	15	2	13	13	1.2	0.2
p112	70	9	42	42	1.7	0.7
pmm2	33	4	20	20	1.7	0.3
pma2	40	5	28	28	1.4	0.4
FG Unclear	44	5	23	23	1.9	0.4
No T. Sym	81	10	26	26	3.1	0.8
Total	806	100	99	100		8.1
Different FGs	270					2.7

Table B1.2 - A Frequency Breakdown of the Primary Children's First Three Restricted Patterns

Frieze Group	% Drawn 1st	% Drawn 2nd	% Drawn 3rd	% of Total Patterns
p111	36	47	41	42
p1m1	1	0	0	2
pm11	26	23	26	21
p1a1	4	4	0	2
p112	14	8	6	9
pmm2	3	2	3	4
pma2	5	4	6	5
Can't Classify	10	11	14	15

Table B1.3 - A 'Style' Classification of the Primary Children's Restricted Pattern Construction

Style	Numeracy	% of Total Patterns	No. of Students	% of Students
Discrete	350	51	88	89
Non-Discrete	185	27	67	68
Touchings	47	7	26	26
Tiling/Filamentary	32	5	17	17
Unsure (?)	69	10	27	27

Table B1.4 - Frieze Group/Style Results of the Primary Children's Restricted Pattern Construction

Frieze Group	Style	Numeracy	% of F G	% of Total Patterns	No. of Students	% of Students	Av. No. Made*	Av. No. Made
p111	PS1	147	43	18.2	61	62	2.4	1.5
	NDP	159	47	19.7	57	58	2.8	1.6
	Tou	19	6	2.4	14	14	1.4	0.2
	Til/Fil	1	0	0.1	1	1	1.0	0.0
	?	16	5	2.0	9	9	1.8	0.2
p1m1	PS3	5	36	0.6	4	4	1.3	0.1
	PS4	4	29	0.5	3	3	1.3	0.0
	NDP	0	0	0.0	0	0		0.0
	Tou	4	29	0.5	4	4	1.0	0.0
	Til/Fil	0	0	0.0	0	0		0.0
	?	1	7	0.1	1	1	1.0	0.0
pm11	PS5	47	28	5.8	30	30	1.6	0.5
	PS6	54	32	6.7	28	28	1.9	0.5
	NDP	17	10	2.1	12	12	1.4	0.2
	Tou	20	12	2.5	13	13	1.5	0.2
	Til/Fil	0	0	0.0	0	0		0.0
	?	31	18	3.8	15	15	2.1	0.3
p1a1	PS2	14	93	1.7	12	12	1.2	0.1
	NDP	1	7	0.1	1	1	1.0	0.0
	Tou	0	0	0.0	0	0		0.0
	Til/Fil	0	0	0.0	0	0		0.0
	?	0	0	0.0	0	0		0.0
p112	PS7	29	41	3.6	23	23	1.3	0.3
	PS8	13	19	1.6	8	8	1.6	0.1
	NDP	5		0.6	5	5	1.0	0.1
	Tou	3	4	0.4	2	2	1.5	0.0
	Til/Fil	18	26	2.2	11	11	1.6	0.2
	?	2	3	0.2	2	2	1.0	0.0
pmm2	PS12	1	3	0.2	1	1	1.0	0.0
	PS13	0	0	0.0	0	0		0.0
	PS14	2	6	0.2	2	2	1.0	0.0
	PS15	18	55	2.2	14	14	1.3	0.2
	NDP	1	3	0.1	1	1	1.0	0.0
	Tou	0	0	0.0	0	0		0.0
	Til/Fil	3	9	0.4	3	3	1.0	0.0
	?	8	24	1.0	6	6	1.3	0.1
pma2	PS9	6	15	0.7	4	3	1.5	0.2
	PS10	4	10	0.5	4	4	1.0	0.0
	PS11	6	15	0.7	5	5	1.2	0.2
	NDP	2	5	0.2	2	2	1.0	0.2
	Tou	1	3	0.1	1	1	1.0	0.0
	Til/Fil	10	25	1.2	10	10	1.0	0.1
	?	11	28	1.4	10	10	1.0	0.1
FG Unclear		44		5.5	23	23	1.9	0.4
No T Sym		81		10.0	26	26	3.1	0.8
Total		808		100.0	99	100		8.1

Table B2.1 - Frieze Group Results of the Secondary Students' Restricted Pattern Construction

Frieze Group	Numeracy	% of Total Patterns	No. of Students	% of Students	Av. No. Made*	Av. No. Made
p111	188	25	64	75	2.9	2.2
p1m1	40	5	31	36	1.3	0.5
pm11	105	14	48	56	2.2	1.2
p1a1	9	1	9	11	1.0	0.1
p112	141	18	63	74	2.2	1.7
pmm2	89	12	49	58	1.8	1.0
pma2	79	10	46	54	1.7	0.9
No T. Sym	84	11	32	38	2.6	1.0
FG Unclear	28	4	20	24	1.2	0.3
Total	763	100	85	100		9.0
Different FGs	304					3.6

Table B2.2 - A Frequency Breakdown of the Secondary Students' First Three Restricted Patterns

Frieze Group	% Drawn 1st	% Drawn 2nd	% Drawn 3rd	% of Total Patterns
p111	15	16	24	25
p1m1	0	7	7	5
pm11	15	11	11	14
p1a1	0	2	0	1
p112	47	29	19	18
pmm2	2	8	13	12
pma2	8	13	16	10
Can't Classify	12	12	11	15

Table B2.3 - A 'Style' Classification of the Secondary Students' Restricted Pattern Construction

Style	Numeracy	% of Total Patterns	No. of Students	% of Students
Discrete	157	24	42	49
Non-Discrete	36	6	21	25
Touching	150	23	55	65
Tiling/Filamentary	264	41	69	81
Unsure (?)	44	7	18	21

Table B2.4 - Frieze Group/Style Results of the Secondary Students' Restricted Pattern Construction

Frieze Group	Style	Numeracy	% of FG	% of Total Patterns	No. of Students	% of Students	Av. No. Made*	Av. No. Made
p111	PS1	70	37	9.2	31	36	2.3	0.8
	NDP	28	15	3.7	17	20	1.6	0.3
	Tou	63	34	8.3	41	48	1.5	0.7
	Til/Fil	20	11	2.6	12	14	1.7	0.2
	?	7	4	0.9	7	8	1.0	0.1
p1m1	PS3	1	3	0.1	1	1	1.0	0.0
	PS4	0	0	0.0	0	0		0.0
	NDP	1	3	0.1	1	1	1.0	0.0
	Tou	13	33	1.7	13	15	1.0	0.2
	Til/Fil	18	45	2.4	13	15	1.4	0.2
	?	7	18	0.9	6	7	1.2	0.1
pm11	PS5	27	26	3.5	13	15	2.1	0.3
	PS6	20	19	2.6	13	15	1.5	0.2
	NDP	3	3	0.4	3	4	1.0	0.0
	Tou	39	37	5.1	28	33	1.4	0.5
	Til/Fil	6	6	0.8	5	6	1.2	0.1
	?	10	10	1.3	9	11	1.1	0.1
p1a1	PS2	4	44	0.5	4	5	1.0	0.0
	NDP	1	11	0.1	1	1	1.0	0.0
	Tou	4	44	0.5	4	5	1.0	0.0
	Til/Fil	0	0	0.0	0	0		0.0
	?	0	0	0.0	0	0		0.0
p112	PS7	12	9	1.6	10	12	1.2	0.1
	PS8	9	6	1.2	8	9	1.1	0.1
	NDP	1	1	0.1	1	1	1.0	0.0
	Tou	4	3	0.5	4	5	1.0	0.0
	Til/Fil	113	80	14.8	51	60	2.2	1.3
	?	2	1	0.3	2	2	1.0	0.0
pmm2	PS12	2	2	0.3	1	1	2.0	0.0
	PS13	0	0	0.0	0	0		0.0
	PS14	1	1	0.1	1	1	1.0	0.0
	PS15	4	4	0.5	3	4	1.3	0.0
	NDP	2	2	0.3	2	2	1.0	0.0
	Tou	24	27	3.1	19	22	1.3	0.3
	Til/Fil	47	53	6.2	31	36	1.5	0.6
	?	9	10	1.2	7	8	1.3	0.1
pma2	PS9	5	6	0.7	4	5	1.3	0.1
	PS10	1	1	0.1	1	1	1.0	0.0
	PS11	1	1	0.1	1	1	1.0	0.0
	NDP	0	0	0.0	0	0		0.0
	Tou	3	4	0.4	3	4	1.0	0.0
	Til/Fil	60	76	7.9	36	42	1.7	0.7
	?	9	11	1.2	7	8	1.3	0.1
FG Unclear		28		3.7	20	24	1.4	0.3
No T. Sym		84		11.0	32	38	2.6	1.0
Total		679		89.0	85	100		8.0

Table B3.1 - Frieze Group Results of the Tertiary Students' Restricted Pattern Construction

Frieze Group	Numeracy	% of Total Patterns	No. of Students	% of Students	Av. No. Made*	Av. No. Made
p111	385	42	69	100	5.6	5.6
p1m1	14	2	9	13	1.6	0.2
pm11	167	18	58	84	2.9	2.4
p1a1	12	1	8	12	1.5	0.2
p112	114	12	52	75	2.2	1.7
pmm2	45	5	27	39	1.7	0.7
pma2	51	6	34	49	1.5	0.7
Unclear	21	2	15	22	1.4	0.3
No T. Sym	116	13	42	61	2.8	1.7
Total	925	100	69	100		13.4
Different FGs	259					3.8

Table B3.2 - A Frequency Breakdown of the Tertiary Students' First Three Restricted Patterns

Frieze Group	% Drawn 1st	% Drawn 2nd	% Drawn 3rd	% of Total Patterns
p111	35	36	34	42
p1m1	0	0	3	2
pm11	23	19	28	18
p1a1	0	3	4	1
p112	19	26	7	12
pmm2	4	0	4	5
pma2	6	1	4	6
Can't Classify	13	15	15	15

Table B3.3 - A 'Style' Classification of the Tertiary Students' Restricted Pattern Construction

Style	Numeracy	% of Total Patterns	No. of Students	% of Students
Discrete	298	38	45	65
Non-Discrete	181	23	48	70
Touching	213	27	40	58
Tiling/Filamentary	79	10	31	45
Unsure (?)	17	2	11	16

Table B3.4 - Frieze Group/Style Results of the Tertiary Students' Restricted Pattern Construction

Frieze Group	Style	Numeracy	% of FG	% of Total Patterns	No. of Students	% of Students	Av. No. Made*	Av. No. Made
p111	PS1	120	31	13.0	48	70	2.5	1.7
	NDP	150	39	16.2	44	64	3.4	2.2
	Tou	107	28	11.6	38	55	2.8	1.6
	Til/Fil	3	1	0.3	3	4	1.0	0.0
	?	5	1	0.5	4	6	1.3	0.1
p1m1	PS3	2	14	0.2	2	3	1.0	0.0
	PS4	3	21	0.3	2	3	1.5	0.0
	NDP	1	7	0.1	1	1	1.0	0.0
	Tou	4	29	0.4	4	6	1.0	0.1
	Til/Fil	2	14	0.2	2	3	1.0	0.0
	?	2	14	0.2	2	3	1.0	0.0
pm11	PS5	50	30	5.4	27	39	1.9	0.7
	PS6	30	18	3.2	19	28	1.6	0.4
	NDP	21	13	2.3	17	25	1.2	0.3
	Tou	61	37	6.6	25	36	2.4	0.9
	Til/Fil	2	1	0.2	2	3	1.0	0.0
	?	3	2	0.3	3	4	1.0	0.0
p1a1	PS2	6	50	0.6	4	6	1.5	0.1
	NDP	1	8	0.1	1	1	1.0	0.0
	Tou	4	33	0.4	3	4	1.3	0.1
	Til/Fil	1	8	0.1	1	1	1.0	0.0
	?	0	0	0.0	0	0		0.0
p112	PS7	31	27	3.4	22	32	1.4	0.4
	PS8	23	20	2.5	16	23	1.4	0.3
	NDP	3	3	0.3	3	4	1.0	0.0
	Tou	5	4	0.5	5	7	1.0	0.1
	Til/Fil	51	45	5.5	23	33	2.2	0.7
	?	1	1	0.1	1	1	1.0	0.0
pmm2	PS12	6	13	0.6	5	7	1.2	0.1
	PS13	0	0	0.0	0	0		0.0
	PS14	2	4	0.2	2	3	1.0	0.0
	PS15	5	11	0.5	5	7	1.0	0.1
	NDP	1	2	0.1	1	1	1.0	0.0
	Tou	26	58	2.8	13	19	2.0	0.4
	Til/Fil	2	4	0.2	2	3	1.0	0.0
	?	3	7	0.3	3	4	1.0	0.0
pma2	PS9	13	25	1.4	12	17	1.1	0.2
	PS10	3	6	0.3	3	4	1.0	0.0
	PS11	4	8	0.4	4	6	1.0	0.1
	NDP	4	8	0.4	4	6	1.0	0.1
	Tou	6	12	0.6	5	7	1.2	0.1
	Til/Fil	18	35	1.9	13	19	1.4	0.3
	?	3	6	0.3	3	4	1.0	0.0
FG Unclear		21		2.3	15	22	1.4	0.3
No T. Sym		116		12.5	42		2.8	1.7
Total		925		100.0	69	100		13.4

Appendix C: Interview Cards

The lists of patterns which follow are the ones which the students were asked to describe. The C1 patterns are from activity (c) in which written descriptions were made. Some Primary students were selected for interviews, and oral explanations were asked for. Each of the 7 activity (c) patterns was mounted on a green card and the interviewees asked to look for any similarities between them. Later, the C2 patterns (mounted on red cards) were shown to the interviewees only and compared with the 'green' patterns. A similar matching exercise was undertaken with the 'green' and the 7 continuous, filamentary (beige) patterns, shown here in appendix C3.

C1 - Discrete Examples of the Seven Frieze Groups (GREEN)[1]

p111 [PS1]

p1m1 [PS3]

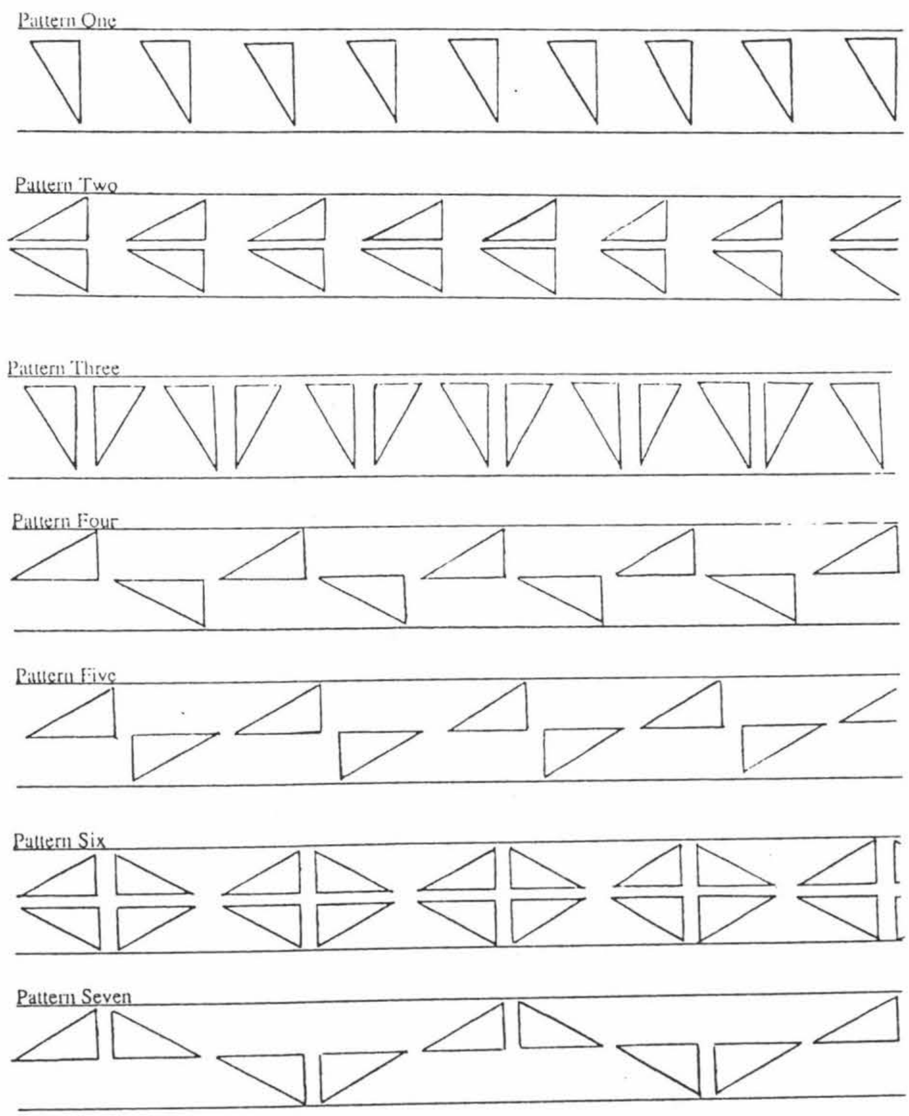
pm11 [PS5]

p1a1 [PS2]

p112 [PS7]

pmm2 [PS12]

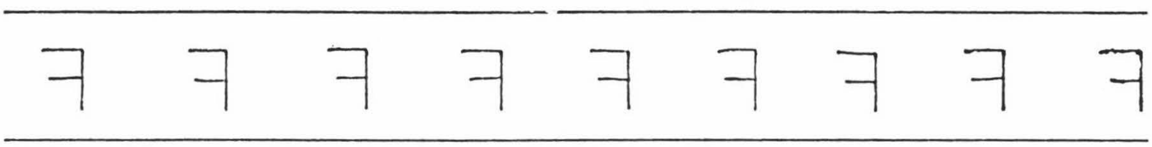
pma2 [PS9]



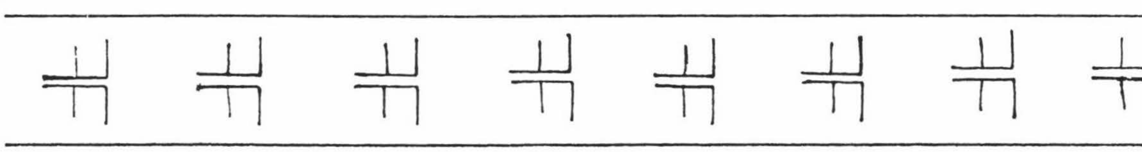
[1] The patterns shown are reduced from their original size.

C2 - Another Set of Discrete Examples of the Seven Frieze Groups (RED)

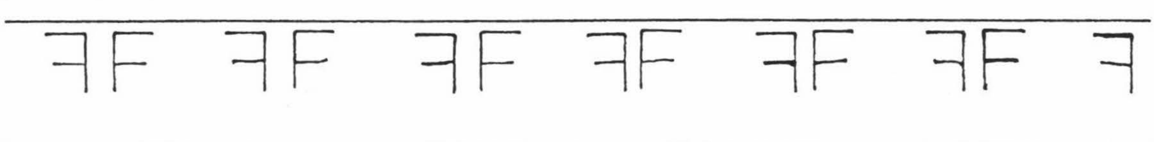
p111 [PS1]



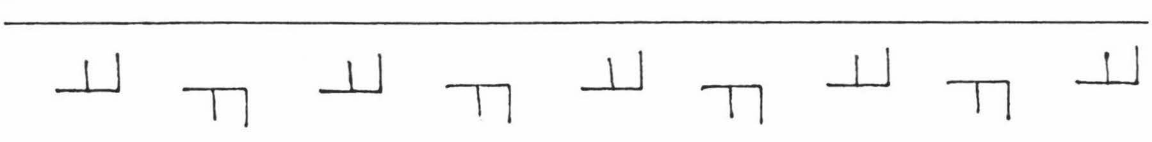
p1m1 [PS3]



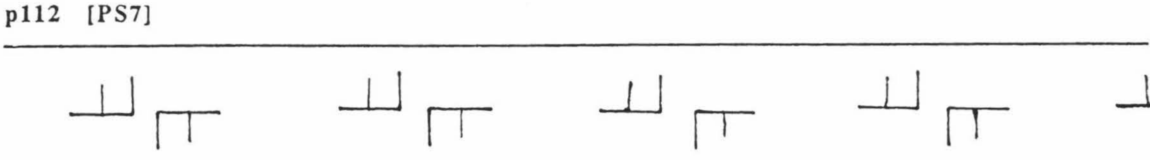
pm11 [PS5]



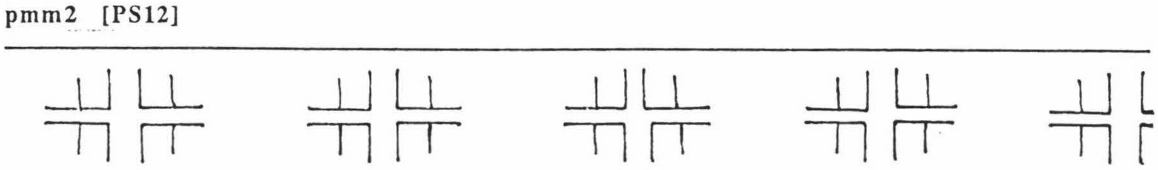
p1a1 [PS2]



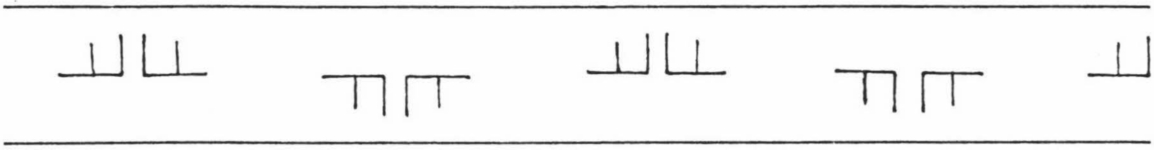
p112 [PS7]



pmm2 [PS12]

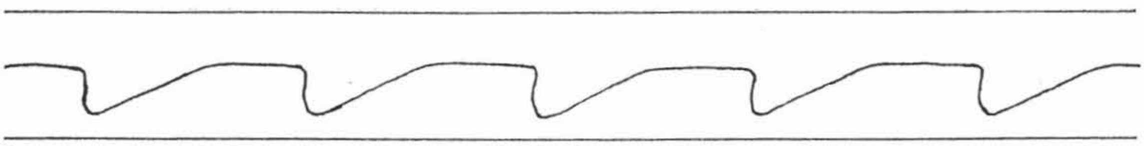


pma2 [PS9]

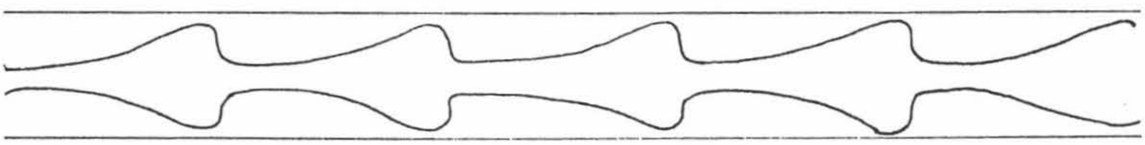


C3 - Continuous, Filamentary Examples of the Seven Frieze Groups (BEIGE)

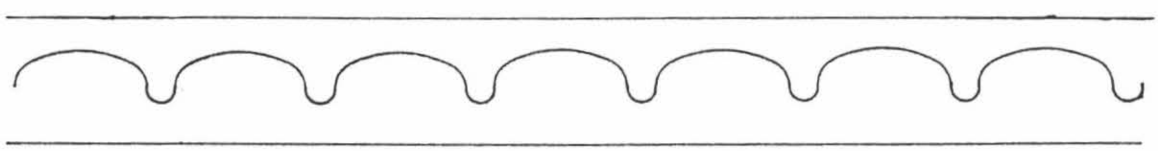
p111



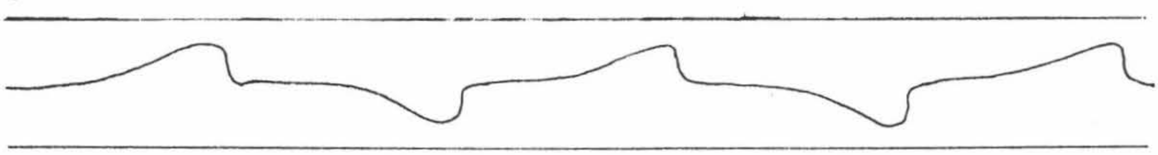
p1m1



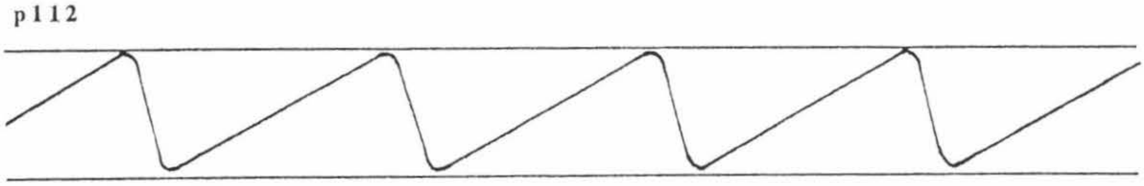
pm11



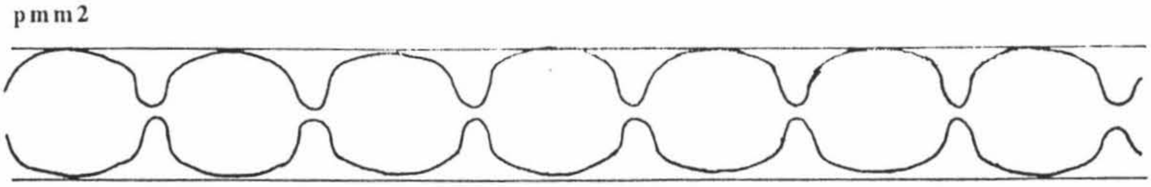
p1a1



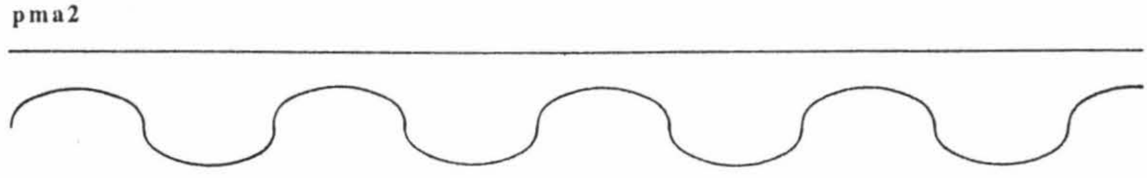
p112



pmm2



pma2



Appendix D: Transformation Geometry Resources

It would be an oversight not to list and describe some of the ideas and resources that can be utilised by teachers in the area of transformation geometry, and in particular, in pattern making or pattern perception. The list of articles below is by no means exhaustive; its purpose is to indicate some of the breadth possible in approaching this topic. Since the emphasis of this thesis is on the nine and ten year old subjects, the activities given below are predominantly aimed at the Primary level, although not exclusively.

D1 Transformations and Symmetry of Finite Objects

1. DIENES, Z. AND GOLDING, E. (1975) *Exploration of Space and Practical Measurement* (6th Ed.), Education Supply Association, Harlow, Essex.

On pages 15-21, the authors outline some activities and games for young children that involve action and attention to reflection and rotation. No theory is introduced. The set up of the games is the sketching of a large shape on the floor with two lines of symmetry, one symmetry line coloured green (say), the other red. A child, standing in the centre with a 'flipper' or 'turner' (a piece of wood similarly marked, but not coloured), directs the other four children (by flipping or rotating the flipper), each of whom is in one of the four regions created by the mirror lines. Developments of this game are made by using mirrors for 'experiments'. They conclude:

> "The games and activities described are meant for opening doors for children to different areas of enquiry, so that when the time comes, the development of the ideas should be based on situations which have become old friends."

2. WOODWARD, E. AND BUCKNER, P. (1987) Reflections and symmetry: a second-grade miniunit. *Arithmetic Teacher*, **35 (2)**, 8-11.

This article describes the activities of a five day miniunit for second-graders, using "Mira's" to explore reflection symmetry. It is a development of Woodward's earlier article in Arithmetic Teacher (volume 24, Feb. 1977, p117). At the end of the miniunit, they believe that each student had a firm grasp of what it means for a figure to have line symmetry and were very enthusiastic about the use of the Mira.

3. EGSGARD, J. (1970) Some ideas in geometry that can be taught from K-6. *Educational Studies in Mathematics*, 478-495.

After outlining activities to explore shape and area, Egsgard describes (on pages 489-493) activities suitable for studying transformation geometry from kindergarten level to grade six (form two). These include the folding of paper shapes, exploiting the tile patterns from the study of area, investigating pattern work in arts and crafts, blot patterns, sliding a cardboard shape (without turning) about a pattern to reinforce congruence and translation. Once the concepts of translation, reflection and rotation have been grasped from various activities, he suggests using various patterns to emphasise the differences between the three. Again, this could be achieved by using a cardboard shape and investigating how it needs to be transformed to get from one position to another. Discussion is also necessary. Examination of the properties of various plane figures could also include these concepts.

4. "Ideas" (1989) What's your name? *Arithmetic Teacher*, **37** (**1**), 26-28.

> "Students are asked to determine numerical and fractional relationships, as well as identify vertical and horizontal symmetries of names drawn from children's literature. They are then asked to compare the attributes of their own name to those of the names of the characters listed"

5. RENSHAW, B. (1986) Symmetry the trademark way. *Arithmetic Teacher*, **34** (**1**), 6-12.

Before examining trademarks, Renshaw explains that students should clearly understand line, rotational (and point, or half-turn) symmetry. Practical activities such as tracing and folding or rotating, or using a Mira can be helpful for these concepts. The tracing/folding or tracing/turning technique is then used in the investigation of trademarks. Questions to explore include: Can one trademark have more than one type of symmetry?

6. JURASCHEK, W. AND MCGLATHERY, G. (1980) Funny Letters: A discrepant event. *Arithmetic Teacher*, **27** (**8**), 43-47.

When a test tube is held horizontally with the word "SULPHURDIOXIDE" facing the observer and is rolled 180° about its horizontal axis, the word SULPHUR is inverted, but DIOXIDE is not. This 'magic' is used as an impelling starting point for exploring the symmetry of letters.

7. CRAIG, J. (1989) Geometry through Logo, *The Mathematics Teacher*, **129**, 36-39.

While this article does not explicitly mention a transformation geometry framework, exercises include the drawing of shapes, repeating, enlarging or rotating them using a logo programme. The motivation this induces in the students is tangible.

8. HOLCOMB, J. (1980) Using geoboards in the primary grades. *Arithmetic Teacher*, **27** **(8)**, 22-25.

Halcomb describes some exercises for Primary school children in counting, area, symmetry and pentominoes using geoboards. Records of symmetric figures can be kept on dot paper. He also recommends investigations of turning and matching figures with two geoboards. His article winds up by noting that not only are geoboards virtually indestructable, many children are motivated by this 'game' approach.

D2 Symmetry and Patterns

1. KAISER, B. (1988) Explorations with tessellating polygons. *Arithmetic Teacher*, **36** **(4)**, 19-24.

This article list three activities with extensions for each. The first involves determining which regular polygons will tesselate on a sheet of paper. The extension suggests exploring the reasons why a shape will or will not tile, providing an excellent application for angle measurement. In addition, it is also interesting to explore non-regular polygons, or using two or more regular polygons. The second activity uses letters of the alphabet. Made into polygons, letters such as C, L, S, and T form tilings. This presents students with the opportunity to apply their knowledge of transformations in describing their pattern constructions. The third activity describes a way of modifying a shape which is known to tessellate so that the resulting figure will also tessellate, yielding Escher-type patterns. An extension of this method for Secondary students can be found in Sheila Haak's (Dec., 1976) article in Mathematics Teacher, **69**, 647-652 entitled "Transformation Geometry and the Artwork of M. C. Escher."

2. DELANEY, K. (1979) A place for space. *Mathematics Teaching*, **86**, xvii-xxvii.

This text emphasizes shape work investigations and is comprised of three articles which outline starting points that worked for some teachers. The first, entitled "Do-it-yourself Islamic patterns" by Kev Delaney and John Dichmont, describes the activity of drawing a design on 2cm square paper, then making a large pattern by repeating it. To the teacher's surprise, most children did not get bored by repeating the basic unit but were fascinated to see the pattern gradually evolve. The analysis of one child's design led to an irrefutable conclusion:

> "we were forced to acknowledge the complexity of the thinking involved, and to recognise the elegance of Natalie's final design."

The second article entitled "Tesselating Circles" by Kev Delaney suggests changing circles (or other non-tesselating figures) into an interesting variety of shapes which do tessellate. One method is by starting with a circle grid and outlining a suitable shape for a motif. Children could make Escher-like patterns by marking the motifs. One difficulty is that it is not always clear whether a shape will tessellate.

The last of the three aticles is called "Thought and Action in a Primary School Classroom" by Michael Armstrong and describes the pleasure and excitement of Sally who made a tetrahedron from ball bearings magnetically held in place. This led her to asking how many balls there were in the stack, and so on. In turn, this led to the writer presenting her with a box of (five-pointed) red sticky paper stars and suggesting that Sally and her friend Helen try arranging them in patterns on paper. The writer reports his own astonishment at the virtuosity that the resulting patterns revealed.

3. KNIGHT, G. (1984a) The geometry of Maori art - rafter patterns. *New Zealand Mathematics Magazine*, **21(3)**, 36-40.

Knight explains how kowhaiwhai (Maori rafter patterns) can be classified mathematically according to seven symmetry classes. He suggests using this classification of kowhaiwhai in a local marae or museum as a project for students, the motivation being for students to relate the mathematics they learn to their cultural heritage.

4. ZASLAVSKY, C. (1990) Symmetry in American Folk Art. *Arithmetic Teacher*, **38 (1)**, 6-12.

This article describes the analysis and construction methods that students could use to make American quilt designs or to make rugs similar to the ones made by the Navajo. The symmetry of motifs used is emphasized. Because studies show that girls are socialised to care more about people, it is argued that these activities which integrate mathematics with culture, art and history may motivate more girls to become interested in mathematics. It is interesting to note that Mary Barnes, in her keynote address entitled *Making Mathematics Girl Friendly* at the *1991 Maths Across the Spectrum* conference in New Zealand, formed a similar conclusion about the exploration of embroidery frieze patterns.

5. SAWADA, D. (1985) Symmetry and tessellations from rotational transformations on transparencies. *Arithmetic Teacher*, **33 (4)**, 12-13.

By superimposing two transparencies, each furnishing a dot grid, interesting 'tessellations' arise. Various transformations of one grid with respect to the other yield different patterns which could form a basis for investigation.

6. "Ideas" (1990) Make your own row patterns. *Arithmetic Teacher*, **37 (6)**, 30.

Students are required to make row patterns from a single shape within the lines provided on a sheet. Students are encouraged to describe what happens to the shape as the pattern is continued.

7. EBA, P. (1979) Space filling with solid polyominoes. *Mathematics in School*, **8 (2)**, 2-5.

Many writers explore ways of filling two-dimensional space with different types of polygonal cells. In contrast, Eba indicates the richness of exploring three-dimensional tesselations by using pentominoes as an example.

8. BIDWELL, J. (1987) Using reflections to find symmetric and asymmetric patterns. *Arithmetic Teacher*, **34 (7)**, 10-15.

Bidwell outlines a way of using four coloured squares, a 3x3 grid, and a mirror to achieve five goals: (a) 'move' the pattern with the mirror; (b) finding patterns with one, two or four lines of symmetry; (c) finding asymmetrical patterns; (d) deciding if patterns are different; and (e) working out all the patterns possible.

9. RANSOM, P. (1988) Interfering with Islamic patterns. *Mathematics in School*, **17 (4)**, 2-6.

A standard method for producing an Islamic pattern on a square lattice of dots is given, followed by variations. Ransom found that pupils find these patterns fairly easy to produce (using tracing paper to check results) and practise transformations such as translation, rotation, and reflection. He also 'discovers', like Suwada (1985) above, that the superimposition of two OHP slides with these dot grids produces interference patterns and suggests that it could be used as an investigation.

10. THOMPSON, C. AND VAN DE WALLE, J. (1985) Patterns and Geometry with Logo. *Arithmetic Teacher*, **32** (**7**), 6-13.

Beginning with concrete objects to make linear patterns with, other non-computer activities are introduced. Using a logo program (listed at the end of the article) students can construct their own figures and then linear patterns. A similar approach is made for the two-dimensional patterns where students use non-computer materials such as geoboards and grid patterns. An explanation of ways of repeating a figure in the logo environment is given to allow the construction of complex figures and two-dimensional patterns.

11. WILLIAMS, H. (1989) Classifying Greek patterns, *Micromath*, **5** (**1**), 22-24.

The focus in this article is on linear patterns. It also discusses all four rigid transformations. A method for classifying the patterns (into the seven frieze groups) is given and a logo program is listed which allows filamentary linear patterns to be constructed.

12. OLIVER, J. (1979) Symmetry and Tessellations. *Mathematics in School*, **8** (**1**), 2-5.

Methods are given for altering a square to produce interesting tilings, which can be used as a basis for exploring the applications of mathematics in art and design.

13. ROBERTSON, A. (1989) *Patterns of Polynesia: New Zealand*. Heinemann Education, Auckland.

General descriptions of the meanings of some Maori patterns, including kowhaiwhai, are given. Activities are outlined for students to construct their own patterns.

Appendix E: Activity Sheets (a), (b) and (c)

This appendix displays the three survey activities used in this study.

Age:

y Sheet (a): *Please fill in as many of the following strips as you can to make different* ***repeating*** *patterns. (Remember:* ***anything*** *goes!)*

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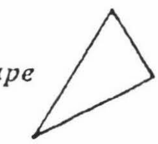
Age:

Sheet(b): *Please fill in as many of the following strips as you can using* ***only*** *the* ***shape*** *(in any way you wish) to make different* ***repeating*** *patterns.*



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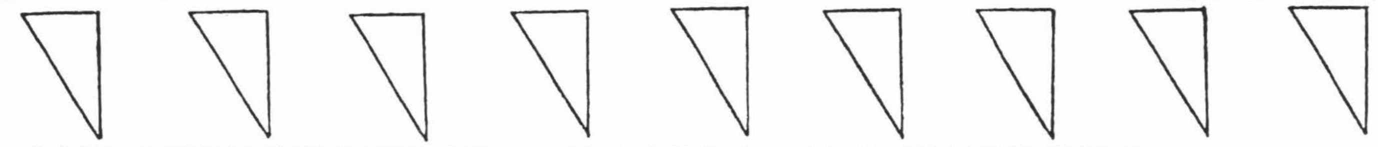
Name: Age:

Activity Sheet (c): ***Describe** the following patterns.*

Pattern One



Description:..

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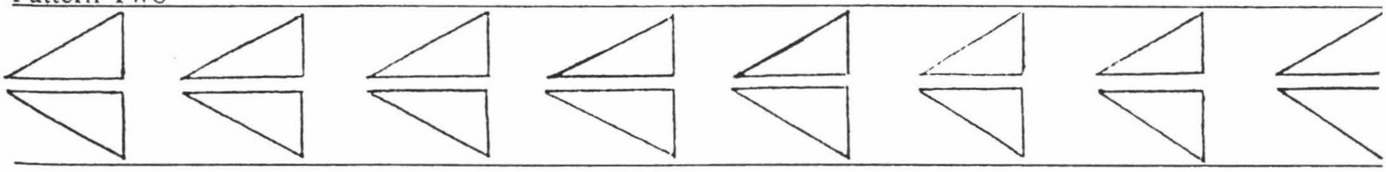
Pattern Two



Description:..

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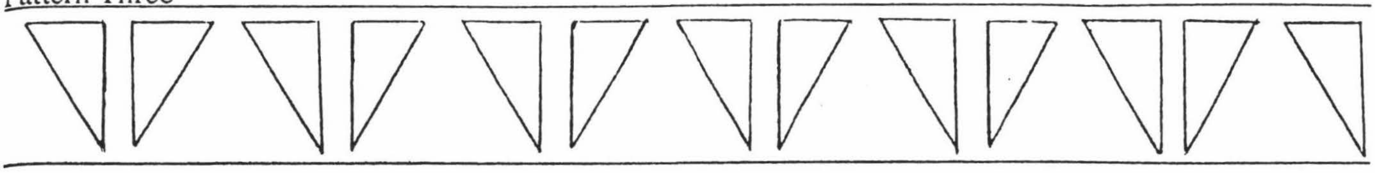
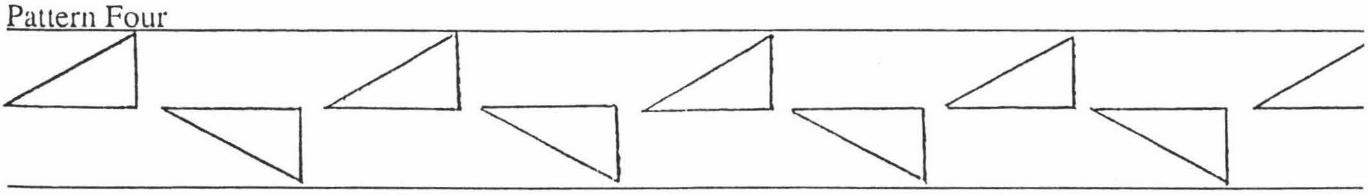
Pattern Three



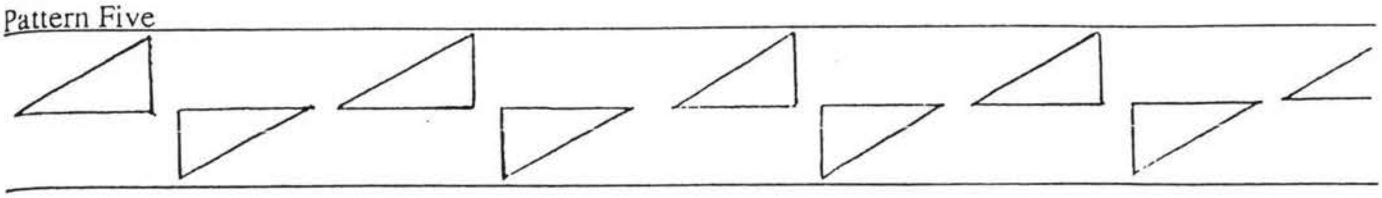
Description:...

Pattern Four



Description:...

Pattern Five



Description:..

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Pattern Six

Description:..

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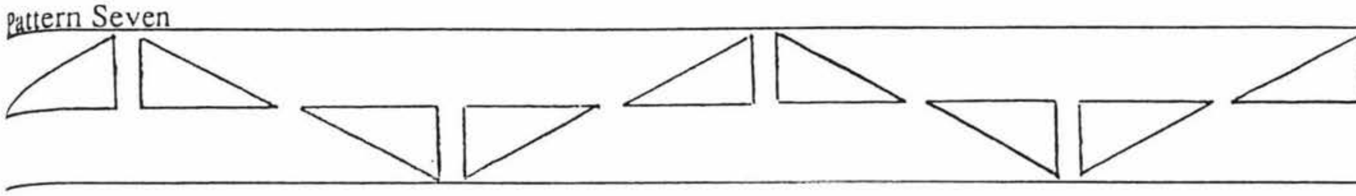
Pattern Seven



Description:

Which patterns did you find the hard to describe? (list their numbers and say why if you can)

Which patterns did you find easy to describe? (list their numbers and say why if you can)

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